J. Austral. Math. Soc. 19 (Series A) (1975), 410–412.

MAXIMAL RIGHT IDEALS OF TRANSFORMATION NEAR-RINGS

MARJORY J. JOHNSON

(Received 17 March 1972)

Communicated by G. E. Wall

The purpose of this note is to characterize maximal right ideals of $T_0(G)$, the (left) mear-ring of transformations from a group $(G, +)$ into itself which leave zero fixed. Using this characterization, we answer some questions raised by Heatherly (1972).

Definitions of (left) near-ring, right ideal of a near-ring, and $T_0(G)$ can be found in either of the references.

We use the following notation. Let G be a group and let C be a subset of G. The cardinality of C is denoted by $|C|$. The complement of C in G is the set $G - C = \{g \in G \mid g \notin C\}$. Let $A(C)$ denote the annihilator right ideal of C; that is, $A(C) = \{\alpha \in T_0(G) \mid (c)\alpha = 0, \text{ for all } c \in C\}$. Finally, using the notation of Johnson (1973), if $\beta \in T_0(G)$, let $B_\beta = \{x \in G \mid (x)\beta = 0\}$.

The following lemma is a restatement of results of Johnson (1973).

LEMMA 1. *Let S be a right ideal of $T_0(G)$. Then $\alpha \in S$ if and only if $A(B_\alpha) \subset S$.*

LEMMA 2. *Let S be a right ideal of $T_0(G)$. Let $\alpha, \beta \in S$. Then there exists $\delta \in S$ such that $B_\delta = B_\alpha \cap B_\beta$.*

PROOF. By Lemma 1 the annihilator right ideals $A(B_\alpha)$ and $A(B_\beta)$ are contained in S. Define maps $\gamma, \eta \in T_0(G)$ by $(x)\gamma = x$ if $x \in G - B_\alpha$, $(x)\gamma = 0$ if $x \in B_\alpha$; $(x)\eta = x$ if $x \in B_\alpha - B_\beta$, $(x)\eta = 0$ if $x \notin B_\alpha - B_\beta$. Since $\gamma \in A(B_\alpha)$ and $\eta \in A(B_\beta)$, then $\gamma, \eta \in S$. Hence $\gamma + \eta \in S$ and $B_{\gamma+\eta} = B_\alpha \cap B_\beta$.

Let $\mathscr{R} = \{R \mid R \text{ is a right ideal of } T_0(G) \text{ and } B_\alpha \cap B_\beta \text{ is infinite, for all } \alpha, \beta \in R\}$. If G is infinite, $\mathscr{R}$ is nonempty, and by Zorn's Lemma, $\mathscr{R}$ contains maximal elements.

THEOREM 3. *S is a maximal right ideal of $T_0(G)$ if and only if either $S = A(\{x\})$ for some nonzero $x \in G$ or S is a maximal element in the collection $\mathscr{R}$.*

PROOF. Heatherly (1972) has shown that if x is a nonzero element of G, then $A(\{x\})$ is a maximal right ideal of $T_0(G)$.

Now suppose S is a maximal element of $\mathscr{R}$. Let K be a right ideal of $T_0(G)$ which properly contains S. Then there exist elements $\alpha, \beta \in K$ such that $B_\alpha \cap B_\beta$ is finite, and hence by Lemma 2 there exists $\delta \in K$ such that B_δ is finite.

Let $|B_\delta| = n$. Suppose $n > 1$. Let $y \in B_\delta$, $y \neq 0$. Define $\omega \in T_0(G)$ by $(y)\omega = y$, $(x)\omega = 0$ if $x \neq y$. The mapping ω is an element of the right ideal $A(G-\{y\})$. It is easy to show that $S + A(G-\{y\}) \in \mathscr{R}$. Since S is maximal in $\mathscr{R}$, then $S = S + A(G-\{y\})$ and so $\omega \in S$. Hence $\delta + \omega \in K$ and $|B_{\delta+\omega}| = n-1$. Repeating the above process, we find an element $\lambda \in K$ such that $B_\lambda = \{0\}$. But then by Lemma 1, $K = T_0(G)$. Therefore, S is a maximal right ideal of $T_0(G)$.

Conversely, suppose S is a maximal right ideal of $T_0(G)$ and suppose $S \neq A(\{x\})$ for any nonzero $x \in G$. We show $S \in \mathscr{R}$. Suppose there exists $\mu \in S$ such that B_μ is a finite subset of G. For $x \in G$, define $\eta_x \in T_0(G)$ by $(x)\eta_x = x$, $(y)\eta_x = 0$ if $y \neq x$. Since S is not an annihilator right ideal, it is easy to show that $\eta_x \in S$, for all $x \in G$. Then we can repeat the above inductive procedure to find an element $\sigma \in S$ such that $B_\sigma = \{0\}$ and hence to show that $S = T_0(G)$. This is a contradiction, since S is a maximal right ideal. Therefore, using Lemma 2, $B_\alpha \cap B_\beta$ is infinite, for all $\alpha, \beta \in S$. Hence, $S \in \mathscr{R}$ and since S is a maximal right ideal, then S must be a maximal element of $\mathscr{R}$.

Heatherly (1972) considered the right ideal $D = \{\alpha \in T_0(G) \mid |\text{support}\,\alpha| < |G|\}$, where support $\alpha = \{x \in G \mid (x)\alpha \neq 0\}$ and where G is an infinite group. He raised the question: Is D a maximal right ideal of $T_0(G)$? Using Theorem 3, it is easy to see that the answer is no. $D \in \mathscr{R}$, but D is not maximal in $\mathscr{R}$. For let H, K be subsets of G such that $H \cap K = \phi$, $G = H \cup K \cup \{0\}$, and $|H| = |K| = |G|$. Then $A(H)$ is a right ideal of $T_0(G)$ and $D + A(H)$ is an element of $\mathscr{R}$ which properly contains D. The following theorem, which answers another question of Heatherly (1972), shows that there is in fact a proper ascending sequence of right ideals of $T_0(G)$ which contain D.

THEOREM 4. *Right ideals of $T_0(G)$ satisfy the ascending chain condition if and only if G is finite.*

PROOF. Suppose G is an infinite group. Define a sequence of sets inductively as follows. Let C_1 be a proper subset of $G - \{0\}$ such that $|C_1| = |G|$. If C_i has been defined, let C_{i+1} be a proper subset of C_i such that $|C_{i+1}| = |G|$ and $|C_i - C_{i+1}| = |G|$. Then $\{0\} \subset A(C_1) \subset A(C_2) \subset \cdots$ is an infinite proper ascending chain of right ideals of $T_0(G)$. Hence, if G satisfies the ascending chain condition on right ideals, then G is finite.

Conversely, if G is finite, then $T_0(G)$ is finite, and thus $T_0(G)$ satisfies the ascending chain condition on right ideals.

References

H. E. Heatherly (1972), 'One-sided ideals in near-rings of transformations', *J. Austral. Math. Soc.* **13**, 171–179.

M. J. Johnson (1973), 'Right ideals and right submodules of transformation near-rings', *J. Algebra* **24**, 386–391.

University of South Carolina
Columbia, South Carolina 29208
U.S.A.