ON RINGS OF SETS

T. P. SPEED

(Received 16 March 1967)

1. Introduction

In the past a number of papers have appeared which give representations of abstract lattices as rings of sets of various kinds. We refer particularly to authors who have given necessary and sufficient conditions for an abstract lattice to be lattice isomorphic to a complete ring of sets, to the lattice of all closed sets of a topological space, or to the lattice of all open sets of a topological space. Most papers on these subjects give the conditions in terms of special elements of the lattice. We thus have completely join-irreducible elements – G. N. Raney [7]; join prime, completely join prime, and supercompact elements – V. K. Balachandran [1], [2]; $\mathscr{N}$-sub-irreducible elements – J. R. Büchi [5]; and lattice bisectors – P. D. Finch [6]. Also meet-irreducible and completely meet-irreducible dual ideals play a part in some representations of G. Birkhoff & O. Frink [4].

What we do in this paper is define a new kind of prime ideal – called an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal – and show that all the above concepts correspond to a particular kind of $\mathfrak{n}$-prime $\mathfrak{m}$-ideal. Here and throughout we mean $\mathfrak{m}$ and $\mathfrak{n}$ to be (possibly infinite) cardinals, always greater than 1. Also the symbol ∞ will be used to denote an arbitrarily large cardinal number. A class of lattices called $(\mathfrak{m}, \mathfrak{n})$-rings of sets is then defined and some theorems proved which cover all the representation theorems mentioned above. It is interesting to note that the elementary methods used in representing distributive lattices carry over completely and yield all these results, although this is hardly obvious when one considers special elements of the lattice.

I wish to express my gratitude to Professor P. D. Finch, whose paper [6] was the inspiration for this work.

2. Notations and Definitions

We assume a familiarity with the elementary notions of lattice theory as outlined in G. Birkhoff [3].

DEFINITION 2.1. A lattice $\mathscr{L} = \langle L; \vee, \wedge \rangle$ is said to be $(\mathfrak{m}, \mathfrak{n})$-*complete* if the join of not more then $\mathfrak{m}$ elements of L belongs to L, and the meet of not more than $\mathfrak{n}$ elements of L belongs to L.

https://doi.org/10.1017/S1446788700006546 Published online by Cambridge University Press

Thus an $(\mathfrak{m}, \mathfrak{n})$-complete lattice may be considered as an algebra with the $\mathfrak{m}$-ary operation of join and the $\mathfrak{n}$-ary operation of meet.

DEFINITION 2.2. An $(\mathfrak{m}, \mathfrak{n})$-complete lattice of sets $\mathscr{L} = \langle L; \vee, \wedge \rangle$ is called an $(\mathfrak{m}, \mathfrak{n})$-*ring of sets* if the $\mathfrak{m}$-ary operation of join corresponds to set union, and the $\mathfrak{n}$-ary operation of meet corresponds to set intersection.

EXAMPLE. The lattice of all open sets of a topological space is an $(\infty, 2)$-ring of sets.

DEFINITION 2.3. An ideal P of the $(\mathfrak{m}, \mathfrak{n})$-complete lattice $\mathscr{L} = \langle L; \vee, \wedge \rangle$ is called an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal if

(i) For $\{x_\gamma : \gamma \in \Gamma\} \subseteqq L$ with $|\Gamma| \leqq \mathfrak{m}$ we have:

$$x_\gamma \in P \;\forall \gamma \in \Gamma \;\Leftrightarrow\; \bigvee_{\gamma \in \Gamma} x_\gamma \in P$$

(ii) For $\{y_\delta : \delta \in \Delta\} \subseteqq L$ with $|\Delta| \leqq \mathfrak{n}$ we have:

$$y_\delta \notin P \;\forall \delta \in \Delta \;\Leftrightarrow\; \bigwedge_{\delta \in \Delta} y \notin P.$$

REMARKS. 1. An ordinary prime ideal is a 2-prime 2-ideal in the above notation.

2. The definition is obviously not the most general possible but it will suffice for the purpose of this paper.

3. If P is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal then $L \setminus P$ is an $\mathfrak{m}$-prime $\mathfrak{n}$-dual ideal with the obvious (dual) definition of the latter.

