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LOCATION OF GEODESICS AND ISOPERIMETRIC INEQUALITIES IN DENJOY DOMAINS

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Abstract We find approximate solutions (chord–arc curves) for the system of equations of geodesics in $\Omega \cap \bar{\mathbb{H}}$ for every Denjoy domain Ω, with respect to both the Poincaré and the quasi-hyperbolic metrics. We also prove that these chord–arc curves are uniformly close to the geodesics. As an application of these results, we obtain good estimates for the lengths of simple closed geodesics in any Denjoy domain, and we improve the characterization in a 1999 work by Alvarez *et al*. on Denjoy domains satisfying the linear isoperimetric inequality.

Keywords: geodesic; quasi-geodesic; chord–arc; Poincaré metric; linear isoperimetric inequality; Denjoy domain

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1. Introduction

Our main aim is to study the geodesics of Denjoy domains, that is, plane domains Ω with $\partial\Omega \subset \mathbb{R}$. These kinds of surface are becoming more and more important in geometric theory of functions since they are a very general type of Riemann surface, yet they are also more manageable than many other types due to their symmetry. For example, Garnett and Jones [**15**] proved the Corona Theorem for Denjoy domains, and Alvarez *et al*. [**2**] obtained a characterization of Denjoy domains that satisfies a linear isoperimetric inequality.

Obtaining the explicit location of the geodesics in a Riemannian surface is not possible except for in a few examples, since in order to do so we must solve a second-order system of two nonlinear differential equations. In the case of a domain with the Poincaré or the quasi-hyperbolic metric, the situation is even worse: on the one hand, usually we do not have an explicit expression for the density of the Poincaré metric, and hence, or for the equations; on the other hand, for the quasi-hyperbolic metric, the coefficients in the differential equations are the derivatives of a non-differentiable function. However, the geodesics are a fundamental object of Riemannian geometry.

We find approximate solutions (chord–arc curves, a very regular kind of quasi-geodesics; see Definition 2.4) for the system of equations of geodesics in $\Omega \cap \bar{\mathbb{H}}$ for every Denjoy domain Ω (see Theorems 4.2 and 4.4). Furthermore, using results on Gromov hyperbolicity (although in general Ω is not Gromov hyperbolic), we also prove that these chord–arc curves are uniformly close to the geodesics (see Theorems 4.5 and 4.6). There exist several papers studying Gromov hyperbolicity of Euclidean domains and Riemann surfaces in general [**3**, **5**, **8**, **19**–**25**, **28**–**35**, **37**] (see also [**9**, **38**, **39**]).

Using these results on chord–arc curves we obtain good estimates for the Poincaré distance of

(i) any couple of points $z, w \in \Omega \cap \mathbb{R}$ (see Theorem 5.2),

(ii) any pair of connected components of $\Omega \cap \mathbb{R}$ (see Theorem 5.3),

(iii) any point $z \in \Omega \cap \mathbb{R}$ and any connected component of $\Omega \cap \mathbb{R}$ (see Theorem 5.5).

In particular, (ii) is equivalent to estimating the length of simple closed geodesics, which is a very interesting and difficult problem for the Poincaré metric.

We obtain these estimates up to multiplicative constants, which are the best possible results for the Poincaré metric, since the sharpest known estimate for the density of the Poincaré metric (see Theorem 2.9) also has this property.

In [**22**] there is a weaker version of Theorems 4.2 and 4.4: but Hästö proved [**22**] that the curves are (a, b)-quasi-geodesics with $b > 0$; although these weaker versions are good enough for the purposes of [**22**], in order to deal with some applications in § 6 we need to work with $(a, 0)$-quasi-geodesics (see Remark 6.7).

As an application of these results, we improve the characterization in [**2**] of the Denjoy domains satisfying the linear isoperimetric inequality (see Theorem 6.8).

Notation

If we do not specify the metric, we always assume that in any Denjoy domain Ω we consider the Poincaré metric. By d_Ω, L_Ω and A_Ω we shall denote, respectively, the distance, the length and the area with respect to the Poincaré metric of Ω.

2. Previous definitions and results

We denote by $\mathbb{H}$ the upper half-plane $\{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ and by $\mathbb{D}$ the unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$. For $D \subset \mathbb{C}$ we denote by ∂D and $\bar{D}$ its boundary and closure, respectively. For $z \in D \subsetneq \mathbb{C}$ we denote by $\delta_D(z)$ the distance to the boundary, $\min_{a \in \partial D} |z - a|$.

The *quasi-hyperbolic metric* in Ω is the distance induced by the density $1/\delta_\Omega(z)$.

