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EXTREMAL VALUES OF $\Delta(x,N) = \sum_{\substack{n<xN \\ (n,N)=1}} 1 - x\varphi(N)$

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ABSTRACT. The function $\Delta(x,N)$ as defined in the title is closely associated via $\Delta(N) = \sup_x |\Delta(x,N)|$ to several problems in the upper bound sieve. It is also known via a classical theorem of Franel that certain conjectured bounds involving averages of $\Delta(x,N)$ are equivalent to the Riemann Hypothesis. We improve the unconditional bounds which have been hitherto obtained for $\Delta(N)$ and show that these are close to being optimal. Several auxiliary results relating $\Delta(Np)$ to $\Delta(N)$, where p is a prime with $p \nmid N$, are also obtained and two new conjectures stated.

Introduction. The function $\Delta(x,N)$ is defined for $x \in \mathbf{R}$ and $N > 1$ by

$$\Delta(x,N) = \sum_{\substack{n\le xN \\ (n,N)=1}} 1 - x\varphi(N)$$

where $\varphi(N)$ is Euler's function. Clearly $\Delta(x,N)$ is periodic, as a function of x, of period 1 with $\Delta(0,N) = 0$ and $\Delta(x,N) = \Delta(\{x\},N)$ where $\{x\} = x - [x]$. Further, if

$$\bar{N} = \prod_{p|N} p,$$

then writing $N = \bar{N}L$, we obtain that

$$\Delta(x,N) = \sum_{\substack{n\le xL\bar{N} \\ (n,\bar{N})=1}} 1 - xL\varphi(\bar{N}) = \Delta(xL,\bar{N}).$$

Hence as far as bounds uniform in x are concerned, we can restrict ourselves to *squarefree* $N > 1$ which will be assumed from now onwards. We shall also always use p and q to indicate prime numbers.

It is easy to see that

$$\Delta(x,N) = -\mu(N)\sum_{d|N}\mu(d)\{xd\}, \tag{1}$$

where μ is the Möbius function and indeed one can also show that

$$\Delta(x,N) = -\sum_{\substack{k \bmod N \\ (k,N)=1}} \left(\left\{x + \frac{k}{N}\right\} - \frac{1}{2}\right).$$

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Certain mean-square estimates for $\Delta(x,N)$ are equivalent to the Riemann Hypothesis. Indeed, as shown by Franel [4], the Riemann Hypothesis is equivalent to the estimate

$$\sum_{n\leq\Phi(N)}\left(q_n-\frac{n}{\Phi(N)}\right)^2=O\left(N^{-1+\varepsilon}\right)$$

where q_n indicates the n-th Farey fraction of order N, $\Phi(N)=\sum_{q\leq N}\varphi(q)$ and $\varepsilon>0$. On noting that

$$\sum_{q\leq N}\Delta(q_n,q)=n-q_n\Phi(N),$$

Franel's equivalence can be rephrased as

$$\sum_{n\leq\Phi(N)}\Big(\sum_{q\leq N}\Delta(q_n,q)\Big)^2=O\left(N^{3+\varepsilon}\right).$$

Further, we also observe that for $N=\prod_{p\leq t}p$, large fluctuations of $\Delta(x,N)$ correspond to an abundance or paucity of integers with smallest prime factor $>t$ over their expected numbers in appropriate intervals. These correspond to limitations in anticipated sieve upper bound estimates in short ranges.

We define

$$\Delta(N)=\sup_x|\Delta(x,N)|.$$

Trivially, we have that

$$|\Delta(x,N)|=\Big|\sum_{d|N}\mu(d)\left(\{xd\}-\frac{1}{2}\right)\Big|\leq\frac{1}{2}\sum_{d|N}1$$

so that $\Delta(N)\leq 2^{\omega(N)-1}$, where $\omega(N)$ is the number of prime factors of N. Vijayaraghavan [11] showed that this is best possible. More precisely, he showed that given any $\varepsilon>0$, $\Delta(N)\geq 2^{\omega(N)-1}-\varepsilon$ for an infinite sequence of N with $\omega(N)\longrightarrow\infty$. For an alternative proof, see also Lehmer [6].