DEFINITION 2.4. An homomorphism ψ between two $(\mathfrak{m}, \mathfrak{n})$-complete lattices is called an $(\mathfrak{m}, \mathfrak{n})$-*homomorphism* if ψ preserves joins of $\mathfrak{m}$ elements and meets of $\mathfrak{n}$ elements.

Note that a lattice of sets is not assumed to have set union and intersection as lattice operations unless stated, although the partial ordering is set inclusion.

3. $(\mathfrak{m}, \mathfrak{n})$-rings of sets

In this section we clarify the notion of $(\mathfrak{m}, \mathfrak{n})$-ring of sets.

PROPOSITION 3.1. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-complete lattice of subsets of a set S. Then $\mathscr{L}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets if and only if for any $s \in S$*

(i) *$s \notin \bigvee\{l \in M : s \notin l\}$ for any $M \subseteqq L$ with $|M| \leqq \mathfrak{m}$*

(ii) *$s \in \bigwedge\{l \in N : s \in l\}$ for any $N \subseteqq L$ with $|N| \leqq \mathfrak{n}$.*

PROOF. If $\mathscr{L}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets, then the $\mathfrak{m}$-ary join and the $\mathfrak{n}$-ary meet operations correspond to set union and intersection respectively. It is thus clear that (i) and (ii) hold in this case.

For the converse we assume (i) and (ii). Observe that we must always have (for $M \subseteqq L$ with $|M| \leqq \mathfrak{m}$)

$$\bigvee \{l : l \in M\} \supseteqq \bigcup \{l : l \in M\}.$$

Now if $s \notin \bigcup\{l : l \in M\}$ then $s \notin l \; \forall l \in M$ and thus by (i) we see that $s \notin \bigvee \{l : l \in M\}$. The reverse inclusion is hence proved and we obtain $\bigvee \{l : l \in M\} = \bigcup\{l : l \in M\}$. Similarly $\bigwedge \{l : l \in N\} \subseteqq \bigcap \{l : l \in N\}$ always holds for $N \subseteqq L$ with $|N| \leqq \mathfrak{n}$, and (ii) implies the reverse inclusion giving

$$\bigwedge \{l : l \in N\} = \bigcap \{l : l \in N\}.$$

The proposition is thus proved.

Our next result is a direct generalisation of G. Birkhoff's theorem for distributive lattices (= (2, 2)-rings of sets), [3] p. 140.

PROPOSITION 3.2. *Let* $\mathscr{L} = \langle L; \vee, \wedge \rangle$ *be an* $(\mathfrak{m}, \mathfrak{n})$*-complete lattice. Then* $\mathscr{L}$ *is isomorphic to an* $(\mathfrak{m}, \mathfrak{n})$*-ring of sets if and only if* $\mathscr{L}$ *has a faithful representation as a subdirect union of a family* $\{\mathscr{L}_\alpha : \alpha \in A\}$ *of replicas of* **2** *in which each projection* $\pi_\alpha : \mathscr{L} \to \mathscr{L}_\alpha$ *is an* $(\mathfrak{m}, \mathfrak{n})$*-homomorphism.*

PROOF. Assume first that $\mathscr{L}$ has a sub-direct union representation with the stated properties. This is equivalent to the existence of an isomorphism ψ of $\mathscr{L}$ onto a lattice $\langle \mathscr{A}, \cup, \cap \rangle$ of subsets of the index set A; explicitly

$$\psi : l \to l\psi = \{\alpha \in A : l\pi_\alpha = 1\}, \quad \mathscr{A} = \{l\psi : l \in L\}.$$

It is clear that $\langle \mathscr{A}; \cup, \cap \rangle$ is a $(\mathfrak{m}, \mathfrak{n})$-complete lattice. We show it is a $(\mathfrak{m}, \mathfrak{n})$-ring of sets. Take $\mathscr{M} \subseteqq \mathscr{A}$ with $|\mathscr{M}| \leqq \mathfrak{m}$, and an arbitrary $\alpha \in A$.

