ON EQUIVALENCE RELATIONS INDUCED BY LOCALLY COMPACT ABELIAN POLISH GROUPS

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Abstract. Given a Polish group G, let $E(G)$ be the right coset equivalence relation $G^\omega/c(G)$, where $c(G)$ is the group of all convergent sequences in G. The connected component of the identity of a Polish group G is denoted by G_0.

Let G, H be locally compact abelian Polish groups. If $E(G) \leq_B E(H)$, then there is a continuous homomorphism $S : G_0 \to H_0$ such that $\ker(S)$ is non-archimedean. The converse is also true when G is connected and compact.

For $n \in \mathbb{N}^+$, the partially ordered set $P(\omega)/\mathrm{Fin}$ can be embedded into Borel equivalence relations between $E(\mathbb{R}^n)$ and $E(\mathbb{T}^n)$.

§1. Introduction. A topological space is *Polish* if it is separable and completely metrizable. For more details in descriptive set theory, we refer to [13]. It is an important application of descriptive set theory to study equivalence relations by using Borel reducibility. Given two Borel equivalence relations E and F on Polish spaces X and Y, respectively, recall that E is *Borel reducible* to F, denoted $E \leq_B F$, if there exists a Borel map $\theta : X \to Y$ such that for all $x, y \in X$,

$$xEy \iff \theta(x)F\theta(y).$$

We denote $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$, and denote $E <_B F$ if $E \leq_B F$ and $F \not\leq_B E$. We refer to [7] for background on Borel reducibility.

Polish groups are important tools in the research on Borel reducibility. A topological group is *Polish* if its topology is Polish. For a Polish group G, the authors [5] defined an equivalence relation $E(G)$ on G^ω by

$$xE(G)y \iff \lim_n x(n)y(n)^{-1} \text{ converges in } G$$

for $x, y \in G^\omega$. We say that $E(G)$ is the *equivalence relation induced by* G. Indeed, $E(G)$ is the right coset equivalence relation $G^\omega/c(G)$, where $c(G)$ is the group of all convergent sequences in G.

In this article, we focus on equivalence relations induced by locally compact abelian Polish groups. Some interesting results have been found in some special cases. For instance, for $c_0, e_0, c_1, e_1 \in \mathbb{N}$, $E(\mathbb{R}^{c_0} \times \mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_1} \times \mathbb{T}^{e_1})$ iff $e_0 \leq e_1$ and $c_0 + e_0 \leq c_1 + e_1$ (cf. [5, Theorem 6.19]).

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Given a group G, the identity element of G is denoted by 1_G. If G is a topological group, the connected component of 1_G in G is denoted by G_0. Recall that a Polish group G is *non-archimedean* if it has a neighborhood base of 1_G consisting of open subgroups.

THEOREM 1.1. *Let G and H be two locally compact abelian Polish groups. If $E(G) \leq_B E(H)$, then there is a continuous homomorphism $S : G_0 \to H_0$ such that* $\ker(S)$ *is non-archimedean.*

By restricting attention to compact connected abelian Polish groups, we prove the following theorem.

THEOREM 1.2 (Rigid Theorem). *Let G be a compact connected abelian Polish group and let H be a locally compact abelian Polish group. Then $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S : G \to H$ such that* $\ker(S)$ *is non-archimedean.*

For every normal space X, denoted by $\dim(X)$ the *covering dimension* of X, where $\dim(X)$ is an integer ≥ -1 or the "infinite number ∞." Let G be an abelian topological group. The topological group $\mathrm{Hom}(G, \mathbb{T})$ is called the *dual group* of G, denoted by $\widehat{G}$ (see Section 4). For any finite dimensional compact abelian Polish group G, $\dim(G) = \mathrm{rank}(\widehat{G})$, the torsion-free rank of $\widehat{G}$ (cf. Lemma 8.13 and Corollary 8.26 of [11]). We say G is *n-dimensional* if $\dim(G) = n$ for some integer n, or *infinite dimensional* if $\dim(G)$ is infinite.

Recall that $\mathbb{T}$ is the multiplicative group of all complex numbers with modulus 1. For finite dimensional compact abelian Polish groups, we obtain the following results.

THEOREM 1.3. *Let G, H be locally compact abelian Polish groups.*

(1) *If G is non-archimedean, then $E(G) \leq_B E_0^\omega$.*
(2) *If G is not non-archimedean, then $E(\mathbb{R}) \leq_B E(G)$.*
(3) *If G is not non-archimedean and G_0 is open, then $E(G) \sim_B E(G_0)$.*
(4) *If n is a positive integer, then $E(\mathbb{T}^n) \leq_B E(G)$ iff $\mathbb{T}^n$ embeds in G.*
(5) *If n is a positive integer and G is compact, then G is n-dimensional iff $E(\mathbb{R}^n) <_B E(G) \leq_B E(\mathbb{T}^n)$.*

Let $\mathcal{P}$ denote the set of all primes. For $P, Q \in \mathcal{P}^\omega$, $Q \preceq P$ means that there is a co-finite subset A of ω and an injection $f : A \to \omega$ such that $Q(n) = P(f(n))$ for each $n \in A$.

For $P \in \mathcal{P}^\omega$, we consider the closed subgroup of $\mathbb{T}^\omega$, named *P-adic solenoid*, $\Sigma_P = \{g \in \mathbb{T}^\omega : \forall l\, (g(l) = g(l+1)^{P(l)})\}$ (cf. [8]).

THEOREM 1.4. *Let P and Q be in $\mathcal{P}^\omega$. Then $E(\Sigma_P) \leq_B E(\Sigma_Q)$ iff $Q \preceq P$.*

The partially ordered set $P(\omega)/\mathrm{Fin}$ is so complicated that every Boolean algebra of size $\leq \omega_1$ embeds into it (see [2]). We usually express that some classes of Borel equivalence relations are extremely complicated under the order of Borel reducibility by showing that $P(\omega)/\mathrm{Fin}$ embeds into them. For instance, Louveau–Velickovic [14] and Yin [18] showed that $P(\omega)/\mathrm{Fin}$ embeds into both LV-equalities and Borel equivalence relations between ℓ_p and ℓ_q, respectively. As an application, we prove that, the partially ordered set $P(\omega)/\mathrm{Fin}$ embeds into the partially ordered set of all $E(G)$'s under the ordering of Borel reducibility.

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THEOREM 1.5. *Let $n \in \mathbb{N}^+$. Then for $A \subseteq \omega$, there is an n-dimensional compact connected abelian Polish group G_A such that $E(\mathbb{R}^n) <_B E(G_A) <_B E(\mathbb{T}^n)$ and for $A, B \subseteq \omega$, we have*

$$A \subseteq^* B \iff E(G_A) \leq_B E(G_B).$$

We also get a sufficient and necessary condition concerning dual groups.

THEOREM 1.6 (Dual Rigid Theorem). *Let G be a compact connected abelian Polish group and let H be a locally compact abelian Polish group. Then $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S^* : \widehat{H} \to \widehat{G}$ such that $\widehat{G}/\mathrm{im}(S^*)$ is a torsion group.*

Notation convention. In this article, the addition operation of any subgroup of $\mathbb{R}^n$ is denoted by $+$ and its identity element is denoted by 0. Unless otherwise specified, for abstract abelian topological groups G, we still use multiplicative notation to express the group operation, and use 1_G to express the identity element of G, since we often consider subgroups of $\mathbb{T}^\omega$.

This article is organized as follows: In Section 2, we prove Theorems 1.1–1.3. In Section 3, we consider P-adic solenoids and prove Theorems 1.4 and 1.5. Finally, In Section 4, we consider dual groups and prove Theorem 1.6.

§2. Locally compact abelian Polish groups.