One can also obtain upper bounds for $\Delta(N)$ with an explicit dependence on the prime factors of N. Suryanarayana [9] proved that

$$\Delta(N)\leq 2^{\omega(N)-1}-\prod_{p|N}\left(1+\frac{1}{p}\right)+1. \tag{2}$$

This is sharp when N is prime. It is an easy consequence of (1) that if $p\nmid N$ then

$$\Delta(x,Np)=\Delta(px,N)-\Delta(x,N), \tag{I}$$

and hence $\Delta(Np)\leq 2\Delta(N)$. Iterating this, we obtain

$$\Delta(N)\leq 2^{\omega(N)-1}\Delta(q)$$

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for any prime factor q of N. Since $\Delta(q) = 1 - 1/q$, we deduce that

$$\Delta(N) \leq 2^{\omega(N)-1}\left(1 - \frac{1}{p_1}\right) \tag{3}$$

where p_1 is the smallest prime factor of N. Apart from the cases $N = 6$ and N prime when both bounds are equal, it is a simple induction exercise to confirm that (3) is always an improvement over (2). In our Theorem 1, we shall improve the bound $\Delta(Np) \leq 2\Delta(N)$ to

$$\Delta(Np) \leq 2\Delta(N) - \frac{1}{p} \qquad (p \nmid N)$$

which leads to an even stronger upper bound for $\Delta(N)$ in which all the prime factors of N play a role. Our Theorem 2 shows that for a certain class of integers N,

$$\Delta(N) \geq 2^{\omega(N)-1} - \frac{2^{\omega(N)}}{p_1 + 1}$$

which essentially differs from (3) by only a factor of 2.

It is a well-known result that

$$\int_0^1 \Delta^2(x, N)\, dx = \frac{1}{12}\,\frac{2^{\omega(N)}\varphi(N)}{N}.$$

Three different proofs of this may be found in Delange [1], van Hamme [10] and Perelli-Zannier [8]. For ease of reference, we include another short proof in Theorem 4(v). As observed in [8], this integral immediately yields that

$$\Delta(N) \geq \left(\frac{1}{12}\frac{2^{\omega(N)}\varphi(N)}{N}\right)^{\frac{1}{2}}.$$

In Theorem 3, we shall exploit the integral in a different manner to obtain the slight sharpening

$$\Delta(N) \geq \left(\frac{1}{12}\,\frac{2^{\omega(N)}\varphi(N)}{N} - \frac{1}{12}\right)^{\frac{1}{2}} + \frac{1}{2}.$$

This bound is actually attained for $N = 2, 3$ and 6.

Our final Theorem 4 consists of auxiliary results and simpler proofs of two known results.

For integers N which are divisible by a prime p, $p \equiv 1 (\mathrm{mod}\, k)$, $k \in \mathbf{N}$, Lehmer [6] showed that for any $a \in \mathbf{Z}$, the number of n in the interval $(aN/k, (a+1)N/k]$ with $(n, N) = 1$ is precisely $\varphi(N)/k$. Necessary and sufficient conditions on N under which this is valid were further investigated by McCarthy [7] and Erdös [2],[3]. In Theorem 4(i), we give a simpler proof of Lehmer's result based on the above identity (**I**). Different applications of this identity combined with a classical theorem of Landau on fractional parts also yield (Theorem 4(ii), (iii)) that

$$\Delta(2N) = \Delta(N)$$

for all odd $N > 1$ and the lower bound for $p \nmid N$,

$$\Delta(Np) \geq \left(1 - \frac{1}{p}\right) \Delta(N).$$

A reasonable conjecture would be that $\Delta(Np) \geq \Delta(N)$ for all $N > 1$ and $p \nmid N$. We also conjecture that if N is the product of the first s primes then

$$\Delta(N) \leq 2^{s-1} \frac{\varphi(N)}{N}$$

and have confirmed this by direct calculation for $s \leq 8$.

Statements of Theorems.

THEOREM 1. *For any squarefree $N > 1$ and a prime p with $p \nmid N$, we have*

$$(i) \qquad \Delta(Np) \leq 2\Delta(N) - \frac{1}{p}$$

In fact, the sharper but more awkward bound

$$(ii) \qquad \Delta(Np) \leq 2\Delta(N) - \frac{(l+1)}{p} \frac{\varphi(N)}{N} + \max\left(0, \frac{\varphi(N)}{Np} + \frac{l\varphi(N)}{N} - 1\right),$$

where $l = \left[\frac{N}{\varphi(N)}\right]$, also holds.

COROLLARIES.