Now $\bigvee \{K \in \mathscr{M} : \alpha \notin K\}$
$= \bigvee \{l\psi \in \mathscr{M} : \alpha \notin l\psi]$ since every $K \in \mathscr{M}$ is of the form $l\psi$, $l \in L$
$= [\bigvee \{l \in M : \alpha \notin l\psi\}]\psi$ where $M = \mathscr{M}\psi^{-1} \subseteqq L$
$= [\bigvee \{l \in M : l\pi_\alpha = 0\}]\psi$ since $\alpha \notin l\psi \equiv l\pi_\alpha = 0$.

Further, $[\bigvee \{l \in M : l\pi_\alpha = 0\}]\pi_\alpha = 0$ since $|M| \leqq \mathfrak{m}$ and the π_α are $(\mathfrak{m}, \mathfrak{n})$-homomorphisms, so that $\alpha \notin \bigvee\{K \in \mathscr{M} : \alpha \in K\}$ for $\mathscr{M} \subseteqq \mathscr{A}$ with $|\mathscr{M}| \leqq \mathfrak{m}$.

Similarly $\bigwedge \{K \in \mathscr{N} : \alpha \in K\}$
$= \bigwedge \{l\psi \in \mathscr{N} : \alpha \in l\psi\}$
$= [\bigwedge \{l \in N : \alpha \in l\psi\}]\psi$
$= [\bigwedge \{l \in N : l\pi_\alpha = 1\}]\psi$ for $\mathscr{N} \subseteqq \mathscr{A}$ and $\alpha \in A$.

This gives $[\bigwedge \{l \in N : l\pi_\alpha = 1\}]\pi_\alpha = 1$ if $|\mathscr{N}| = |N| \leqq \mathfrak{n}$

since the π_α are $(\mathfrak{m}, \mathfrak{n})$-homomorphisms, so that $\alpha \in \bigwedge\{K \in \mathscr{N} : \alpha \in K\}$, and we have shown that (i) and (ii) of Proposition 3.1 are satisfied. Hence $\mathscr{L}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets.

https://doi.org/10.1017/S1446788700006546 Published online by Cambridge University Press

For the converse assume $\mathscr{L}$ is isomorphic to an $(\mathfrak{m}, \mathfrak{n})$-ring of sets $\mathscr{L}'$. Then $\mathscr{L}'$ has a representation as a subdirect union of replicas of **2** and the working above readily reverses to establish the fact that the π_α are $(\mathfrak{m}, \mathfrak{n})$-homomorphisms.

4. $\mathfrak{n}$-prime $\mathfrak{m}$-ideals

We now discuss the notion of $\mathfrak{n}$-prime $\mathfrak{m}$-ideal. The first result is straightforward but the corollary is used to establish the equivalence between our ideals and the various concepts mentioned in the introduction. These concepts are not defined here – we refer to the papers concerned – for this reason the corollary is presented without proof.

PROPOSITION 4.1. *Let* $\mathscr{L} = \langle L; \vee, \wedge \rangle$ *be an* $(\mathfrak{m}, \mathfrak{n})$*-complete lattice. Then there is a one-one correspondence between*

(i) $\mathfrak{n}$*-prime* $\mathfrak{m}$*-ideals,*

(ii) $\mathfrak{m}$*-prime* $\mathfrak{n}$*-dual ideals,*

(iii) $(\mathfrak{m}, \mathfrak{n})$*-homomorphisms onto* **2**.

PROOF. It has already been remarked that (i) and (ii) are in one-one correspondence. Let $\psi : \mathscr{L} \to \mathbf{2}$ be an $(\mathfrak{m}, \mathfrak{n})$-homomorphism onto **2**. Then it is easy to see that $\{1\}\psi^{-1}$ is an $\mathfrak{m}$-prime $\mathfrak{n}$-dual ideal and $\{0\}\psi^{-1}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal. Conversely if P is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal, we may define a map $\pi : \mathscr{L} \to \mathbf{2}$ by setting $l\pi = 0$ or 1 according as $l \in P$ or $l \notin P$. π may be checked to be an $(\mathfrak{m}, \mathfrak{n})$-homomorphism and our proposition is proved.