DEFINITION 2.1 [5, Definition 6.1]. Let G be a Polish group. We define equivalence relation $E_*(G)$ on G^ω as, for $x, y \in G^\omega$,

$$xE_*(G)y \iff \lim_n x(0)x(1)\dots x(n)y(n)^{-1}\dots y(1)^{-1}y(0)^{-1} \text{ converges.}$$

The following is an easy but important observation.

PROPOSITION 2.2. *Let G be a Polish group. Then $E(G) \sim_B E_*(G)$.*

PROOF. To see that $E(G) \leq_B E_*(G)$, for $x \in G^\omega$, we define $\theta(x) \in G^\omega$ as

$$\theta(x)(n) = \begin{cases} x(0), & n = 0, \\ x(n-1)^{-1}x(n), & n > 0. \end{cases}$$

Then θ witnesses that $E(G) \leq_B E_*(G)$.

To show the converse, for $x \in G^\omega$, we define $\vartheta(x) \in G^\omega$ as

$$\vartheta(x)(n) = x(0)x(1)\dots x(n).$$

Then ϑ witnesses that $E_*(G) \leq_B E(G)$. ⊣

In this article, we focus on abelian Polish groups. For abelian Polish groups G, it is more convenient to take $E_*(G)$ as research object than $E(G)$.

Some reducibility results are obtained in [5]. Since we will use them again and again in this article, for the convenience of readers, we list them as follows.

PROPOSITION 2.3 [5, Proposition 3.4]. *Let G, H be two Polish groups. If G is topologically isomorphic to a closed subgroup of H, then $E(G) \leq_B E(H)$.*

A metric d on a group G is called *two-sided invariant* if $d(ghl, gkl) = d(h, k)$ for all $g, h, k, l \in G$. We say that a Polish group G is TSI if it admits a compatible two-sided invariant metric. Any abelian Polish group is TSI (cf. [7, Exercise 2.1.4]).

LEMMA 2.4 [5, Theorem 6.5]. *Let G, H, K be three TSI Polish groups. Suppose $\psi : G \to H$ and $\varphi : H \to K$ are continuous homomorphisms with $\varphi(\psi(G)) = K$ such that* $\ker(\varphi \circ \psi)$ *is non-archimedean. If the interval* $[0, 1]$ *embeds into H, then $E(G) \leq_B E(H)$.*

LEMMA 2.5 [5, Theorem 6.13]. *Let G, H be TSI Polish groups such that H is locally compact. If $E(G) \leq_B E(H)$, then there exist an open normal subgroup G_c of G and a continuous map $S : G_c \to H$ with $S(1_G) = 1_H$ such that, for $x, y \in G_c^\omega$, if $\lim_n x(n)y(n)^{-1} = 1_G$, then*

$$xE_*(G_c)y \iff S(x)E_*(H)S(y),$$

where $S(x)(n) = S(x(n)), S(y)(n) = S(y(n))$ for each $n \in \omega$.

In particular, if $G = G_c$ and the interval $[0, 1]$ *embeds in H, then the converse is also true.*

REMARK 2.6. Since G_c in the preceding lemma is an open subgroup, it is also closed. So $G_0 \subseteq G_c$ as it is connected. Since S is continuous, we have $S(G_0) \subseteq H_0$. Moreover, for all $x, y \in G_0^\omega$, if $\lim_n x(n)y(n)^{-1} = 1_G$, we have

$$xE_*(G_0)y \iff S(x)E_*(H_0)S(y).$$

The next lemma plays the key role in the proof of Theorem 2.8.

LEMMA 2.7. *Let G and H be two abelian Polish groups such that*:

(1) *H is locally compact,*
(2) *$H_0 \subseteq \mathbb{R}^\omega \times \mathbb{T}^\omega$,*
(3) *there is a nonzero continuous homomorphism $f : \mathbb{R}^m \to G$ for some $m \in \mathbb{N}^+$.*

If $E_*(G) \leq_B E_*(H)$, then there is a continuous map $S : G_0 \to H_0$ such that the map S restricted on $f(\mathbb{R}^m)$ is a homomorphism to H_0.

PROOF. First, from Remark 2.6, we can obtain a continuous map $S : G_0 \to H_0$ with $S(1_{G_0}) = 1_{H_0}$ such that, for $x, y \in G_0^\omega$, if $\lim_n x(n)y(n)^{-1} = 1_{G_0}$, then

$$xE_*(G_0)y \iff S(x)E_*(H_0)S(y),$$

where $S(x)(n) = S(x(n)), S(y)(n) = S(y(n))$ for each $n \in \omega$.

Since $H_0 \subseteq \mathbb{R}^\omega \times \mathbb{T}^\omega$, without loss of generality we may assume that $h(2k) \in \mathbb{R}$ and $h(2k+1) \in \mathbb{T}$ for all $h \in H_0$. For $k \in \omega$, we define continuous homomorphisms $\phi^{2k} : H_0 \to \mathbb{R}$ and $\phi^{2k+1} : H_0 \to \mathbb{T}$ by $\phi^j(h) = h(j)$.

Now fix $g_0, g_1 \in f(\mathbb{R}^m)$ and find $a_0, a_1 \in \mathbb{R}^m$ such that $f(a_0) = g_0$ and $f(a_1) = g_1$. For $t \in [0, 1]$ and $l \in \{1, 2\}$, define $a^l(t) \in \mathbb{R}^m$ as

$$a^l(t) = \begin{cases} a_0 + t(a_1 - a_0), & l = 1, \\ t(a_0 + a_1), & l = 2. \end{cases}$$

By the following claim, we can easily construct a continuous function $F_j^l : [0, 1] \to \mathbb{R}$ for each $l \in \{1, 2\}$ and $k \in \omega$ such that

$$F_{2k}^l(t) = \phi^{2k}(S(f(a^l(t)))), \quad \exp(\mathrm{i}F_{2k+1}^l(t)) = \phi^{2k+1}(S(f(a^l(t)))). \qquad (*)$$

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The nontrivial part of the construction, i.e., $j = 2k + 1$, follows from a more general claim.

CLAIM 1. *Given a continuous function* $\gamma : [0, 1] \to \mathbb{T}$ *and* $t_0 \in [0, 1]$ *with* $\exp(\mathrm{i}s_0) = \gamma(t_0)$ *for some* $s_0 \in \mathbb{R}$, *there exists a continuous function* $\widetilde{\gamma} : [0, 1] \to \mathbb{R}$ *such that* $\exp(\mathrm{i}\widetilde{\gamma}(t)) = \gamma(t)$ *and* $\widetilde{\gamma}(t_0) = s_0$.

PROOF. Note that the map $t \mapsto \exp(\mathrm{i}t)$ is a covering map from $\mathbb{R}$ to $\mathbb{T}$, and the interval $[0, 1]$ is simply connected (see Definitions A2.1 and Proposition A2.8 of [11]). So such a $\widetilde{\gamma}$ exists (cf. [11, Definition A2.6]).

For the convenience of readers, we briefly explain the construction of $\widetilde{\gamma}$. Since the map $t \mapsto \exp(it)$ is a local homeomorphism, by the continuity of γ, for each $u \in [0, 1]$, there is an open interval J_u containing u and a continuous function $\widetilde{\gamma_u} : J_u \cap [0, 1] \to \mathbb{R}$ such that $\sup_{t,t' \in J_u \cap [0,1]} |\gamma(t) - \gamma(t')| < \frac{1}{2}$ and $\exp(\mathrm{i}\widetilde{\gamma_u}(t)) = \gamma(t)$ for $t \in J_u \cap [0, 1]$. Note that $\exp(\mathrm{i}(\widetilde{\gamma_u}(t) + 2p\pi)) = \exp(\mathrm{i}\widetilde{\gamma_u}(t))$ for each $p \in \mathbb{Z}$. By the compactness of $[0, 1]$, there are $u_0, u_1, \dots, u_q \in [0, 1]$ such that $[0, 1] \subseteq \bigcup_{0 \le i \le q} J_{u_i}$. We can find $0 = p_0, p_1, \dots, p_q \in \mathbb{Z}$ such that for each $t \in J_{u_i} \cap J_{u_j} \cap [0, 1]$, we have $\widetilde{\gamma_{u_i}}(t) + 2p_i\pi = \widetilde{\gamma_{u_j}}(t) + 2p_j\pi$. Then for $t \in [0, 1] \cap J_{u_i}$, let $\widetilde{\gamma}'(t) = \widetilde{\gamma_{u_i}}(t) + 2p_i\pi$. In the end, we put $\widetilde{\gamma}(t) = \widetilde{\gamma}'(t) - \widetilde{\gamma}'(t_0) + s_0$. It is obvious that $\exp(\mathrm{i}\widetilde{\gamma}(t)) = \gamma(t)$ and $\widetilde{\gamma}(t_0) = s_0$. ⊣

Note that $S(f(a^2(0))) = 1_H$. We can assume that $F^2_j(0) = 0$ for each j.

