THE LATTICE OF ALL TOPOLOGIES IS COMPLEMENTED

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In (**3**), J. Hartmanis raised the question whether the lattice of all topologies in a given set is complemented and gave the affirmative answer for the case of a finite set. H. Gaifman (**2**), has extended this result to denumerable sets. Using Gaifman's paper, Anne K. Steiner (**4**) has proved that the lattice is always complemented. Our aim in this article is to give an alternative proof, independent of Gaifman's results. So far, Steiner's proof has not been available to the author.

Throughout this paper X is a fixed set. The letters A, V, W denote subsets of X; $\mathfrak{A}$, $\mathfrak{B}$ are topologies in X; α, β, $\gamma, \ldots$ are ordinal numbers. We call A^c the complement of $A \subset X$.

A topology $\mathfrak{A}$ in X is identified with the family of all $\mathfrak{A}$-open subsets of X. If $\mathfrak{A}$ is a topology in X and $A \subset X$, then $\mathfrak{A}|A = \{V \cap A\colon V \in \mathfrak{A}\}$ is a topology in A.

For two topologies $\mathfrak{A}$, $\mathfrak{B}$ in X we put $\mathfrak{A} \leqq \mathfrak{B}$ if every $\mathfrak{A}$-open set is $\mathfrak{B}$-open, i.e., if $\mathfrak{A} \subset \mathfrak{B}$. Thus, as is well known, the set $\mathrm{Top}(X)$ of all topologies in X becomes a complete lattice, its largest element being the discrete topology, while its smallest element is the topology $\{\emptyset, X\}$.

The g.l.b. of $\mathfrak{A}, \mathfrak{B} \in \mathrm{Top}(X)$ is $\mathfrak{A} \cap \mathfrak{B}$, their l.u.b. is the topology that has $\{V \cap W\colon V \in \mathfrak{A}, W \in \mathfrak{B}\}$ as a basis. Thus, $\mathfrak{A}$, $\mathfrak{B}$ are complements of each other if and only if the following two conditions are satisfied.

(I) If $V \in \mathfrak{A} \cap \mathfrak{B}$ and $V \neq \emptyset$, then $V = X$.

(II) For every $x \in X$ there exist $V \in \mathfrak{A}$, $W \in \mathfrak{B}$ with $\{x\} = V \cap W$.

Let Y be a set, let α, γ be ordinal numbers, $\alpha < \gamma$. For $\beta \geqq \alpha$, $\beta < \gamma$, let y_β be an element of Y. Then the symbol $[y_\alpha, y_\gamma)$ will denote the well-ordered sequence of the y_β with $\alpha \leqq \beta < \gamma$ (which, strictly speaking, is a map of the interval $[\alpha, \gamma)$ into Y). Note that y_γ does not necessarily exist. Sometimes we shall also use $[y_\alpha, y_\gamma)$ to denote the *set* $\{y_\beta\colon \alpha \leqq \beta < \gamma\}$.

LEMMA. *Let* $\mathfrak{A} \in \mathrm{Top}(X)$. *Then* X *has an* $\mathfrak{A}$*-dense subset* A *such that* $\mathfrak{A}|A$ *has a complement in* $\mathrm{Top}(A)$.

Proof. Consider the well-ordered sequences $[x_1, x_\gamma)$ in X that have the property: for each $\alpha < \gamma$, x_α does not lie in the $\mathfrak{A}$-closure of $[x_1, x_\alpha)$. By Zorn's lemma, there is a maximal sequence $A = [x_1, x_\gamma)$ of this kind. Clearly, this A is $\mathfrak{A}$-dense in X. In the rest of this proof we identify β and x_β. (Note that