COROLLARY (Special Cases). Under the conditions of the proposition, with the appropriate values of $\mathfrak{m}$ and $\mathfrak{n}$, there is a one-one correspondence between the objects in the following groups.

A. $(\mathfrak{m} = 2, \mathfrak{n} = \infty)$

(i) prime principal dual ideals

(ii) join prime elements (V. K. Balachandran [2]); lattice bisectors (P. D. Finch [6]); $\mathscr{N}$-sub-irreducible elements for a certain $\mathscr{N}$ (J. R. Büchi [5]).

(iii) $(2, \infty)$-homomorphism onto **2**; lower complete homomorphisms onto **2** (P. D. Finch [6]).

B. $(\mathfrak{m} = \infty, \mathfrak{n} = 2)$

(i) ∞-prime dual ideals; completely prime dual ideals (G. Birkhoff & O. Frink [4]).

(ii) prime principal ideals

(iii) $(\infty, 2)$-homomorphisms onto **2**.

https://doi.org/10.1017/S1446788700006546 Published online by Cambridge University Press

C. $(\mathfrak{m} = \infty, \mathfrak{n} = \infty)$

(i) completely prime principal dual-ideals

(ii) completely join prime elements (V. K. Balachandran [2]); super-compact elements (V. K. Balachandran [1]); completely join irreducible elements (G. N. Raney [7]).

(iii) (∞, ∞)-homomorphisms onto **2**; complete homomorphisms onto **2** (G. N. Raney [7]).

LEMMA 4.2. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ and $\mathscr{L}' = \langle L', \vee, \wedge \rangle$ be two $(\mathfrak{m}, \mathfrak{n})$-complete lattices. Suppose there is an $(\mathfrak{m}, \mathfrak{n})$-homomorphism*

$$\pi : \mathscr{L} \to \mathscr{L}'.$$

Then if P' is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}'$, $P = P'\pi^{-1}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}$.

PROOF. P is well known to be an ideal of $\mathscr{L}$. We first show that P is an $\mathfrak{m}$-ideal. Let $\{l_\gamma : \gamma \in \Gamma\} \subseteqq P$ be such that $|\Gamma| \leqq \mathfrak{m}$. Then

$$(\bigvee_{\gamma \in \Gamma} l_\gamma)\pi = \bigvee_{\gamma \in \Gamma} l_\gamma \pi$$

and since $l_\gamma \pi \in P'$, $\forall \gamma \in \Gamma$, $\bigvee_{\gamma \in \Gamma} l_\gamma \pi \in P$, and we deduce that $\bigvee_{\gamma \in \Gamma} l_\gamma \in P = P'\pi^{-1}$.

Finally we show that P is $\mathfrak{n}$-prime. Suppose $\{l_\delta : \delta \in \Delta\} \subseteqq L$ is such that $|\Delta| \leqq \mathfrak{n}$ and $l_\delta \notin P \;\; \forall \delta \in \Delta$.

Then $(\bigwedge_{\delta \in \Delta} l_\delta)\pi = \bigwedge_{\delta \in \Delta} l_\delta \pi$ and since $l_\delta \notin P \;\; \forall \delta \in \Delta$ we have $l_\delta \pi \notin P'$ $\forall \delta \in \Delta$. Thus, since P' is $\mathfrak{n}$-prime, $\bigwedge_{\delta \in \Delta} l_\delta \pi \notin P'$ and so $\bigwedge_{\delta \in \Delta} l_\delta \notin P = P'\pi^{-1}$. The result is proved.

LEMMA 4.3. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-ring of sets, subsets of a set $\mathscr{X}$. Then for any $x \in \mathscr{X}$, $P_x = \{l \in L : x \notin l\}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}$.*

PROOF. P_x is clearly an ideal of $\mathscr{L}$. We show it is an $\mathfrak{m}$-ideal.