Next we claim that F^l_j are linear functions.

CLAIM 2. $F^l_j(t) = F^l_j(0) + t(F^l_j(1) - F^l_j(0))$ *for* $t \in [0, 1]$.

PROOF. We only verify the claim for $l = 1$. It is similar for $l = 2$.

Fix $j_0 \in \omega$. Define $\gamma : [0, 1] \to \mathbb{R}$ as $\gamma(t) = F^1_{j_0}(t) - F^1_{j_0}(0) - t(F^1_{j_0}(1) - F^1_{j_0}(0))$. Note that γ is continuous and $\gamma(0) = \gamma(1) = 0$. We only need to prove that $\gamma(t) = 0$ for all $t \in (0, 1)$.

If not, without loss of generality we may assume that $\gamma(t_0) > 0$ for some $t_0 \in (0, 1)$. Similar to the proof of [5, Lemma 6.17], we can find $0 < \xi_0 < \xi_1 < \xi_2 < \cdots < \xi < 1$ such that $\gamma(\xi_k) = \frac{k+1}{k+2}\gamma(t_0)$ for each $k \in \omega$, and $1 > \zeta_0 > \zeta_1 > \zeta_2 > \cdots > \zeta > 0$, $K \in \omega$ such that, for $k \ge K$, we have

$$\xi - \xi_k > \zeta_k - \zeta > \xi - \xi_{k+1},$$

$\lim_k \xi_k = \xi$, $\lim_k \zeta_k = \zeta$, $\gamma(\xi) = \gamma(t_0)$, and $\gamma(\zeta) > \gamma(\zeta_k)$ for each k.

Note that $f : \mathbb{R}^m \to G$ is a nonzero continuous homomorphism. For $p \in \omega$, we set

$$x(p) = \begin{cases} f(a^1(\xi)), & p = 2k, \\ f(a^1(\zeta)), & p = 2k+1, \end{cases} \qquad y(p) = \begin{cases} f(a^1(\xi_k)), & p = 2k, \\ f(a^1(\zeta_k)), & p = 2k+1. \end{cases}$$

From the alternating series test, the following series:

$$(\xi - \xi_0) + (\zeta - \zeta_0) + \cdots + (\xi - \xi_k) + (\zeta - \zeta_k) + \cdots$$

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is convergent. Then

$$\begin{aligned} & x(0)x(1)\dots x(2k)y(2k)^{-1}\dots y(1)^{-1}y(0)^{-1} \\ = \ & x(0)y(0)^{-1}x(1)y(1)^{-1}\dots x(2k)y(2k)^{-1} \\ = \ & f(((\xi-\xi_0)+(\zeta-\zeta_0)+\cdots+(\xi-\xi_k))(a_1-a_0)). \end{aligned}$$

Since f is continuous and $\lim_p x(p)y(p)^{-1} = 1_G$, we have $xE_*(G)y$. And hence, by Remark 2.6, we have $S(x)E_*(H)S(y)$.

On the other hand, we have

$$\sum_k(\gamma(\xi)-\gamma(\xi_k)+\gamma(\zeta)-\gamma(\zeta_k)) \geq \sum_k(\gamma(\xi)-\gamma(\xi_k)) = \sum_k \frac{\gamma(t_0)}{k+2} = \infty.$$

Note that

$$\begin{aligned} & F^1_{j_0}(\xi)-F^1_{j_0}(\xi_k)+F^1_{j_0}(\zeta)-F^1_{j_0}(\zeta_k) \\ = \ & \gamma(\xi)-\gamma(\xi_k)+\gamma(\zeta)-\gamma(\zeta_k)+(\xi-\xi_k+\zeta-\zeta_k)(F^1_{j_0}(1)-F^1_{j_0}(0)). \end{aligned}$$

If $j_0 = 2i$, then

$$\begin{aligned} & \phi^{2i}(S(x(0))S(x(1))\dots S(x(2k))S(y(2k))^{-1}\dots S(y(1))^{-1}S(y(0))^{-1}) \\ = \ & F^1_{j_0}(\xi)-F^1_{j_0}(\xi_0)+F^1_{j_0}(\zeta)-F^1_{j_0}(\zeta_0)+\cdots+F^1_{j_0}(\xi)-F^1_{j_0}(\xi_k). \end{aligned}$$

Thus $S(x)E_*(H)S(y)$ fails. We get a contradiction. If $j_0 = 2i+1$, following similar arguments, we can also get a contradiction. This complete the proof of the claim. ⊣

Now by Claim 2 and $F^2_j(0) = 0$, we know that

$$F^1_j(1/2) = F^1_j(0) + (F^1_j(1)-F^1_j(0))/2 = (F^1_j(0)+F^1_j(1))/2,$$

$$F^2_j(1/2) = F^2_j(1)/2.$$

By comparing equation $(*)$ before Claim 1, it follows that

$$S(f(a^1(1/2)))^2 = S(f(a_0))S(f(a_1)) = S(g_0)S(g_1),$$

$$S(f(a^2(1/2)))^2 = S(f(a_0+a_1)) = S(g_0g_1).$$

Since $a^1(1/2) = a^2(1/2)$, we have $S(g_0)S(g_1) = S(g_0g_1)$.

So, the map $S : f(\mathbb{R}^m) \to H_0$ is a continuous homomorphism. ⊣

Let us recall the structure of Hausdorff locally compact abelian groups. Let G be a Hausdorff locally compact abelian group, then G is topologically isomorphic to the group $\mathbb{R}^n \times H$, where H is a locally compact abelian group containing a compact open subgroup (cf. [10, Theorem 24.30]). Moreover, if G is also connected, then it is a direct product of a compact connected abelian group K and the group $\mathbb{R}^n$ (cf. [10, Theorem 9.14]). Any locally compact connected metrizable abelian group can be embedded as a closed subgroup of $\mathbb{R}^n \times \mathbb{T}^\omega$. In particular, all compact metrizable abelian groups can be embedded in $\mathbb{T}^\omega$ (see page 119 of [1]). G is said to be *solenoidal* if there is a continuous homomorphism $f : \mathbb{R} \to G$ such that $f(\mathbb{R})$ is dense in G (see [10, (9.2)]). It is well known that a compact metrizable abelian group is solenoidal iff it is connected (see page 13 and Proposition 5.16 of [1]). Thus for each locally compact connected metrizable abelian group G, there is a

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continuous homomorphism $f: \mathbb{R}^m \to G$ which satisfies $\overline{f(\mathbb{R}^m)} = G$. For more details on locally compact abelian groups, we refer to [1, 10].

By applying Lemma 2.7 for locally compact abelian Polish groups, we get the following result.