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$x_\alpha \neq x_\beta$ as soon as $\alpha \neq \beta$.) Then every $[1, \alpha)$ is $(\mathfrak{A}|A)$-closed, so that every $[\alpha, \gamma)$ is in $\mathfrak{A}|A$. For $\alpha \in A$ put $\alpha' = \min\{\beta: \alpha < \beta \leqq \gamma; [\alpha, \beta) \in \mathfrak{A}|A\}$. There is an ordinal number δ and a sequence $[\alpha_1, \alpha_{\delta+1})$ of ordinal numbers such that $\alpha_1 = 1$, $\alpha_{\nu+1} = (\alpha_\nu)'$ if $\nu + 1 \leqq \delta$, $\alpha_\nu = \sup\{\alpha_\lambda: \lambda < \nu\}$ if ν is a limit ordinal $<\delta$, and $\gamma = \sup\{\alpha_\lambda: \lambda < \delta\}$. Then the intervals $[\alpha_\lambda, \alpha_{\lambda+1})$ form a disjoint covering of A, and $[\alpha_\lambda, \alpha_{\lambda+1}) \in \mathfrak{A}|A$ for each λ. Let $\mathfrak{B}_A$ consist of the empty set and all the sets $V \subset A$ that have the property: for every $\lambda < \delta$,

$$V \cap [\alpha_\lambda, \alpha_{\lambda+1})$$

is of the form $[\alpha_\lambda, \beta)$, where $\beta > \alpha_\lambda$.

Since $\mathfrak{B}_A$ is closed under arbitrary unions and intersections, it is a topology in A.

(I) If $V \in \mathfrak{B}_A \cap (\mathfrak{A}|A)$ and $V \neq \emptyset$, then, if $\lambda < \delta$, $V \cap [\alpha_\lambda, \alpha_{\lambda+1}) = [\alpha_\lambda, \beta)$ for some $\beta > \alpha_\lambda$. Since $[\alpha_\lambda, \alpha_{\lambda+1}) \in \mathfrak{A}|A$ we have $[\alpha_\lambda, \beta) \in \mathfrak{A}|A$, so that $\beta \geqq (\alpha_\lambda)' = \alpha_{\lambda+1}$. Thus, $V \supset [\alpha_\lambda, \alpha_{\lambda+1})$ for all $\lambda < \delta$ and hence, $V = A$.

(II) Let $\beta \in A$ and let $[\alpha_\nu, \alpha_{\nu+1})$ be the interval in which β falls. Put $V = [\beta, \alpha_{\nu+1})$ and $W = [\alpha_\nu, \beta + 1) \cup \{\alpha_\lambda: \lambda < \delta\}$. Then

$$V = [\alpha_\nu, \alpha_{\nu+1}) \cap [\beta, \gamma),$$

hence $V \in \mathfrak{A}|A$. Obviously, $W \in \mathfrak{B}_A$ and $V \cap W = \{\beta\}$.

From (I) and (II) we infer that $\mathfrak{B}_A$ is a complement of $\mathfrak{A}|A$ in $\mathrm{Top}(A)$. Thus we have proved the lemma.

THEOREM. $\mathrm{Top}(X)$ *is complemented.*

Proof. Let $\mathfrak{A} \in \mathrm{Top}(X)$. From the preceding lemma we deduce the existence of a well-ordered sequence $[X_1, X_\mu)$ of non-empty subsets of X such that

(1) The X_α form a disjoint covering of X,

(2) For each $\alpha < \mu$, $\mathfrak{A}|X_\alpha$ has a complement $\mathfrak{B}_\alpha$ in $\mathrm{Top}(X_\alpha)$,

(3) For each α, X_α is dense in $\bigcup\{X_\beta: \beta \geqq \alpha\}$ in the topology induced by $\mathfrak{A}$.

Let δ be the first $\beta > 1$ such that $\bigcup\{X_\gamma: \gamma < \beta\} \in \mathfrak{A}$. Let $\mathfrak{B}$ be the set of all those $V \subset X$ for which

(a) $V \cap X_\alpha \in \mathfrak{B}_\alpha$ for each α,

(b) If $V \cap X_1 \neq \emptyset$, then $\bigcup\{X_\beta: \beta \geqq \delta\} \subset V$.

This $\mathfrak{B}$ is a topology in X. We shall show it to be a complement of $\mathfrak{A}$.

(I) Let $V \in \mathfrak{A} \cap \mathfrak{B}$, $V \neq \emptyset$. By (a) there is a non-empty set $\Sigma \subset [1, \mu)$ such that $V = \bigcup\{X_\alpha: \alpha \in \Sigma\}$.