Let $\{l_\gamma : \gamma \in \Gamma\} \subseteqq P_x$ be such that $|\Gamma| \leqq \mathfrak{m}$. Since $x \notin l_\gamma$ for $\gamma \in \Gamma$, Proposition 3.1 (i) tells us that $x \notin \bigvee_{\gamma \in \Gamma} l_\gamma$ or $\bigvee_{\gamma \in \Gamma} l_\gamma \in P_x$.

Similarly let $\{l_\delta : \delta \in \Delta\} \subseteqq L$ be such that $|\Delta| \leqq \mathfrak{n}$ and $l_\delta \notin P_x \forall \delta \in \Delta$. Then Proposition 3.1 (ii) tells us that $x \in \bigwedge_{\delta \in \Delta} l_\delta$ or $\bigwedge_{\delta \in \Delta} l_\delta \notin P_x$.

We have thus proved P_x is $\mathfrak{n}$-prime and so it is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal.

5. Representation of lattices by $(\mathfrak{m}, \mathfrak{n})$-rings of sets

In this section we give a fundamental representation theorem and then show all such representations are of this form.

PROPOSITION 5.1. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-complete lattice and $\mathscr{P} = \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n})$ the set of all $\mathfrak{n}$-prime $\mathfrak{m}$-ideals of $\mathscr{L}$. We assume $\mathscr{P} \neq \Box$. Let $\mathscr{X}$ denote a non-empty subset of $\mathscr{P}$ and define a lattice $\mathscr{R}_{\mathscr{X}} = \langle R; \vee, \wedge \rangle$ by*

$\mathscr{R}_{\mathfrak{X}} = \mathscr{L}\rho$ *where* $\rho = \rho_{\mathfrak{X}}$ *is defined by* $\rho : \mathscr{L} \to \mathscr{R}_{\mathfrak{X}}$, $l\rho = \{P \in \mathscr{P} : l \notin P\}$. *Then* $\mathscr{R}_{\mathfrak{X}}$ *is an* $(\mathfrak{m}, \mathfrak{n})$*-ring of sets and* ρ *is an* $(\mathfrak{m}, \mathfrak{n})$*-homomorphism.*

PROOF. We show that ρ is an $(\mathfrak{m}, \mathfrak{n})$-homomorphism and it will then follow that $\mathscr{R}_{\mathfrak{X}}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets. Take $\{l_\gamma : \gamma \in \Gamma\} \subseteqq L$ with $|\Gamma| \leqq \mathfrak{m}$.

Since

$$l_\gamma \leqq \bigvee_{\gamma \in \Gamma} l_\gamma$$

we deduce that

$$l_\gamma\rho \supseteqq (\bigvee_{\gamma \in \Gamma} l_\gamma)\rho$$

and hence

$$\bigcup_{\gamma \in \Gamma} l_\gamma\rho \supseteqq (\bigvee_{\gamma \in \Gamma} l_\gamma)\rho.$$

Now if $P \in \bigcup_{\gamma \in \Gamma} l_\gamma\rho$, then $l_\gamma \notin P$ for some $\gamma \in \Gamma$. Thus $\bigvee_{\gamma \in \Gamma} l_\gamma \notin P$ and hence $P \in (\bigvee_{\gamma \in \Gamma} l_\gamma)\rho$. We have proved ρ preserves joins of $\mathfrak{m}$ elements.

Next take $\{l_\delta : \delta \in \Delta\} \subseteqq L$ with $|\Delta| \leqq \mathfrak{n}$:

$$\bigwedge_{\delta \in \Delta} l_\delta \leqq l_\delta \quad \forall \delta \in \Delta$$

and so $(\bigwedge_{\delta \in \Delta} l_\delta)\rho \supseteqq l_\delta\rho$, giving

$$(\bigwedge_{\delta \in \Delta} l_\delta)\rho \supseteqq \bigcap_{\delta \in \Delta} l_\delta\rho.$$

For the reverse inclusion take $P \in (\bigwedge_{\delta \in \Delta} l_\delta)\rho$. Then $\bigwedge_{\delta \in \Delta} l_\delta \notin P$ and so, since P is $\mathfrak{n}$-prime, we must have $l_\delta \notin P \ \forall \delta \in \Delta$;

Thus $P \in \bigcap_{\delta \in \Delta} l_\delta\rho$ and we have

$$(\bigwedge_{\delta \in \Delta} l_\delta)\rho = \bigcap_{\delta \in \Delta} l_\delta\rho.$$

ρ is now proved to be an $(\mathfrak{m}, \mathfrak{n})$-homomorphism and the statements in the proposition all follow.