THEOREM 2.8. *Let G and H be two locally compact abelian Polish groups. If $E(G) \leq_B E(H)$, then there is a continuous homomorphism $S : G_0 \to H_0$ such that* $\ker(S)$ *is non-archimedean.*

PROOF. If $E(G) \leq_B E(H)$, then $E_*(G) \leq_B E_*(H)$. Without loss of generality we may assume that G_0 is nontrivial. First note that H_0 can be embedded into $\mathbb{R}^n \times \mathbb{T}^\omega$. Thus we may assume without loss of generality that $H_0 \subseteq \mathbb{R}^\omega \times \mathbb{T}^\omega$. Let f be a continuous homomorphism from $\mathbb{R}^m$ to G_0 with $\overline{f(\mathbb{R}^m)} = G_0$. Then by Lemma 2.7 there exists a continuous map $S : G_0 \to H_0$ such that the map S restricted on $f(\mathbb{R}^m)$ is a homomorphism to H_0. Since $f(\mathbb{R}^m)$ is dense in G_0, we see that S is a homomorphism from G_0 to H_0.

Then we only need to check that $\ker(S)$ is non-archimedean. Assume toward a contradiction that $\ker(S)$ is not non-archimedean.

Note that $\ker(S)$ is an abelian Polish group. Fix a compatible two-sided invariant metric on $\ker(S)$. Let $V_k \subseteq \ker(S)$, $k \in \omega$ be an open symmetric neighborhood base of $1_{\ker(S)} = 1_G$ with $\lim_k \operatorname{diam}(V_k) = 0$. Then there exists a $k_0 \in \omega$ such that V_{k_0} does not contain any open subgroup of $\ker(S)$. Since V_k is symmetric, $\bigcup_m V_k^m$ is an open subgroup of $\ker(S)$, so $\bigcup_m V_k^m \nsubseteq V_{k_0}$ for each k. Thus we can find an $m_k \in \omega$ and $g_{k,0}, \ldots, g_{k,m_k-1} \in V_k$ such that $g_{k,0}g_{k,1} \ldots g_{k,m_k-1} \notin V_{k_0}$.

Denote $M_{-1} = 0$ and $M_k = m_0 + m_1 + \cdots + m_k$ for $k \in \omega$. Now for $n \in \omega$, define

$$x(n) = \begin{cases} g_{k,j}, & n = M_{k-1} + j, 0 \leq j < m_k, \\ 1_G, & \text{otherwise.} \end{cases}$$

Therefore $x E_*(G) 1_{G^\omega}$ fails. Note that we have $\lim_n x(n) = 1_G$ and $S(x(n)) = 1_H$ for each n. So it is trivial that $S(x) E_*(H_0) S(1_{G^\omega})$, where $S(x)(n) = S(x(n))$, contradicting Lemma 2.5. ⊣

In particular, if G is compact connected, then the converse of Theorem 2.8 is also true.

THEOREM 2.9 (Rigid Theorem). *Let G be a compact connected abelian Polish group and let H be a locally compact abelian Polish group. Then $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S : G \to H$ such that* $\ker(S)$ *is non-archimedean.*

PROOF. Let S be a continuous homomorphism from G to H such that $\ker(S)$ is non-archimedean. Since G is compact, $S(G)$ is a compact, thus closed subgroup of H. So we have $E(S(G)) \leq_B E(H)$.

Note that $S(G)$ is also a compact connected abelian Polish group. Let f be a continuous homomorphism $f : \mathbb{R} \to S(G)$ such that $\overline{f(\mathbb{R})} = S(G)$. Then $\ker(f)$ is a proper closed subgroup of $\mathbb{R}$. Hence $\ker(f)$ is a discrete group. This gives that the interval $[0, 1]$ embeds in $S(G)$. Then by Lemma 2.4, we get that $E(G) \leq_B E(S(G)) \leq_B E(H)$.

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On the other hand, if $E(G) \leq_B E(H)$, by Theorem 2.8, there is a continuous homomorphism $S : G_0 \to H_0$ such that $\ker(S)$ is non-archimedean. Since G is connected, we have $G = G_0$. ⊣

COROLLARY 2.10. *Let G be a compact connected abelian Polish group and let H be a locally compact abelian Polish group. Suppose $H_0 \cong \mathbb{R}^n \times K$, where K is a compact connected abelian group. Then $E(G) \leq_B E(H)$ iff $E(G) \leq_B E(K)$.*

PROOF. ($\Leftarrow$) part is trivial, since $E(K) \leq_B E(H_0) \leq_B E(H)$.

($\Rightarrow$). If $E(G) \leq_B E(H)$, then there exists a continuous homomorphism $S : G \to H$ such that $\ker(S)$ is non-archimedean. So $S(G)$ is a connected compact subgroup of H, thus $S(G) \subseteq H_0$. Without loss of generality, we assume that $H_0 = \mathbb{R}^n \times K$. Let $\pi : H_0 \to \mathbb{R}^n$ and $\pi_K : H_0 \to K$ be canonical projections. Then $\pi(S(G))$ is a compact subgroup of $\mathbb{R}^n$, so $\pi(S(G)) = \{0\}$. It follows that $\ker(\pi_K \circ S) = \ker(S)$. Applying Theorem 2.9 on $\pi_K \circ S : G \to K$, we have $E(G) \leq_B E(K)$. ⊣

Recall that a topological group G is *totally disconnected* if $G_0 = \{1_G\}$. For any locally compact abelian Polish group, it is totally disconnected iff it is non-archimedean (cf. [11, Theorem 1.34]).

For every normal space X, denoted by $\dim(X)$ the *covering dimension* of X, where $\dim(X)$ is an integer ≥ -1 or the "infinite number ∞." We omit the definition of covering dimension since it is very complicated (see page 54 of [6]). We recall the following useful facts concerning compact abelian group G: $\dim(G) = n < \infty$ iff G has a totally disconnected closed subgroup Δ such that $G/\Delta \cong \mathbb{T}^n$ iff there is a compact totally disconnected subgroup N of G and a continuous surjective homomorphism $\varphi : N \times \mathbb{R}^n \to G$ which has a discrete kernel (see Theorem 8.22 and Corollary 8.26 of [11]). In this case, we say that G is *finite dimensional* (cf. [11, Definitions 8.23]). Clearly, $\dim(G) = 0$ iff G is totally disconnected. For more details on compact abelian groups, see [11].

Now we recall two equivalence relations E_0^ω and $E(M;0)$ (see [4, Definition 3.2]). The equivalence relation E_0^ω on $2^{\omega\times\omega}$ defined by

$$xE_0^\omega y \iff \forall k \, \exists m \, \forall n \geq m \, (x(n,k) = y(n,k)).$$

Fix a metric space M. The equivalence relation $E(M;0)$ on M^ω defined by

$$xE(M;0)y \iff \lim_n d(x(n), y(n)) = 0.$$

From the above discussions, we can establish the following theorem.

THEOREM 2.11. *Let G, H be locally compact abelian Polish groups.*

(1) *If G is non-archimedean, then $E(G) \leq_B E_0^\omega$.*
(2) *If G is not non-archimedean, then $E(\mathbb{R}) \leq_B E(G)$.*
(3) *If G is not non-archimedean and G_0 is open, then $E(G) \sim_B E(G_0)$.*
(4) *If n is a positive integer, then $E(\mathbb{T}^n) \leq_B E(G)$ iff $\mathbb{T}^n$ embeds in G.*
(5) *If n is a positive integer and G is compact, then G is n-dimensional iff $E(\mathbb{R}^n) <_B E(G) \leq_B E(\mathbb{T}^n)$.*

PROOF. (1) It follows from [5, Theorem 3.5(3)].