Assume $\alpha \in \Sigma$, $\beta \leqq \alpha$. We claim that $\beta \in \Sigma$. In fact, in the topology $\mathfrak{A}$ induces in $\bigcup\{X_\gamma: \gamma \geqq \beta\}$, X_β is dense, and $V \cap \bigcup\{X_\gamma: \gamma \geqq \beta\}$ is open and non-empty; thus $X_\beta \cap V \neq \emptyset$, and $\beta \in \Sigma$. Hence, Σ is of the form $[1, \epsilon)$. Since $V \in \mathfrak{A}$, $\epsilon \geqq \delta$. It follows from (b) that $V = X$.

(II) Take $x \in X$. There is an α with $x \in X_\alpha$. By definition of $\mathfrak{B}_\alpha$ there are $V_1 \in \mathfrak{A}|X_\alpha$ and $W_1 \in \mathfrak{B}_\alpha$ such that $\{x\} = V_1 \cap W_1$. There is a $V_2 \in \mathfrak{A}$ with $V_1 = V_2 \cap X_\alpha$. If $\alpha = 1$, we put

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$$V = V_2 \cap \bigcup\{X_\beta: \beta < \delta\} \quad \text{and} \quad W = W_1 \cup \bigcup\{X_\beta: \beta \geqq \delta\};$$

if $\alpha > 1$, take $V = V_2$ and $W = W_1$.

In either case, $V \in \mathfrak{A}$, $W \in \mathfrak{B}$, and $\{x\} = V \cap W$.

Let us call a topology $\mathfrak{A}$ in X *complete* if $\{V^c: V \in \mathfrak{A}\}$ is a topology, i.e., if the open sets are closed under arbitrary intersections. This is the case if and only if every element of X has a smallest $\mathfrak{A}$-neighbourhood. A closer inspection of the preceding proof reveals that every $\mathfrak{A} \in \mathrm{Top}(X)$ has a complement that is complete; for, in the proof, one has only to replace the word "complement" by "complete complement". The complete topologies in X form a subset $\mathrm{C\,Top}(X)$ of $\mathrm{Top}(X)$, and one easily proves it to be a sublattice. It contains the smallest and the largest elements of $\mathrm{Top}(X)$. Hence, $\mathrm{C\,Top}(X)$ is complemented. Now the complete topologies can be interpreted as quasi-orders in the following way. (A quasi-order in X is a binary, reflexive, and transitive relation on X.) For $\mathfrak{A} \in \mathrm{C\,Top}(X)$ we define the binary relation $\mathfrak{A}^*$ on X by

$$x\,\mathfrak{A}^*\,y \quad \text{if } x \in V \in \mathfrak{A} \Rightarrow y \in V \qquad (x, y \in X).$$

Thus, $x\,\mathfrak{A}^*\,y$ if and only if y is contained in the smallest $\mathfrak{A}$-neighbourhood of x. Then $\mathfrak{A}^*$ is a quasi-order. Conversely, if $\mathfrak{a}$ is a quasi-order in X, there is exactly one $\mathfrak{A} \in \mathrm{C\,Top}(X)$ with $\mathfrak{A}^* = \mathfrak{a}$, viz.,

$$\mathfrak{A} = \{V: V \subset X, \text{ if } x \in V \text{ and } x\,\mathfrak{a}\,y, \text{ then } y \in V\}.$$

Thus there is a one-to-one correspondence between the elements of $\mathrm{C\,Top}(X)$ and the quasi-orders in X. Further, we notice that

$$\mathfrak{A} \leqq \mathfrak{B} \qquad (\mathfrak{A}, \mathfrak{B} \in \mathrm{C\,Top}(X))$$

is equivalent to

"If $x\,\mathfrak{B}^*\,y$, then $x\,\mathfrak{A}^*\,y$."

For quasi-orders $\mathfrak{a}$, $\mathfrak{b}$, let us define $\mathfrak{a} \leqq \mathfrak{b}$ by

$$x\,\mathfrak{b}\,y \Rightarrow x\,\mathfrak{a}\,y \qquad (x, y \in X).$$

Then we have proved the following corollary.

COROLLARY. *The quasi-orders in any set X form a complemented lattice.*

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