Our next result is basic.

PROPOSITION 5.2. *Let* $\mathscr{L} = \langle L; \vee, \wedge \rangle$ *be a* $(\mathfrak{m}, \mathfrak{n})$*-complete lattice, and* ϕ *a* $(\mathfrak{m}, \mathfrak{n})$*-homomorphism of* $\mathscr{L}$ *onto a* $(\mathfrak{m}, \mathfrak{n})$*-ring of sets* $\mathscr{K} = \langle K; \cup, \cap \rangle$, *subsets of some set* $\mathscr{Y}$. *Then there is a nonempty subset* $\mathfrak{X}$ *of* $\mathscr{P} = \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n})$ *and an isomorphism* $\theta : \mathscr{K} \to \mathscr{R}_{\mathfrak{X}}$ *such that* $\phi \circ \theta = \rho_{\mathfrak{X}}$.

PROOF. Let us first look at $\mathscr{K}$. Since $\mathscr{K}$ is a $(\mathfrak{m}, \mathfrak{n})$-ring of subsets of $\mathscr{Y}$, $P_y = \{k \in K : y \notin k\}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{K}$ by Lemma 4.3. Also, since ϕ is a $(\mathfrak{m}, \mathfrak{n})$-homomorphism of $\mathscr{L}$ onto $\mathscr{K}$, $P_y\phi^{-1}$ is a $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}$ by Lemma 4.2.

Define $\mathfrak{X} \subseteqq \mathscr{P}$ by $\mathfrak{X} = \{P_y\phi^{-1} : y \in \mathscr{Y}\}$. In the statement of the proposition $\mathscr{R}_{\mathfrak{X}}$ and $\rho = \rho_{\mathfrak{X}}$ are defined as in Proposition 5.1. It remains to check that θ defined by $\phi \circ \theta = \rho_{\mathfrak{X}}$ is an isomorphism of $\mathscr{K}$ onto $\mathscr{R}_{\mathfrak{X}}$.

(i) θ is well defined. For suppose $l_1\phi = l_2\phi$ for $l_1, l_2 \in L$. Then

$$\{y \in \mathscr{Y} : l_1\phi \notin P_y\} = \{y \in \mathscr{Y} : l_2\phi \notin P_y\}$$

and so
$$\{y \in \mathscr{Y} : l_1 \notin P_y\phi^{-1}\} = \{y \in \mathscr{Y} : l_2 \notin P_y\phi^{-1}\}.$$

Thus
$$\{P \in \mathscr{X} : l_1 \notin P\} = \{P \in \mathscr{X} : l_2 \notin P\}$$

and so
$$l_1\rho = l_2\rho.$$

(ii) θ is an injection. For suppose $l_1\phi\theta = l_2\phi\theta$. Then $l_2\rho = l_2\rho$ by definition of θ, and the lines above reverse completely to prove $l_1\phi = l_2\phi$.

(iii) θ is clearly a surjection, for $\rho_{\mathfrak{X}}$ is a surjection and so is ϕ.

(iv) We finally check that θ is an homomorphism. Take $k_1, k_2 \in K$ such that $k_i = l_i\phi$. Then

$$k_1 \vee k_2 = l_1\phi \vee l_2\phi = (l_1 \vee l_2)\phi$$

whence

$$\begin{aligned}(k_1 \vee k_2)\theta &= (l_1 \vee l_2)\phi \circ \theta = (l_1 \vee l_2)\rho = l_1\rho \vee l_2\rho \\ &= (l_1)\phi\theta \vee (l_2)\phi\theta = k_1\theta \vee k_2\theta.\end{aligned}$$

Similarly $(k_1 \wedge k_2)\theta = k_1\theta \wedge k_2\theta$ and θ is established to be an isomorphism. The proposition is thus proved.