(2) Note that G is not totally disconnected (cf. [11, Theorem 1.34]), so G_0 contains at least two points. We have $G_0 \cong \mathbb{R}^n \times K$, where K is a compact connected

abelian group. If $n > 0$, it is trivial that $E(\mathbb{R}) \leq_B E(G)$. By Proposition 2.3, $E(K) \leq_B E(G_0) \leq_B E(G)$. Thus we may assume that G is compact connected and $G \subseteq \mathbb{T}^\omega$. Note that there is a continuous homomorphism $f : \mathbb{R} \to G$ such that $\overline{f(\mathbb{R})} = G$. For $g \in G \subseteq \mathbb{T}^\omega$ and $p \in \omega$, let $\phi_p(g) = g(p)$. Since G contains at least two points, we can find $p_0 \in \omega$ such that $\phi_{p_0}(f(\mathbb{R})) \neq \{1_\mathbb{T}\}$, so $\phi_{p_0}(f(\mathbb{R})) = \mathbb{T}$. By [11, Corollary 8.24], the interval $[0,1]$ embeds in G. Then by Lemma 2.4, we have $E(\mathbb{R}) \leq_B E(G)$.

(3) By [10, Section 24.45], we have $G \cong G_0 \times G/G_0$. Since G_0 is open, G/G_0 is countable and discrete. By [5, Corollary 3.6], this implies that $E(G_0 \times G/G_0) \sim_B E(G_0)$ and thus $E(G) \sim_B E(G_0)$.

(4) The "if" part follows Proposition 2.3. Assume that $E(\mathbb{T}^n) \leq_B E(G)$. By Theorem 2.8 and [7, Corollary 2.3.4], there is a closed subgroup Δ of $\mathbb{T}^n$ such that the group $\mathbb{T}^n/\Delta$ can be embedded in G, where Δ is non-archimedean. It is obvious that $\mathbb{T}^n/\Delta$ is a locally connected, connected and compact abelian Polish group. By [1, Proposition 8.17], $\mathbb{T}^n/\Delta \cong \mathbb{T}^n$.

(5) If $n = \dim(G)$, then we have $(N \times \mathbb{R}^n)/\Delta_1 \cong G$ and $G/\Delta_2 \cong \mathbb{T}^n$, where N, Δ_1, and Δ_2 are totally disconnected, and hence are non-archimedean. Then Proposition 2.3 and Lemma 2.4 imply that

$$E(\mathbb{R}^n) \leq_B E(N \times \mathbb{R}^n) \leq_B E(G) \leq_B E(\mathbb{T}^n).$$

So we only need to show that $E(G) \not\leq_B E(\mathbb{R}^n)$. To see this, assume toward a contradiction that $E(G) \leq_B E(\mathbb{R}^n)$. By Theorem 2.8, there exists a continuous homomorphism $S : G_0 \to \mathbb{R}^n$ such that $\ker(S)$ is non-archimedean. Note that $\mathbb{R}^n$ has no nontrivial compact connected subgroup. So this implies that $S(G_0) = \{0\}$, contradicting that $\ker(S)$ is non-archimedean.

On the other hand, suppose $E(\mathbb{R}^n) <_B E(G) \leq_B E(\mathbb{T}^n)$. Let $m = \dim(G)$. By (1) we have $m > 0$. Assume for contradiction that $m = \infty$, then there exists a continuous homomorphism $S : G_0 \to \mathbb{T}^n$ such that $\ker(S)$ is non-archimedean. Then we have $\dim(G_0/\ker(S)) = \infty$, and hence $[0,1]^\omega$ embeds into $G_0/\ker(S)$ (cf. [11, Corollary 8.24]). By [7, Corollary 2.3.4], S induces an embedding from $G_0/\ker(S)$ to $\mathbb{T}^n$. So $[0,1]^\omega$ embeds into $\mathbb{T}^n$, contradicting that n is finite. Therefore, we have $0 < m < \infty$, and hence $E(\mathbb{R}^m) <_B E(G) \leq_B E(\mathbb{T}^m)$. Then [5, Theorem 6.19] gives $m = n$. ⊣

Remark 2.12. Let G and H be two locally compact abelian Polish groups. Suppose that G_0 is an open subgroup of G, and that G_0 is compact or $G_0 \cong \mathbb{R}$. Then Theorems 2.8, 2.9, and 2.11(2),(3) imply that $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S : G_0 \to H_0$ such that $\ker(S)$ is non-archimedean. This generalizes Rigid Theorem, i.e., Theorem 2.9. We don't know whether this can be generalized to all locally compact abelian Polish groups.

Question 2.13. *Does the converse of Theorem 2.8 hold for all locally compact abelian Polish groups*?

Question 2.14. *Let G be a locally compact abelian Polish group. If G is not non-archimedean, does $E(G) \sim_B E(G_0)$*?

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§3. *P*-adic solenoids. Let $P = (P(0), P(1), \dots)$ be a sequence of integers greater than 1. Recall that the *P-adic solenoid* S_P is defined by

$$S_P = \{g \in \mathbb{T}^\omega : \forall l \, (g(l) = g(l+1)^{P(l)})\}.$$

In particular, if for each i, $P(i)$ is a prime number, then the P-adic solenoid is denoted by Σ_P (cf. [8]). Let $\mathcal{P}$ denote the set of all primes. The group S_P is topologically isomorphic to $\Sigma_{P'}$ for some $P' \in \mathcal{P}^\omega$ satisfying that $P(l) = P'(i_l) \dots P'(i_{l+1}-1)$, where $0 = i_0 < i_1 < \cdots < i_l < \cdots$. For example, we have $S_{(4,6,8,9,\dots,9,\dots)} \cong \Sigma_{(2,2,2,3,2,2,2,3,3,\dots,3,3,\dots)}$.

It is well known that, the group Σ_P is a compact connected abelian group which is neither locally connected (cf. [8]), nor arcwise connected (see [1, Theorem 8.27]). Every nontrivial proper closed subgroup H of a P-adic solenoid is totally disconnected (cf. [12, Proposition 2.7]), and thus H is non-archimedean. Clearly, Σ_P is a 1-dimensional and metrizable group.

Denote $\Omega = \{\mathbb{R}, \mathbb{T}, \Sigma_P : P \in \mathcal{P}^\omega\}$.

LEMMA 3.1. *Let $m, n \in \mathbb{N}^+$ and let $G_1, G_2, \dots, G_m, H_1, H_2 \dots, H_n \in \Omega$. Then the following are equivalent*:

(1) $E(G_1 \times G_2 \times \cdots \times G_m) \leq_B E(H_1 \times H_2 \times \cdots \times H_n)$.
(2) *There is a injective map $\theta^* : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ such that $E(G_i) \leq_B E(H_{\theta^*(i)})$ for $1 \leq i \leq m$.*

In particular, $E(G_1^m) \leq_B E(H_1^n)$ iff $m \leq n$ and $E(G_1) \leq_B E(H_1)$.

PROOF. (2) $\Rightarrow$ (1) is obvious. We only prove (1) $\Rightarrow$ (2).

Denote $G = G_1 \times G_2 \times \cdots \times G_m$ and $H = H_1 \times H_2 \times \cdots \times H_n$. For $1 \leq i \leq m$, let e^i be the canonical injection of G_i into $G_1 \times \cdots \times G_m$, i.e., $e^i(g) = (1_{G_1}, \dots, 1_{G_{i-1}}, g, 1_{G_{i+1}}, \dots, 1_{G_m})$.

Suppose $E(G) \leq_B E(H)$. Since G and H are both connected, by Theorem 2.8, there is a continuous homomorphism $S : G \to H$ such that $\ker(S)$ is non-archimedean. For each $1 \leq j \leq n$, let π_j be the canonical projection from H onto H_j.

Note that, except for $\mathbb{R}$, all groups in Ω are compact. By rearranging, we may assume that there is an $i_0 \leq m$ such that, G_i is compact for $1 \leq i \leq i_0$, and $G_i = \mathbb{R}$ for $i_0 < i \leq m$.