(i) For primes p and q with $p > q \geq 3$,

$$\Delta(pq) \leq 2\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right).$$

(ii) For any $s \in \mathbf{N}$ and distinct primes $p_s > p_{s-1} > \cdots > p_1$,

$$\Delta(p_1 \ldots p_s) \leq 2^{s-1} - \sum_{i=1}^{s} \frac{2^{s-i}}{p_i}.$$

If $p_1 = 2$ and $s \geq 2$, this can be sharpened to

$$\Delta(p_1 \ldots p_s) \leq 2^{s-2} - \sum_{i=2}^{s} \frac{2^{s-1-i}}{p_i}.$$

REMARKS. (a) The two inequalities in Theorem 1 are, in fact, equalities when $N = 2$ and p is any odd prime.

(b) The bound in Corollary (i) is an equality when $q = 3$ and $p \equiv 1 (\mathrm{mod}\, 6)$ (*cf.* Theorem 4(iv)).

(c) Corollary (ii) is obtained by using Theorem 1(i). By using Theorem 1(ii) instead, we can obtain a slight improvement in this corollary. Indeed, further small improvements can be obtained by incorporating Corollary (i) into the argument.

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(d) Corollary (ii) shows that given s primes $p_1 < \cdots < p_s$ in some interval $[X,(1+\varepsilon)X]$, where $\varepsilon > 0$, we have that

$$\Delta(p_1 \dots p_s) \le 2^{s-1} - \frac{1}{(1+\varepsilon)}\frac{2^s}{p_1} + \frac{1}{(1+\varepsilon)p_1}.$$

THEOREM 2. *Let $k \in \mathbf{N}$ and let N be composed of primes p with $p \equiv -1 (\mathrm{mod}\, k)$. Then*

$$\Delta(N) \ge 2^{\omega(N)-1}\left(\frac{k-2}{k}\right).$$

In particular, given any prime p, all N with smallest prime factor p and with all other prime factors q satisfying $q \equiv -1\big(\mathrm{mod}(p+1)\big)$ has

$$\Delta(N) \ge 2^{\omega(N)-1}\left(1 - \frac{2}{p+1}\right).$$

THEOREM 3. *For any $N > 1$, we have*

(*i*)
$$\frac{1}{\varphi(N)}\sum_{i=1}^{\varphi(N)} \Delta^2\left(\frac{a_i}{N}, N\right) = \frac{1}{12} 2^{\omega(N)}\frac{\varphi(N)}{N} + \frac{1}{6}$$

(*ii*)
$$\Delta(N) \ge \left(\frac{1}{12}2^{\omega(N)}\frac{\varphi(N)}{N} - \frac{1}{2}\right)^{\frac{1}{2}} + \frac{1}{2}.$$

THEOREM 4.

*(i) (*LEHMER*) Let N be a squarefree integer which is divisible by a prime p, $p \equiv 1 (\mathrm{mod}\, k)$ and $k \in \mathbf{N}$. Then for any $a \in \mathbf{Z}$,*

$$\sum_{\substack{\frac{aN}{k} < n \le \frac{(a+1)N}{k} \\ (n,N)=1}} 1 = \frac{1}{k}\varphi(N).$$

(ii) $\Delta(2N) = \Delta(N)$ *for any odd* $N > 1$.

(iii) $\Delta(Np) \ge \left(1 - \frac{1}{p}\right)\Delta(N)$ *for any $N \in \mathbf{N}$ and prime p with $p \nmid N$.*

(iv) $\Delta(3p) = \begin{cases} \frac{4}{3} - \frac{2}{p}, & p \equiv -1 (\mathrm{mod}\, 6) \\ \frac{4}{3}\left(1 - \frac{1}{p}\right), & p \equiv 1 (\mathrm{mod}\, 6). \end{cases}$

(v) For any $N > 1$,

$$\int_0^1 \Delta^2(x,N)\,dx = \frac{1}{12}\frac{2^{\omega(N)}\varphi(N)}{N}.$$

Preliminary Discussion. Let $1 = a_1 < a_2 < \cdots < a_{\varphi(N)} = N-1$ be the $\varphi(N)$ integers in $[1,N]$ which are coprime to N. For convenience, we shall also define $a_0 = 0$ and $a_{\varphi(N)+1} = N$. Note that the relation $N - a_i = a_{\varphi(N)-i+1}$ is true for all i, $0 \le i \le \varphi(N)+1$.

We shall refer to points a/N with $(a,N) = 1$ as N-*nodal* so that, in $[0,1]$, these are precisely the points a_i/N, $1 \le i \le \varphi(N)$.

