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DISJOINT CONJUGATES OF CYCLIC SUBGROUPS OF FINITE GROUPS

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In an earlier paper (2) we considered the following question "If S is a cyclic subgroup of a finite group G and $S \cap F(G) = 1$, where $F(G)$ is the Fitting subgroup of G, does there necessarily exist a conjugate S^x of S in G with $S \cap S^x = 1$?" and we gave an affirmative answer for G simple or soluble. In this paper we answer the question affirmatively in general (in fact we prove a somewhat stronger result (Theorem 3)). We give an example of a group G with a cyclic subgroup S such that (i) no nontrivial subgroup of S is normal in G and (ii) no x exists for which $S \cap S^x = 1$.

The notation is standard with the following additions: if X is a finite set, $|X|$ denotes the cardinality of X, and if n is a natural number and p a prime, n_p denotes the p-share of n.

We begin with

Lemma 1. *Let G be a finite nilpotent group, and let $S_1, \ldots, S_n$ be proper subgroups of G for which there exist distinct primes $p_1, \ldots, p_n$ such that, for each i, there exists a prime q_i dividing $|G|$ such that $[G:S_i]_{q_i} \equiv 0$ or 1 mod p_i.*

Then $G \neq \cup_{i=1}^{n} S_i$

Proof. We prove the result by induction on $|G|$. Suppose first that q is a prime divisor of $|G|$ and that G is not a q-group. Let $G = A \times B$, where A, B are q-, q'-groups respectively, and, for each i, let $S_i = S_i(A) \times S_i(B)$ as a subgroup of $A \times B$.

Let

$$X = \{i | S_i(A) \neq A \quad \text{and} \quad [A : S_i(A)] \equiv 0 \text{ or } 1 \bmod p_i\}$$

and

$$Y = \{i | S_i(B) \neq B \text{ and there exists a prime divisor } q_i \text{ of } |B| \text{ such that } [B:S_i(B)]_{q_i} \equiv 0 \text{ or } 1 \bmod p_i\}$$

Note that $X \cup Y = \{1, 2, \ldots, n\}$.

By induction, $A \neq \cup_{i \in X} S_i(A)$ and $B \neq \cup_{i \in Y} S_i(B)$. Let

$$a \in A - \cup_{i \in X} S_i(A),\ b \in B - \cup_{i \in Y} S_i(B).$$

Then $(a, b) \notin S_j$ for any j $(1 \leq j \leq n)$, and the result follows.

Hence we may assume that G is a q-group for some prime q, and thus for $p_i \neq q$, $[G: S_i] = q^{k_i} > 1$ and $q^{k_i} \equiv 1 \bmod p_i$. For p a prime different from

q let $m(p)$ be the order of $q \bmod p$. By Lemma 2 of (2),

$$1/q + \sum_{\substack{p \neq q \\ p \text{ prime}}} (1/q^{m(p)}) < 1.$$

This implies that $\Sigma_{i=1}^{n} |S_i| < |G|$ and the result is proved.

Note that Theorem 1 of (2) is a consequence of this lemma.

The next result is due to Oscar E. Barriga (1).

Lemma 2. *Let G be a finite group, N a normal subgroup of G with $F(N) = 1$, and S a cyclic subgroup of G with $C_S(N) = 1$. Then there exists $x \in N$ with $S \cap S^x = 1$.*

We now prove our main result:-

Theorem 3. *Let G be a finite group and S a cyclic subgroup of G. Suppose that no nontrivial subgroup of S is a normal subgroup of $F(G)$. Then there exists $g \in G$ with $S \cap S^g = 1$.*

Proof. We use induction on $|G| + |S|$. Clearly, we may assume that $|S|$ is square-free and that $S = S_1 \times \cdots \times S_t$, where S_i has a prime order p_i.

We observe first that if S_i does not centralise $F(G)$, then $N_{F(G)}(S_i) \neq F(G)$—this is true by hypothesis if $S_i \leq F(G)$, while if $S_i \not\leq F(G)$, then $C_{F(G)}(S_i) = N_{F(G)}(S_i)$. Let L be a Sylow q-subgroup of $F(G)$ for some prime q. Then $|L - N_L(S_i)|$ is divisible by p_i, and thus $[L : N_L(S_i)] \equiv 0$ or $1 \bmod p_i$.