For any $1 \leq i \leq i_0$, since $\ker(S)$ is non-archimedean, there exists j satisfying that $\pi_j(S(e^i(G_i))) \neq \{1_{H_j}\}$. Note that H_j has no nontrivial proper connected compact subgroup. It follows that $\pi_j(S(e^i(G_i))) = H_j$. Now we construct a bipartite graph $G[X, Y]$ as follows. Let $X = \{G_1, G_2, \dots, G_{i_0}\}$,

$$Y = \{H_j : \exists i \, (1 \leq i \leq i_0 \text{ and } \pi_j(S(e^i(G_i))) = H_j)\}.$$

For $G_i \in X$ and $H_j \in Y$, we put an edge between G_i and H_j if $\pi_j(S(e^i(G_i))) = H_j$. Given $K \subseteq X$, we denote the set of all neighbors of the vertices in K by $N(K)$.

Next we show that $|N(K)| \geq |K|$ for all $K \subseteq X$. Given $K \subseteq X$, denote

$$G^K = \{x \in G : x(i) = 1_{G_i} \text{ for all } G_i \notin K\},$$

$$H^{N(K)} = \{z \in H : z(j) = 1_{H_j} \text{ for all } H_j \notin N(K)\}.$$

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Then the restriction of S on G^K is a continuous homomorphism to $H^{N(K)}$. By Theorem 2.9, $E(G^K) \leq_B E(H^{N(K)})$. Again by Theorem 2.11(5), this implies $E(\mathbb{R}^{|K|}) \leq_B E(\mathbb{T}^{|N(K)|})$. Then [5, Theorem 6.19] gives $|N(K)| \geq |K|$.

By Hall's theorem (cf. [3, Theorem 16.4]), there is a injective map $\theta^* : \{1, 2, \dots, i_0\} \to \{1, 2, \dots, n\}$ such that $\pi_{\theta^*(i)}(S(e^i(G_i))) = H_{\theta^*(i)}$. Since every proper closed subgroup of G_i is non-archimedean, from Theorem 2.9, we have $E(G_i) \leq_B E(H_{\theta^*(i)})$.

In the end, since $\dim(G) = m$ and $\dim(H) = n$, by Theorem 2.11(5), we have $E(\mathbb{R}^m) \leq_B E(\mathbb{T}^n)$. So $m \leq n$. Since $E(\mathbb{R}) \leq_B E(H_j)$ for each j, we can trivially extend θ^* to an injection from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$ such that $E(G_i) \leq_B E(H_{\theta^*(i)})$ for each i. ⊣

Let P and Q be in $\mathcal{P}^\omega$. We write $Q \preceq P$ provided there is a co-finite subset A of ω and an injection $f : A \to \omega$ such that $Q(n) = P(f(n))$ for each $n \in A$ (for more details, see [8, 9, 16]).

Lemma 3.2 (folklore). *Let P and Q be in $\mathcal{P}^\omega$. Then the following are equivalent*:

(1) *There is a nonzero continuous homomorphism $f : \Sigma_P \to \Sigma_Q$.*
(2) *There is a surjective continuous homomorphism $g : \Sigma_P \to \Sigma_Q$.*
(3) *There is a surjective continuous map $h : \Sigma_P \to \Sigma_Q$.*
(4) *$Q \preceq P$.*

Proof. (2) ⇒ (1) and (2) ⇒ (3) are obvious. (1) ⇒ (2) follows immediately from the fact that each nontrivial proper closed subgroup of a P-adic solenoid is totally disconnected. The equivalence of (3) and (4) follows from [16, Theorem 4.4].

It remains to show (3) ⇒ (1). Let h be a surjective continuous map from Σ_P to Σ_Q. Without loss of generality assume that $h(1_{\Sigma_P}) = 1_{\Sigma_Q}$. Then there exists a continuous homomorphism $f : \Sigma_P \to \Sigma_Q$ such that h is homotopic to f (cf. [17, Corollary 2]). Since Σ_Q is not arcwise connected, $\ker(f) \neq \Sigma_P$. ⊣

Theorem 3.3. *Let P and Q be in $\mathcal{P}^\omega$. Then $E(\Sigma_P) \leq_B E(\Sigma_Q)$ iff $Q \preceq P$ iff there is a nonzero continuous homomorphism $f : \Sigma_P \to \Sigma_Q$.*

Proof. Note that every nontrivial proper closed subgroup of Σ_P is non-archimedean. Then this follows from Theorem 2.9 and Lemma 3.2. ⊣

Let Fin denote the set of all finite subsets of ω. For $A, B \subseteq \omega$, we use $A \subseteq^* B$ to denote $A \setminus B \in \mathrm{Fin}$.

We prove that, for $n \in \mathbb{N}^+$, the partially ordered set $P(\omega)/\mathrm{Fin}$ can be embedded into Borel equivalence relations between $E(\mathbb{R}^n)$ and $E(\mathbb{T}^n)$.

Lemma 3.4. *Let P be in $\mathcal{P}^\omega$. Then $E(\mathbb{R}) <_B E(\Sigma_P) <_B E(\mathbb{T})$.*

Proof. By Theorem 2.11(5), we have that $E(\mathbb{R}) <_B E(\Sigma_P) \leq_B E(\mathbb{T})$.

Assume toward a contradiction that $E(\mathbb{T}) \leq_B E(\Sigma_P)$. From Theorem 2.11(4), $\mathbb{T}$ embeds in Σ_P. This is impossible, since Σ_P is not arcwise connected and every proper closed subgroup of Σ_P is non-archimedean. ⊣

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For $P \in \mathcal{P}^\omega$ and $\gamma \in \mathcal{P}$, we define $t^P(\gamma) \in \omega \cup \{\omega\}$ as

$$t^P(\gamma) = \begin{cases} \omega, & \exists^\infty j \in \omega\,(P(j) = \gamma), \\ |\{j : P(j) = \gamma\}|, & \text{otherwise.} \end{cases}$$

Given $P, Q \in \mathcal{P}^\omega$, denote

$$D(P, Q) = \{\gamma \in \mathcal{P} : t^P(\gamma) < t^Q(\gamma)\}.$$

From the definition of $Q \preceq P$, we can easily see that

$$E(\Sigma_P) \leq_B E(\Sigma_Q) \iff Q \preceq P \iff \sum_{\gamma \in D(P,Q)} (t^Q(\gamma) - t^P(\gamma)) \text{ is finite.}$$

LEMMA 3.5. *Let $P, Q \in \mathcal{P}^\omega$ with $E(\Sigma_Q) \leq_B E(\Sigma_P)$. Suppose that $D(P, Q)$ is infinite. Then for $A \subseteq \omega$, there is a group Σ_{P_A} such that $E(\Sigma_Q) <_B E(\Sigma_{P_A}) <_B E(\Sigma_P)$ and for $A, B \subseteq \omega$, we have*

$$A \subseteq^* B \iff E(\Sigma_{P_A}) \leq_B E(\Sigma_{P_B}).$$

PROOF. Enumerate $D(P, Q)$ as $d_0 < d_1 < d_2 < \dots$. Let $P_0^* \in \mathcal{P}^\omega$ such that $P_0^*(i) = d_{3i}$ for all $i \in \omega$.