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From the definition of $\Delta(x, N)$, we have that

$$\Delta\left(\frac{a_i}{N}, N\right) = i - a_i\frac{\varphi(N)}{N}, \quad 0 \le i \le \varphi(N),$$
$$\Delta\left(\frac{a_{i+1}}{N}, N\right) = \Delta\left(\frac{a_i}{N}, N\right) + 1 - (a_{i+1} - a_i)\frac{\varphi(N)}{N}, \quad 0 \le i < \varphi(N),$$

and that if $\frac{a_i}{N} \le x < \frac{a_{i+1}}{N}$, $0 \le i \le \varphi(N)$, then

$$\Delta(x, N) = \Delta\left(\frac{a_i}{N}, N\right) - \left(x - \frac{a_i}{N}\right)\varphi(N).$$

These observations imply that $\Delta(x, N)$ is a piecewise linear function of x with each line-segment in $\left[a_i/N, a_{i+1}/N\right)$ having gradient $-\varphi(N)$ and that in the bounds

$$-\Delta(N) \le \Delta(x, N) \le \Delta(N)$$

equality is attained in the upper bound for some N-nodal point x while the lower bound is, in fact, a strict inequality. Note also that if x is N-nodal then we have the sharper lower bound

$$\Delta(x, N) = 1 + \lim_{t \longrightarrow x-} \Delta(t, N) \ge -\Delta(N) + 1.$$

The relation $\Delta\left(\frac{a_i}{N}, N\right) = -\Delta\left(\frac{N-a_i}{N}, N\right) + 1$ shows, in fact, that

$$\inf_{1 \le i \le \varphi(N)} \Delta\left(\frac{a_i}{N}, N\right) = -\Delta(N) + 1.$$

Proofs of Theorems. We begin with the proof of Theorem 4 because it contains some of the results which are required in the subsequent theorems.

PROOF OF THEOREM 4. (i) Write $N = pM$ where $p \nmid M$ and $p \equiv 1 (\operatorname{mod} k)$. Identity (**I**) implies that for any $a, 0 \le a \le k - 1$,

$$\Delta\left(\frac{a}{k}, N\right) = \Delta\left(\frac{pa}{k}, M\right) - \Delta\left(\frac{a}{k}, M\right) = \Delta\left(\frac{a}{k}, M\right) - \Delta\left(\frac{a}{k}, M\right) = 0.$$

and, clearly, this also holds for $a = k$. Hence

$$0 = \Delta\left(\frac{a+1}{k}, N\right) - \Delta\left(\frac{a}{k}, N\right) = \sum_{\substack{\frac{aN}{k} < n \le \frac{(a+1)N}{k} \\ (n,N)=1}} 1 - \frac{1}{k}\varphi(N).$$

This proves (i).

(ii) For any $N > 1$, we have that

$$\Delta(x, N) = -\mu(N)\sum_{d|N}\mu(d)\left(\{xd\} - \frac{1}{2}\right).$$

Hence for $(l,N)=1$,

$$\sum_{n=0}^{l-1}\Delta\left(\frac{u+n}{l},N\right) = -\mu(N)\sum_{d|N}\mu(d)\sum_{n=0}^{l-1}\left(\left\{\frac{ud}{l}+\frac{nd}{l}\right\}-\frac{1}{2}\right)$$
$$= -\mu(N)\sum_{d|N}\mu(d)\sum_{n=0}^{l-1}\left(\left\{\frac{ud}{l}+\frac{n}{l}\right\}-\frac{1}{2}\right).$$

The inner sum is $\{ud\}-\frac{1}{2}$ (see *e.g.* Landau [5], p. 170). We therefore deduce that for any $(l,N)=1$ and $u\in\mathbf{R}$,

$$\sum_{n=0}^{l-1}\Delta\left(\frac{u+n}{l},N\right)=\Delta(u,N). \tag{4}$$

Using (4) with $l=2$ and N odd together with identity (**I**), we have that

$$\Delta\left(\frac{u}{2},N\right)=\Delta(u,N)-\Delta\left(\frac{u+1}{2},N\right)=\Delta\left(\frac{u+1}{2},2N\right).$$

By varying u through an interval of length 2, we deduce that the set of values of $\Delta(x,N)$ and that of $\Delta(x,2N)$ is the same and (ii) follows.

(iii) Using (4) with $l=p$ where $p\nmid N$ and identity (**I**), we have that

$$\sum_{n=0}^{p-1}\Delta\left(\frac{u+n}{p},Np\right) = \sum_{n=0}^{p-1}\Delta(u,N)-\sum_{n=0}^{p-1}\Delta\left(\frac{u+n}{p},N\right)=p\Delta(u,N)-\Delta(u,N)$$
$$= (p-1)\Delta(u,N).$$

Choosing u so that $\Delta(u,N)=\Delta(N)$, we deduce that

$$(p-1)\Delta(N)\le p\Delta(Np)$$

which implies (iii).