Next, we note that if no S_i centralises $F(G)$, then the theorem follows from Lemma 1—the hypotheses of Lemma 1 are satisfied by the subgroups $N_{F(G)}(S_i)$, using the argument of the last paragraph. Hence we may assume that there exists $s \geq 1$ such that

$$C_S(F(G)) = S_1 S_2 \ldots S_s.$$

In particular, we can see that $C_G(F(G))$ is non abelian, since otherwise $C_G(F(G)) = Z(F(G))$ and $S_i \lhd F(G)$ $(1 \leq i \leq s)$. Let T_1 be a minimal nonabelian normal subgroup of G contained in $C_G(F(G))$ and, if $C_G(T_1 F(G))$ is nonabelian, let T_2 be a minimal normal nonabelian subgroup of G contained in $C_G(T_1 F(G))$.

Proceeding by induction, having found $T_1, \ldots, T_c$ with $T_{i+1} \leq C_G(T_1 \ldots T_i F(G))$ $(1 \leq i \leq c-1)$ we consider $C_G(T_1 \ldots T_c F(G))$, and if this is nonabelian, we let T_{c+1} be a minimal nonabelian normal subgroup of G contained in $C_G(T_1 \ldots T_c F(G))$. We note that $T_{i+1} \not\leq T_1 \ldots T_i$ and hence there exists an integer $d \geq 1$ such that if $T = T_1 \ldots T_d$, $C_G(F(G)T)$ is abelian and thus $C_G(F(G)T) = Z(F(G))$. Also $F(T) = Z(T)$. Now $T_i' = T_i$ for all i, since otherwise T_i' is abelian and thus $T_i' \leq F(T) = Z(T)$ and $T_i \leq F(T) = Z(T)$, a contradiction. Thus $T' = T$.

We now observe that if $[S_j, T] \leqslant Z(T)$, then $[S_j, T, T] = [T, S_j, T] = 1$ so by the Three Subgroups Lemma, $[T, T, S_j] = 1$, and thus $[T, S_j] = 1$ (since $T' = T$).

If $C_S(T) = 1$, then, since $F(T/Z(T)) = 1$, the theorem follows from Lemma 2 applied to the group $ST/Z(T)$. Hence we may assume that $C_S(T) \neq 1$. Note that since $C_G(F(G)T) = Z(F(G))$, no S_i centralises both $F(G)$ and T. Thus we may assume that $C_S(T) = S_{s+1} \dots S_u$ for some $u \geqslant s+1$.

Applying Lemma 2 to the group $S_0T/Z(T)$ where $S_0 = \Pi_{i=1}^{s} S_i \, \Pi_{j=u+1}^{t} S_j$ and noting that $C_{S_0Z(T)/Z(T)}[T/Z(T)] = 1$, we find that there exists $y \in T$ such that $[S_0Z(T)]^y \cap S_0Z(T) = Z(T)$. Let $S_{00} = \Pi_{j=s+1}^{t} S_j$. Then $C_{S_{00}}(F(G)) = 1$ and $\{N_{F(G)}(S_j) | j > s\}$ satisfies the hypotheses of Lemma 1. Hence there exists $x \in F(G)$ such that $S_{00}^x \cap S_{00} = 1$.

We now prove that $S \cap S^{xy} = 1$. For suppose $S_j^{xy} = S_j$. If $1 \leqslant j \leqslant s$, this gives $S_j^y = S_j$ and thus $[S_jZ(T)]^y = S_jZ(T)$ and since $S_j \not\leqslant Z(T)$, this contradicts our choice of y. If $s+1 \leqslant j \leqslant u$, then $S_j^x = S_j$ contradicting our choice of x. Suppose then that $j > u$. Let $S_j = \langle w \rangle$. Then $w^{xy} = w^m$ for some integer m and thus $[w, xy] = w^{m-1} \in S_j \cap F(G)T$. If $S_j \not\leqslant F(G)T$ this implies that $w^{m-1} = 1$, and thus $[w, y] \in F(G) \cap T = Z(T)$ and $[S_jZ(T)]^y = S_jZ(T)$, contrary to our choice of y. Suppose finally that $S_j \leqslant F(G)T$. Let $S_j = \langle a_jb_j \rangle$ where $a_j \in F(G)$, $b_j \in T$ are p_j-elements. Then $(a_jb_j)^{xy} = (a_jb_j)^r$ for some integer r, and thus $a_j^{-r}a_j^x = b_j^r b_j^{-y} \in F(G) \cap T = Z(T)$. In particular

$$a_j^{-r}a_j^x \in Z(F(G)) \text{ and } a_j^{-(r-1)} \in [a_j, x]^{-1}Z(F(G)).$$