For $L, M \in \mathcal{P}^\omega$, we define an element $L \oplus M \in \mathcal{P}^\omega$ as

$$(L \oplus M)(n) = \begin{cases} L(k), & n = 2k, \\ M(k), & n = 2k + 1. \end{cases}$$

It is clear that

$$t^{L \oplus M}(\gamma) = \begin{cases} \omega, & t^L(\gamma) = \omega \text{ or } t^M(\gamma) = \omega, \\ t^L(\gamma) + t^M(\gamma), & \text{otherwise.} \end{cases}$$

Given a set $A \subseteq \omega$, define $P_A \in \mathcal{P}^\omega$ as follows. If $\omega \setminus A$ is finite, put $P_A = P_0^* \oplus P$. Then

$$t^{P_A}(\gamma) = \begin{cases} t^P(\gamma) + 1, & \gamma = d_{3i}, i \in \omega, \\ t^P(\gamma), & \text{otherwise.} \end{cases}$$

If $\omega \setminus A$ is infinite, enumerate it as $a_0 < a_1 < a_2 < \dots$. Define $P_A^* \in \mathcal{P}^\omega$ as $P_A^*(j) = d_{1+3a_j}$ for $j \in \omega$, and put $P_A = P_A^* \oplus (P_0^* \oplus P)$. Then

$$t^{P_A}(\gamma) = \begin{cases} t^P(\gamma) + 1, & (\gamma = d_{3i}, i \in \omega) \text{ or } (\gamma = d_{1+3a}, a \in (\omega \setminus A)), \\ t^P(\gamma), & \text{otherwise.} \end{cases}$$

Next we show that $E(\Sigma_Q) <_B E(\Sigma_{P_A}) <_B E(\Sigma_P)$ for all $A \subseteq \omega$.

First, since $t^P(\gamma) \leq t^{P_A}(\gamma)$ for all $\gamma \in \mathcal{P}$, we have $D(P_A, P) = \emptyset$. So $P \preceq P_A$, and hence $E(\Sigma_{P_A}) \leq_B E(\Sigma_P)$.

Since $E(\Sigma_Q) \leq_B E(\Sigma_P)$, by Theorem 3.3, we have $P \preceq Q$, and hence

$$\sum_{\gamma \in D(Q,P)} (t^P(\gamma) - t^Q(\gamma)) \text{ is finite.}$$

Note that $t^{P_A}(\gamma) = t^P(\gamma) + 1$ only occurs when $t^Q(\gamma) > t^P(\gamma)$ holds, in which case we always have $\gamma \notin D(Q, P_A)$. So we have $D(Q, P_A) = D(Q, P)$ and $t^{P_A}(\gamma) = t^P(\gamma)$ for all $\gamma \in D(Q, P_A)$. This gives $E(\Sigma_Q) \leq_B E(\Sigma_{P_A})$.

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Since $d_{3i} \in D(P, P_A)$ for $i \in \omega$, $D(P, P_A)$ is infinite, so $E(\Sigma_P) \not\leq_B E(\Sigma_{P_A})$. Similarly, since $t^{P_A}(d_{2+3i}) = t^P(d_{2+3i}) < t^Q(d_{2+3i})$, we have $d_{2+3i} \in D(P_A, Q)$ for $i \in \omega$, so $E(\Sigma_{P_A}) \not\leq_B E(\Sigma_Q)$.

Given $A, B \subseteq \omega$, note that $A \subseteq^* B$ iff $(\omega \setminus B) \setminus (\omega \setminus A) = (A \setminus B)$ is finite. We will check that $A \subseteq^* B$ iff $P_B \preceq P_A$. We consider four cases as follows: (1) If both $\omega \setminus A$ and $\omega \setminus B$ are finite, then we have $A \subseteq^* B$ and $P_A = P_B = P_0^* \oplus P$. (2) If $\omega \setminus A$ is infinite and $\omega \setminus B$ is finite, then we have $A \subseteq^* B$ and $P_B = P_0^* \oplus P \preceq P_A^* \oplus (P_0^* \oplus P) = P_A$, since $t^{P_B}(\gamma) \leq t^{P_A}(\gamma)$ for all $\gamma \in \mathcal{P}$. (3) If $\omega \setminus A$ is finite and $\omega \setminus B$ is infinite, then $A \not\subseteq^* B$ and $P_B = P_B^* \oplus (P_0^* \oplus P) \not\preceq P_0^* \oplus P = P_A$, since $t^{P_A}(d_{1+3b}) < t^{P_B}(d_{1+3b})$ for $b \in (\omega \setminus B)$. (4) If both $\omega \setminus A$ and $\omega \setminus B$ are infinite, then $t^{P_A}(\gamma) < t^{P_B}(\gamma)$ iff $\gamma = d_{1+3b}$ for some $b \in (\omega \setminus B) \setminus (\omega \setminus A) = (A \setminus B)$. Moreover, $t^{P_B}(d_{1+3b}) = t^P(d_{1+3b}) + 1 = t^{P_A}(d_{1+3b}) + 1$ for all $b \in (A \setminus B)$. So

$$\sum_{\gamma \in D(P_A, P_B)} (t^{P_B}(\gamma) - t^{P_A}(\gamma)) = |A \setminus B|,$$

and hence $A \subseteq^* B$ iff $P_B \preceq P_A$.

Again by Theorem 3.3, we have $A \subseteq^* B$ iff $E(\Sigma_{P_A}) \leq_B E(\Sigma_{P_B})$. ⊣

THEOREM 3.6. *Let $n \in \mathbb{N}^+$. Then for $A \subseteq \omega$, there is an n-dimensional compact connected abelian Polish group G_A such that $E(\mathbb{R}^n) <_B E(G_A) <_B E(\mathbb{T}^n)$ and for $A, B \subseteq \omega$, we have*

$$A \subseteq^* B \iff E(G_A) \leq_B E(G_B).$$

PROOF. It follows from Theorem 2.11(5) and Lemmas 3.1, 3.4, and 3.5. ⊣

§4. Dual groups. Let G and H be two abelian topological groups. Denote the class of all continuous homomorphisms of G to H by $\mathrm{Hom}(G, H)$, which is an abelian group under pointwise addition. We always equip $\mathrm{Hom}(G, H)$ with compact-open topology. The abelian topological group $\mathrm{Hom}(G, \mathbb{T})$ is called the *dual group* of G, denoted by $\widehat{G}$ (cf. [11, Definition 7.4]).

Let $(A, +)$ be an abelian group whose identity element denoted by 0_A. We say that $(A, +)$ is a *torsion group* if each element of A is finite order. We say that $(A, +)$ is *torsion-free* if $n \cdot g \neq 0_A$ for all $g \in A$ with $g \neq 0_A$ and $n \in \mathbb{N}^+$. A subset X of A is *free* if any equation $\sum_{x \in X} n_x \cdot x = 0_A$ implies $n_x = 0$ for all $x \in X$. The *torsion-free rank* of A, written $\mathrm{rank}(A)$, is the cardinal number (uniquely determined) of any maximal free subset of A.

Each Hausdorff locally compact abelian group G is reflexive, thus it is topologically isomorphic to the double dual group $\widehat{\widehat{G}}$ (cf. [11, Theorem 7.63]). A Hausdorff locally compact abelian group is compact and metrizable iff its dual group is a countable discrete group (cf. Proposition 7.5(i) and Theorem 8.45 of [11]). Let G be a Hausdorff compact abelian group, then G is connected iff $\widehat{G}$ is torsion-free; and G is totally disconnected iff $\widehat{G}$ is torsion (cf. [11, Corollary 8.5]). For any finite dimensional compact abelian Polish group G, the covering dimension of A is equal to $\mathrm{rank}(\widehat{G})$ (cf. Lemma 8.13 and Corollary 8.26 of [11]).

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If H is a subset of an abelian topological group G, then the subgroup

$$H^{\perp} = \{\gamma \in \widehat{G} : \forall x \in H\ (\gamma(x) = 1_{\mathbb{T}})\}$$

is called the *annihilator* of H in $\widehat{G}$ (cf. [11, Definition 7.12]).

Now we focus on compact connected abelian Polish groups.