(iv) For any a with $1\le a<3p$ and $(a,3p)=1$, identity (**I**) yields

$$\Delta\left(\frac{a}{3p},3p\right)=\Delta\left(\frac{a}{3},3\right)-\Delta\left(\frac{a}{3p},3\right).$$

It follows directly from the definition of $\Delta(x,3)$ that

$$\Delta\left(\frac{a}{3},3\right)=\begin{cases}1/3, & a\equiv 1(\operatorname{mod}3)\\ 2/3, & a\equiv 2(\operatorname{mod}3)\end{cases}$$

and that

$$\Delta\left(\frac{a}{3p},3\right)=\left[\frac{a}{p}\right]-\frac{2a}{3p}.$$

We deduce that if $a\equiv 2(\operatorname{mod}3)$ and $a<p$ then

$$\Delta\left(\frac{a}{3p},3p\right)=\frac{2}{3}+\frac{2a}{3p}$$

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and hence that if $p \equiv 1 \pmod 6$ then

$$\Delta\left(\frac{p-2}{3p}, 3p\right) = \frac{4}{3}\left(1 - \frac{1}{p}\right),$$

and if $p \equiv -1 \pmod 6$ then

$$\Delta\left(\frac{p-3}{3p}, 3p\right) = \frac{4}{3} - \frac{2}{p}.$$

We now show that these are indeed the largest values of $\Delta(x, 3p)$. Clearly, this is indeed the case if $a \equiv 2 \pmod 3$ and $a < p$. If $a \equiv 1 \pmod 3$ then

$$\Delta\left(\frac{a}{3p}, 3p\right) = \frac{1}{3} - \left[\frac{a}{p}\right] + \frac{2a}{3p} \le \frac{1}{3} + \frac{2(p-1)}{3p} = 1 - \frac{2}{3p} < \frac{4}{3} - \frac{2}{p}$$

for any $p \ge 5$ and so is smaller than either of the above candidates for $\Delta(3p)$.

If $a \equiv 2 \pmod 3$ and $2p \le a < 3p$ then

$$\Delta\left(\frac{a}{3p}, 3p\right) = -\frac{4}{3} + \frac{2a}{3p} < \frac{2}{3}$$

and this is also smaller. Finally, if $a \equiv 2 \pmod 3$ and $p \le a < 2p$ then

$$\Delta\left(\frac{a}{3p}, 3p\right) = -\frac{1}{3} + \frac{2a}{3p} \le \begin{cases} 1 - \frac{2}{p}, & p \equiv 1 \pmod 6 \\ 1 - \frac{4}{3p}, & p \equiv -1 \pmod 6 \end{cases}$$

which are smaller as well. This completes the proof of (iv).

(v) Since $\Delta(x, N) = -\mu(N) \sum_{d|N} \mu(d)(\{xd\} - \frac{1}{2})$, using a classical result of Franel [4], we have that

$$\int_0^1 \Delta^2(x, N)\, dx = \sum_{d_1|N, d_2|N} \mu(d_1)\mu(d_2) \int_0^1 \left(\{xd_1\} - \frac{1}{2}\right)\left(\{xd_2\} - \frac{1}{2}\right) dx$$

$$= \frac{1}{12} \sum_{d_1|N, d_2|N} \mu(d_1)\mu(d_2) \frac{(d_1, d_2)^2}{d_1 d_2}. \tag{5}$$

Writing $r = (d_1, d_2)$, $d_1 = \delta_1 r$, $d_2 = \delta_2 r$, the above sum is

$$\sum_{r|N} \sum_{\substack{\delta_1|N/r, \delta_2|N/r \\ (\delta_1, \delta_2) = 1}} \frac{\mu(\delta_1)\mu(\delta_2)}{\delta_1 \delta_2} = \sum_{r|N} \sum_{d|N/r} \frac{\mu(d)\tau(d)}{d} = \sum_{r|N} \sum_{d|r} \frac{\mu(d)\tau(d)}{d}.$$

The function $f(r) = \sum_{d|r} \mu(d)\tau(d)/d$ is multiplicative with $f(p) = 1 - 2/p$. Further, the function $g(N) = \sum_{r|N} f(r)$ is also multiplicative with

$$g(p) = 1 + f(p) = 2\left(1 - \frac{1}{p}\right).$$

Hence, for squarefree N, $g(N) = 2^{\omega(N)} \varphi(N)/N$. We deduce from (5) that

$$\int_0^1 \Delta^2(x, N)\, dx = \frac{1}{12} g(N) = \frac{1}{12} \frac{2^{\omega(N)} \varphi(N)}{N}$$

as required. Using $\Delta(x, N) = \Delta(xL, \bar{N})$ as noted in the introduction, it follows easily that the result holds even if N is not squarefree.