We close with a theorem which determines when faithful representations exist. For the theorem, let $\mathscr{P}^d(\mathscr{L}; \mathfrak{m}, \mathfrak{n})$ denote the set of all $\mathfrak{m}$-prime $\mathfrak{m}$-dual ideals of $\mathscr{L}$.

THEOREM 5.3. *Let* $\mathscr{L} = \langle L; \vee, \wedge\rangle$ *be an* $(\mathfrak{m}, \mathfrak{n})$*-complete lattice. Then the following are equivalent*:

(i) $\mathscr{L}$ *is isomorphic with an* $(\mathfrak{m}, \mathfrak{n})$*-ring of sets.*

(ii) $(l] = \bigcap \{P \in \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in P\}$ *for all* $l \in L$.

(iii) $[l) = \bigcap \{D \in \mathscr{P}^d(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in D\}$ *for all* $l \in L$.

PROOF. Assume $\mathscr{L}$ is isomorphic with an $(\mathfrak{m}, \mathfrak{n})$-ring of sets. Then by Proposition 5.2 there must be a set $\mathscr{X} \subseteqq \mathscr{P}$ such that $\rho_{\mathfrak{X}}$ is one-one. Thus we see that the map $l \to \{P \in \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in P\}$ is also one-one and hence

$$(l] = \bigcap \{P \in \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in P\}.$$

So (i) $\Rightarrow$ (ii).

It is clear that (ii) and (iii) are equivalent. Let us assume (ii). Then the map ρ is seen to be one-one and so $\mathscr{L}$ has a faithful representation as an $(\mathfrak{m}, \mathfrak{n})$-ring of subsets of $\mathscr{P}$. The proof of the theorem is now complete.

REMARK. We do not deduce all possible corollaries. It suffices to illustrate the method by taking $\mathfrak{m} = \mathfrak{n} = \infty$ and deducing the result.

Corollary. (G. N. Raney [7], V. K. Balachandran [1]). *A complete lattice $\mathscr{L}$ is isomorphic with a complete ring of sets if and only if $\mathscr{L}$ possesses a join basis of completely join irreducibles.*

Proof. Take $\mathfrak{m} = \mathfrak{n} = \infty$ in Theorem 5.3 parts (i) and (ii). An ∞-prime ∞-dual ideal is equivalent to a completely prime principal dual ideal and its generator is thus a completely join irreducible element. Since the intersection of a family of principal dual ideals is the principal dual ideal generated by the join of the generators of the family, we see that (iii) tells us that for any $l \in L$

$[l) = \bigcap_{\nu \in V} [j_\nu) = [\bigvee_\nu j_\nu)$ where the j_ν are completely join irreducible. This is equivalent to $l = \bigvee j_\nu$ and our Corollary is proved.

References

[1] V. K. Balachandran, "A characterisation of $\Sigma\Delta$-rings of subsets', *Fundamenta Mathematica* 41 (1953), 38—41.

[2] V. K. Balachandran, 'On complete lattices and a problem of Birkhoff and Frink,' *Proc. Amer. Math. Soc.* 6 (1953), 548—553.

[3] G. Birkhoff, *Lattice Theory* (Amer. Math. Soc. Colloq. New York 1948).

[4] G. Birkhoff & O. Frink, 'Representations of lattices by sets', *Trans. Amer. Math. Soc.* 64 (1948), 299—316.

[5] J. R. Büchi, 'Representations of complete lattices by sets', *Portugaliae Mathematicae* 11 (1952), 151—167.

[6] P. D. Finch, 'On the lattice equivalence of topological spaces', *Journ. Aust. Math. Soc.* 6 (1966), 495—511.

[7] G. N. Raney, 'Completely distributive complete lattices', *Proc. Amer. Math. Soc.* 3 (1952), 677—680.

https://doi.org/10.1017/S1446788700006546 Published online by Cambridge University Press