Let $F_0 = F(G)$, $F_1 = [F_0, F_0]$, $F_2 = [F_0, F_1], \dots$ be the lower central series for $F(G)$. If $a_j \in Z(F(G))$, then $a_jb_j \in Z(F(G))$ and S_j centralises $F(G)$, contradicting $j > u$. Hence there exists a maximum integer n for which $a_j \in F_nZ(F_0)$. Then $a_j^{r-1} \in F_{n+1}Z(F_0)$, so $\langle a_j^{r-1} \rangle \neq \langle a_j \rangle$ and thus p_j divides $r-1$ and since $a_j^{p_j} = b_j^{-p_j}$, $a_j^{r-1} \in Z(T)$. Also

$$b_j^{-(r-1)} \in Z(T) \text{ and } (a_jb_j)^{-1}(a_jb_j)^y \in Z(T).$$

Thus $[S_jZ(T)]^y = S_jZ(T)$ and this contradicts our choice of y. The proof of the theorem is now complete.

Corollary 1. *Let G be a finite group and S a cyclic subgroup of G with $S \cap F(G) = 1$. Then there exists $x \in G$ with $S \cap S^x = 1$.*

Corollary 2. *Let G be a finite group and w an automorphism of G such that no nontrivial subgroup of $\langle w \rangle$ fixes $F(G)[C_G(F(G))]'$ pointwise. Then $\langle w \rangle$ has a regular orbit on G (i.e. there exists $g \in G$ such that*

$$|\{gw^i | i \in Z\}| = |\langle w \rangle|).$$

Proof. Let H be the natural semidirect product $G\langle w \rangle$. The result

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follows from the theorem if we can show that no nontrivial subgroup of $\langle w\rangle$ is normal in $F(H)$.

Suppose then that $\langle w_1\rangle$ is a nontrivial subgroup of $\langle w\rangle$ with $\langle w_1\rangle \lhd F(H)$. Then $[F(G), w_1] \leq F(G) \cap \langle w_1\rangle = 1$, and

$$[C_G(F(G)), \langle w_1\rangle] \leq F(H) \cap C_G(F(G)) = Z(F(G)).$$

Thus $[C_G(F(G)), \langle w_1\rangle, C_G(F(G))] = 1$. So, by the Three Subgroups Lemma, $[[C_G(F(G))]', w_1] = 1$. This completes the proof.

Note. This result is of interest since $F(G)[C_G(F(G))]'$ is a characteristic subgroup of G containing its centraliser.

The proof of the next result is easy, using Corollary 1 of Theorem 3, and is omitted.

Theorem 4. *Let G be a group with $|G| < 3600$ and let S be a cyclic subgroup of G. Suppose that no nontrivial subgroup of S is normal in G. Then there exists $x \in G$ with $S \cap S^x = 1$.*

We now give the example referred to in the introduction. Let $M = A \times B \times C$ be the direct product of the elementary abelian groups A, B, C of orders 4, 9, and 25 respectively. Let $\{x_1, x_2\}$, $\{y_1, y_2\}$, $\{z_1, z_2\}$ be generating sets for A, B, C respectively.

Let v, w be the automorphisms of H defined by

$$\begin{array}{lll} x_1v = x_2; & x_2v = x_1; & y_1v = y_1; \\ y_2v = y_2; & z_1v = z_2; & z_2v = z_1; \\ x_1w = x_1; & x_2w = x_2; & y_1w = y_2; \\ y_2w = y_1; & z_1w = z_2; & z_2w = z_1; \end{array}$$

Note that $\langle v, w\rangle$ is the four group. Let G be the semidirect product $H\langle v, w\rangle$ and let $S = \langle x_1\rangle\langle y_1\rangle\langle z_1\rangle$, $S_1 = \langle x_1\rangle$, $S_2 = \langle y_1\rangle$ and $S_3 = \langle z_1\rangle$. Then $|G| = 3600$ and $G = \cup_{i=1}^{3} N_G(S_i)$ and $[G : N_G(S_i)] = 2$, $i = 1, 2, 3$. The automorphism of G induced by conjugation by $x_1y_1z_1$ has no regular orbit on G.

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