PROOF OF THEOREM 1. Let a with $(a, Np) = 1$ and $1 \le a < Np$ be chosen such that

$$\Delta(Np) = \Delta\left(\frac{a}{Np}, Np\right).$$

By identity (**I**), we have that

$$\Delta(Np) = \Delta\left(\frac{a}{N}, N\right) - \Delta\left(\frac{a}{Np}, N\right). \tag{6}$$

Since $(a, N) = 1$, $\{a/N\}$ is N-nodal but clearly a/Np is not N-nodal. We can therefore define $i \in \mathbf{N}$, $1 \le i \le \varphi(N) + 1$, such that

$$\frac{a_{i-1}}{N} < \frac{a}{Np} < \frac{a_i}{N}.$$

This implies that $a < pa_i$ and so we can write $a = pa_i - r$ with $r \in \mathbf{N}$.

We shall prove the validity of both

$$\Delta(Np) \le 2\Delta(N) - \frac{r}{Np}\varphi(N) \tag{7}$$

and, if $r \le Np/\varphi(Np)$,

$$\Delta(Np) \le 2\Delta(N) - 1 + \frac{r}{Np}\varphi(Np) \tag{8}$$

We begin by considering the case $i = \varphi(N) + 1$ on its own. Here $a_i = N$ and hence $a = pN - r$ so that

$$\Delta\left(\frac{a}{Np}, N\right) = \Delta(1, N) + \left(1 - \frac{a}{Np}\right)\varphi(N) = \frac{r\varphi(N)}{Np}$$

so that we deduce immediately from (6) that (7) is true. Note also that in this case

$$\begin{aligned}\Delta(Np) &= \Delta\left(\frac{a}{Np}, Np\right) \le \Delta(1, Np) + \left(1 - \frac{a}{Np}\right)\varphi(Np) \\ &= \frac{r\varphi(Np)}{Np} \le 2\Delta(N) - 1 + \frac{r\varphi(Np)}{Np},\end{aligned}$$

since $\Delta(N) \ge 1/2$ for $N > 1$. This proves (8).

We may therefore assume from now onward that $1 \le i \le \varphi(N)$. Hence, using our preliminary observations,

$$\begin{aligned}\Delta\left(\frac{a}{Np}, N\right) &= \Delta\left(\frac{a_i}{N}, N\right) - 1 + \left(\frac{a_i}{N} - \frac{a}{Np}\right)\varphi(N) \\ &= \Delta\left(\frac{a_i}{N}, N\right) - 1 + \frac{r}{Np}\varphi(N)\end{aligned}$$

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so that (6) implies that

$$\begin{aligned}\Delta(Np) &= \Delta\left(\frac{a}{N}, N\right) - \Delta\left(\frac{a_i}{N}, N\right) + 1 - \frac{r}{Np}\varphi(N)\\ &\le \Delta(N) - \big(-\Delta(N)+1\big) + 1 - \frac{r}{Np}\varphi(N),\end{aligned}$$

since a_i/N is N-nodal. This implies (7).

On the other hand, identity (**I**) implies that

$$\Delta\left(\frac{a+r}{Np}, Np\right) = \Delta\left(\frac{pa_i}{N}, N\right) - \Delta\left(\frac{a_i}{N}, N\right)$$

and hence

$$\Delta\left(\frac{a+r}{Np}, Np\right) \le \Delta(N) - \big(-\Delta(N)+1\big) = 2\Delta(N) - 1. \tag{9}$$