THEOREM 4.1 (Dual Rigid Theorem). *Let G be a compact connected abelian Polish group and let H be a locally compact abelian Polish group. Then $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S^* : \widehat{H} \to \widehat{G}$ such that $\widehat{G}/\mathrm{im}(S^*)$ is a torsion group.*

PROOF. ($\Rightarrow$). We assume that $E(G) \leq_B E(H)$. By Theorem 2.8, there is a continuous homomorphism $S : G \to H$ such that $\ker(S)$ is non-archimedean. This implies that there is a homomorphism S^* from $\widehat{H}$ to $\widehat{G}$ such that $\ker(S) \cong \mathrm{im}(S^*)^{\perp}$ (cf. [1, P.22 and P.23(a)]). By [11, Lemma 7.13(ii)], we have that $\ker(S) \cong (\widehat{G}/\mathrm{im}(S^*))\widehat{\ }$, and hence $\widehat{\ker(S)} \cong \widehat{G}/\mathrm{im}(S^*)$. Since $\ker(S)$ is non-archimedean, thus is totally disconnected, so $\widehat{G}/\mathrm{im}(S^*)$ is a torsion group.

($\Leftarrow$). Since $G \cong \widehat{\widehat{G}}$ and $H \cong \widehat{\widehat{H}}$, we can define $S : G \to H$ via $(S^*)^* : \widehat{\widehat{G}} \to \widehat{\widehat{H}}$ (cf. [10, (24.41)]). Then the similar arguments as the preceding paragraph give the desired result. ⊣

COROLLARY 4.2. *Let G be a compact connected abelian Polish group and let H be a locally compact abelian Polish group. If $E(G) \leq_B E(H)$, then there is a nonzero continuous homomorphism $S^* : \widehat{H} \to \widehat{G}$.*

PROOF. It follows from Theorem 4.1 and that $\widehat{G}$ is non-torsion. ⊣

EXAMPLE 4.3. $\widehat{\mathbb{T}} \cong \mathbb{Z}$ (cf. [10, Examples 23.27(a)]). Fix a $P \in \mathcal{P}^{\omega}$, then $\widehat{\Sigma_P} \cong \left\{\frac{m}{P(0)P(1)\dots P(n)} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ (see [10, (25.3)]). In view of Corollary 4.2, we get $E(\mathbb{T}) \nleq_B E(\Sigma_P)$ again.

Recall that $\widehat{\mathbb{Q}} \cong S_{(2,3,4,5,6,\dots)}$ (see [10, (25.4)]). We have the following.

COROLLARY 4.4. *Let G be an n-dimensional compact abelian Polish group with $n \in \mathbb{N}^+$. Then $E((\widehat{\mathbb{Q}})^n) \leq_B E(G)$.*

PROOF. By [11, Theorem 8.22(4)], $G_0 \cong (\widehat{\mathbb{Q}})^n/\Delta$, where Δ is a compact totally disconnected subgroup of $(\widehat{\mathbb{Q}})^n$. Again by Theorem 2.9, this means that $E((\widehat{\mathbb{Q}})^n) \leq_B E(G_0)$, and thus $E((\widehat{\mathbb{Q}})^n) \leq_B E(G)$. ⊣

From the arguments above, if Γ is a countable discrete torsion-free abelian group, then $\widehat{\Gamma}$ is a compact connected abelian Polish group.

REMARK 4.5. Let G be a compact connected Polish group with $E(\mathbb{R}^n) \leq_B E(G) \leq_B E(\mathbb{T}^n)$ for some $n > 0$. By Theorem 2.11(5), $\dim(G) = n$, so $\mathrm{rank}(\widehat{G}) = n$. Thus $\widehat{G}$ is isomorphic to a subgroup of $\mathbb{Q}^n$ (cf. [7, Exercise 13.4.3]). In particular, if $n = 1$, we have either $G \cong \mathbb{T}$ or there exists a $P \in \mathcal{P}^{\omega}$ such that $G \cong \Sigma_P$.

The following proposition shows that, if $n > 1$, the structure of G can be more complicated.

PROPOSITION 4.6. *There is a* 2*-dimensional compact connected Polish group* G *such that* $E(G) \not\leq_B E(\Sigma_{P_0} \times \Sigma_{P_1} \times \cdots \times \Sigma_{P_n})$ *for* $n \in \mathbb{N}$ *and each* $P_i \in \mathcal{P}^\omega$. *Moreover, if* $|\{i \in \omega : P(i) = 2\}| < \infty$, *then* $E(\Sigma_P) \not\leq_B E(G)$.

PROOF. Pontryagin has constructed a countable torsion-free abelian group $\Gamma \subseteq \mathbb{Q}^2$ whose rank is two (cf. [15, Example 2]). Then $\widehat{\Gamma}$ is a 2-dimensional compact connected abelian Polish group. The group Γ defined by its generators $\eta, \xi_i, (i = 0, 1, 2 \ldots)$ and relations,

$$2^{k_{i+1}}\xi_{i+1} = \xi_i + \eta, \tag{$**$}$$

where $i \in \omega$ and $k_i \in \mathbb{N}^+$ such that $\sup\{k_i : i \in \omega\} = \infty$.

Put $G = \widehat{\Gamma}$. We claim that $E(G) \not\leq_B E(\Sigma_{P_0} \times \Sigma_{P_1} \times \cdots \times \Sigma_{P_n})$. Otherwise, by Corollary 4.2 and [10, Theorem 23.18], there exists $i \leq n$ such that there is a nonzero continuous homomorphism f from $\widehat{\Sigma_{P_i}}$ to $\widehat{G}$. Note that for any $a \in \widehat{\Sigma_{P_i}}$, there are infinitely many positive integers n such that the equation $nx = a$ has a solution. But any element in Γ does not admit such property. This implies that $f(\widehat{\Sigma_{P_i}}) = \{1_\Gamma\}$ contradicting that f is a nonzero homomorphism.

Now assume that $E(\Sigma_P) \leq_B E(G)$ for some $P \in \mathcal{P}^\omega$. We show that $\{i \in \omega : P(i) = 2\}$ is infinite. By Corollary 4.2, there is a nonzero homomorphism f from $\widehat{G}$ to $\widehat{\Sigma_P}$. Without loss of generality we may assume $\widehat{G} = \Gamma$ and $\widehat{\Sigma_P} = \left\{\frac{m}{P(0)P(1)\ldots P(n)} : m \in \mathbb{Z}, n \in \mathbb{N}\right\} \subseteq \mathbb{Q}$. From $(**)$, a straightforward calculation shows that

$$2^{k_1+k_2+\cdots+k_i}\xi_i = \xi_0 + \eta(1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_{i-1}}).$$

So we have

$$2^{k_1+k_2+\cdots+k_i} f(\xi_i) = f(\xi_0) + f(\eta)(1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_{i-1}}).$$

Note that $\lim_i 2^{-(k_1+k_2+\cdots+k_i)} f(\xi_0) = 0$ and

$$\frac{1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_{i-1}}}{2^{k_1+k_2+\cdots+k_i}} f(\eta) \leq \frac{f(\eta)}{2^{k_i-1}} \to 0 \quad (i \to \infty).$$

This implies that $\lim_i f(\xi_i) = 0$.

Let $f(\xi_0) = a/b$ and $f(\eta) = c/d$ for some integers a, b, c, d with $c, d > 0$. Note that $2^{k_{i+1}} f(\xi_{i+1}) = f(\xi_i) + f(\eta)$. Since f is a nonzero homomorphism, there can be at most one $f(\xi_i) = 0$. For large enough i, we have $f(\xi_i) \neq 0$. So there exist integers m_i, m_i', c', d', l_i with $m_i, m_i' \neq 0$ and $c', d' > 0$ such that

$$f(\xi_i) = \frac{m_i}{2^{k_1+k_2+\cdots+k_i}cd} = \frac{m_i'}{2^{l_i}c'd'},$$

where m_i' and $2^{l_i}c'd'$ are coprime and $c'|c$, $d'|d$. It follows that

$$|f(\xi_i)| \geq \frac{1}{2^{l_i}c'd'} \geq \frac{1}{2^{l_i}cd} \to 0 \quad (i \to \infty).$$

So $l_i \to \infty$ as $i \to \infty$, and hence $\{i \in \omega : P(i) = 2\}$ is infinite. ⊣

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