Since $i \le \varphi(N)$, we have that

$$\frac{a}{Np} < \frac{a_i}{N} \le 1 - \frac{1}{N} < 1 - \frac{1}{Np} = \frac{Np-1}{Np}$$

and hence a/Np is not the largest Np-nodal point in (0,1). Denoting by b/Np the least Np-nodal point larger than a/Np, the definition of a/Np implies that

$$0 \ge \Delta\left(\frac{b}{Np}, Np\right) - \Delta\left(\frac{a}{Np}, Np\right) = 1 - \frac{(b-a)\varphi(Np)}{Np}$$

and hence $b - a \ge Np/\varphi(Np)$. Since $(a+r)/Np$ is not Np-nodal, we deduce that if $r \le Np/\varphi(Np)$ then

$$\frac{a}{Np} < \frac{a+r}{Np} < \frac{b}{Np}.$$

For such r, we use (9) to infer that

$$\begin{aligned}\Delta(Np) &= \Delta\left(\frac{a}{Np}, Np\right) = \Delta\left(\frac{a+r}{Np}, Np\right) + \frac{r}{Np}\varphi(Np)\\ &\le 2\Delta(N) - 1 + \frac{r}{Np}\varphi(Np).\end{aligned}$$

This proves (8) and hence completes the proof of (7) and (8).

We now prove (i).

If $r \ge N/\varphi(N)$ then (7) immediately yields

$$\Delta(Np) \le 2\Delta(N) - \frac{1}{p}.$$

If, on the other hand, $r < N/\varphi(N)$ then certainly $r < Np/\varphi(Np)$ so that (8) yields

$$\Delta(Np) \le 2\Delta(N) - 1 + \frac{N}{\varphi(N)} \cdot \frac{1}{Np}\varphi(Np) = 2\Delta(N) - \frac{1}{p}.$$

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This completes the proof of (i).

We now prove (ii). Put $l = \left[N/\varphi(N)\right]$.

If $r \geq l+1$ then (7) implies that

$$\Delta(Np) \leq 2\Delta(N) - \frac{(l+1)\varphi(N)}{Np}.$$

If $r \leq l$ then certainly $r < Np/\varphi(Np)$ so that (8) yields

$$\Delta(Np) \leq 2\Delta(N) - 1 + \frac{l}{Np}\varphi(Np) = 2\Delta(N) - (l+1)\frac{\varphi(N)}{Np} + \frac{\varphi(N)}{Np} + \frac{l\varphi(N)}{N} - 1.$$

Hence, in any case,

$$\Delta(Np) \leq 2\Delta(N) - (l+1)\frac{\varphi(N)}{Np} + \max\left(0, \frac{\varphi(N)}{Np} + \frac{l\varphi(N)}{N} - 1\right)$$

as required.

This completes the proof of Theorem 1.

PROOF OF COROLLARIES. In Theorem 1(ii), put $N = q \geq 3$. Then $l = 1$ and so we obtain

$$\begin{aligned}\Delta(pq) &\leq 2\Delta(q) - \frac{2}{pq}(q-1) + \max\left(0, \frac{q-1}{pq} + \frac{q-1}{q} - 1\right)\\ &= 2\left(1 - \frac{1}{q}\right) - \frac{2}{p}\left(1 - \frac{1}{q}\right) = 2\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\end{aligned}$$

as required for Corollary (i). Corollary (ii) follows on iterating Theorem 1(i). If $p_1 = 2$, we just use Theorem 4(ii) to note that $\Delta(p_1 \ldots p_s) = \Delta(p_2 \ldots p_s)$ before iterating Theorem 1(i).

PROOF OF THEOREM 2. We use induction on $\omega(N)$ to first show that

$$\Delta\left(\frac{a}{k}, N\right) = -\mu(N)2^{\omega(N)}\left(\frac{a}{k} - \frac{1}{2}\right)$$

for any a, $1 \leq a \leq k-1$.

If $p \equiv -1 (\operatorname{mod} k)$ then

$$\Delta\left(\frac{a}{k}, p\right) = \frac{a}{k} - \left\{\frac{pa}{k}\right\} = \frac{a}{k} - \left(1 - \frac{a}{k}\right) = 2\left(\frac{a}{k} - \frac{1}{2}\right),$$

and so the result is true for $\omega(N) = 1$. Suppose that it is true for some N whose prime factors q satisfy $q \equiv -1 (\operatorname{mod} k)$ and let p be another prime with $p \equiv -1 (\operatorname{mod} k)$ and $p \nmid N$. By identity (**I**),

$$\Delta\left(\frac{a}{k}, Np\right) = \Delta\left(\frac{pa}{k}, N\right) - \Delta\left(\frac{a}{k}, N\right).$$

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Since $\left\{\frac{pa}{k}\right\} = \frac{k-a}{k}$, the induction hypothesis implies that

$$\begin{aligned}\Delta\left(\frac{a}{k}, Np\right) &= -\mu(N)2^{\omega(N)}\left(\frac{k-a}{k} - \frac{1}{2} - \left(\frac{a}{k} - \frac{1}{2}\right)\right)\\ &= -\mu(Np)2^{\omega(Np)}\left(\frac{a}{k} - \frac{1}{2}\right)\end{aligned}$$

as required. Hence

$$\Delta(N) \geq \left|\Delta\left(\frac{k-1}{k}, N\right)\right| = 2^{\omega(N)-1}\left(\frac{k-2}{k}\right).$$

PROOF OF THEOREM 3. For the proof of Theorem 3, we shall need an elementary lemma which we state in a general context since it may be of independent interest.

LEMMA. *Let* $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ *be* l *points in* $(0, 1)$ *and define for any* $x \in [0, 1]$,

$$\Delta(x) = \sum_{\substack{i \\ \alpha_i \leq x}} 1 - xl.$$

Then

$$\frac{1}{l}\sum_{i=1}^{l} \Delta^2(\alpha_i) = \int_0^1 \Delta^2(x)\,dx + \frac{1}{6} - \left(\sum_{i=1}^{l} \alpha_i - \frac{l}{2}\right).$$

PROOF. Define $\alpha_0 = 0$ and $\alpha_{l+1} = 1$. Observe that if $\alpha_i \leq x < \alpha_{i+1}$, then $\Delta(x) = i - xl$. Hence

$$\begin{aligned}\int_0^1 \Delta^2(x)\,dx &= \sum_{i=0}^{l} \int_{\alpha_i}^{\alpha_{i+1}} \Delta^2(x)\,dx\\ &= \sum_{i=0}^{l} i^2(\alpha_{i+1} - \alpha_i) - l\sum_{i=0}^{l} i(\alpha_{i+1}^2 - \alpha_i^2) + \frac{l^2}{3}\sum_{i=0}^{l}(\alpha_{i+1}^3 - \alpha_i^3).\\ &= \frac{l^2}{3} + \sum_{i=1}^{l} \alpha_i - 2\sum_{i=1}^{l} i\alpha_i + l\sum_{i=1}^{l} \alpha_i^2.\end{aligned} \tag{10}$$

Further, since $\Delta(\alpha_i) = i - \alpha_i l$,

$$\sum_{i=1}^{l} \Delta^2(\alpha_i) = \sum_{i=1}^{l}\left(i^2 - 2i\alpha_i l + l^2\alpha_i^2\right) = \frac{l(l+1)(2l+1)}{6} - 2l\sum_{i=1}^{l} i\alpha_i + l^2\sum_{i=1}^{l} \alpha_i^2. \tag{11}$$

Comparing (10) and (11), we deduce that

$$\frac{1}{l}\sum_{i=1}^{l} \Delta^2(\alpha_i) = \int_0^1 \Delta^2(x)\,dx - \sum_{i=1}^{l} \alpha_i + \frac{l}{2} + \frac{1}{6}$$

as required.

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COROLLARY. *If, in addition, the points α_i are symmetric about $\frac{1}{2}$ then*

$$\frac{1}{l}\sum_{i=1}^{l}\Delta^2(\alpha_i) = \int_0^1 \Delta^2(x)\,dx + \frac{1}{6}.$$

For $N > 1$, we apply the above corollary with $\alpha_i = \frac{a_i}{N}$, $1 \le i \le \varphi(N)$, and use Theorem 4(v) to obtain Theorem 3(i).

Since

$$\sum_{i=1}^{\varphi(N)} \Delta\left(\frac{a_i}{N}, N\right) = \sum_{i=1}^{\varphi(N)}\left(i - a_i\frac{\varphi(N)}{N}\right) = \frac{\varphi(N)}{2}, \tag{12}$$

we deduce that

$$\frac{1}{\varphi(N)}\sum_{i=1}^{\varphi(N)}\left(\Delta\left(\frac{a_i}{N}, N\right) - \frac{1}{2}\right)^2 = \frac{1}{12}2^{\omega(N)}\frac{\varphi(N)}{N} - \frac{1}{12}. \tag{13}$$

Since $\inf \Delta\left(\frac{a_i}{N}, N\right) = -\sup \Delta\left(\frac{a_i}{N}, N\right) + 1$, we deduce from (13) that

$$\left(\Delta(N) - \frac{1}{2}\right)^2 \ge \frac{1}{12}2^{\omega(N)}\frac{\varphi(N)}{N} - \frac{1}{12}.$$

Theorem 3(ii) now follows on observing that (12) implies that $\Delta(N) \ge \frac{1}{2}$.

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