Recall that a domain $\Omega \subset \mathbb{C}$ is said to be *non-exceptional* if it has at least two finite boundary points.The universal cover of such a domain is the unit disc $\mathbb{D}$. In Ω we can define the Poincaré metric, i.e. the metric obtained by projecting the metric $ds = 2|dz|/(1-|z|^2)$ of the unit disc by any universal covering map $\pi \colon \mathbb{D} \to \Omega$. Equivalently, we

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can project the metric $ds = |dz|/\operatorname{Im} z$ of the upper half-plane $\mathbb{H}$. Therefore, any simply connected subset of Ω is isometric to a subset of $\mathbb{D}$. With this metric, Ω is a geodesically complete Riemannian manifold with constant curvature -1; in particular, Ω is a geodesic metric space. The *Poincaré metric* is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant 1 when we consider the respective Poincaré metrics.

We denote by λ_Ω the density of the hyperbolic metric in Ω. It is well known that, for all domains $\Omega_1 \subseteq \Omega_2$, we have $\lambda_{\Omega_1}(z) \geqslant \lambda_{\Omega_2}(z)$ for every $z \in \Omega_1$.

A *Denjoy domain* $\Omega \subset \mathbb{C}$ is a domain whose boundary is contained in the real axis. As mentioned in § 1, Denjoy domains are becoming increasingly relevant to Geometric Function Theory (see, for example, [**1**, **2**, **15**, **17**]).

Definition 2.1. If $\gamma\colon [a,b] \to X$ is a continuous curve in a metric space (X,d), the *length* of γ is

$$L(\gamma) := \sup\biggl\{\sum_{i=1}^{n} d(\gamma(t_{i-1}),\gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b\biggr\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s)) = |t-s|$ for every $s,t \in [a,b]$. We say that X is a *geodesic metric space* if, for every $x,y \in X$, there exists a geodesic joining x and y; we denote by xy any such geodesic (since we do not require the uniqueness of geodesics, this notation is ambiguous, but also convenient).

Definition 2.2. Consider a geodesic metric space X. If $x_1,x_2,x_3 \in X$, a *geodesic triangle* $T = \{x_1,x_2,x_3\}$ is the union of three geodesics x_1x_2, x_2x_3 and x_3x_1. We say that T is δ-*thin* if, for every $x \in x_ix_j$, we have that $d(x, x_jx_k \cup x_kx_i) \leqslant \delta$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin.

Example 2.3.

(i) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic (see, for example, [**16**, p. 29]).

(ii) Every complete simply connected Riemannian manifold with sectional curvature that is bounded from above by $-k$, with $k > 0$, is hyperbolic (see, for example, [**16**, p. 52]).

(iii) Every tree with edges of arbitrary length is 0-hyperbolic (see, for example, [**16**, p. 29]).

Definition 2.4. A function between two metric spaces $f\colon X \to Y$ is an (a,b)-*quasi-isometry*, $a \geqslant 1$, $b \geqslant 0$, if

$$\frac{1}{a} d_X(x_1,x_2) - b \leqslant d_Y(f(x_1),f(x_2)) \leqslant a d_X(x_1,x_2) + b \quad \text{for every } x_1,x_2 \in X.$$

An (a,b)-*quasi-geodesic* in X is an (a,b)-quasi-isometry between an interval of $\mathbb{R}$ and X.

A map f between an interval I of $\mathbb{R}$ and X is *a-chord–arc* if

$$L_X(f|_{[x_1,x_2]}) \leqslant a d_X(x_1,x_2) \quad \text{for every } [x_1,x_2] \subseteq I.$$

Chord–arc curves play a key role in harmonic analysis and in geometry. It is clear that the a-chord–arc curves with their arc-length parametrization are $(a,0)$-quasi-geodesics; they are a very special type of 'very regular' quasi-geodesics (note that a quasi-geodesic can be discontinuous).

Definition 2.5. Let us consider $\varepsilon > 0$, a metric space X and subsets $Y, Z \subseteq X$. The set $N_\varepsilon(Y) := \{x \in X : (x,Y) \leqslant \varepsilon\}$ is called the *ε-neighbourhood* of Y in X. The *Hausdorff distance* of Y to Z is defined by

$$H(Y,Z) := \inf\{\varepsilon > 0 : Y \subseteq N_\varepsilon(Z),\ Z \subseteq N_\varepsilon(Y)\}.$$

The following is a beautiful and useful result.

Theorem 2.6 (Ghys and de la Harpe [16, p. 87]). *For each $\delta \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$, there exists a constant H_0 that depends only on δ, a and b, with the following property.*

Let us consider a δ-hyperbolic geodesic metric space X and an (a,b)-quasi-geodesic g starting in x and finishing in y. If γ is a geodesic joining x and y, then $H(g,\gamma) \leqslant H_0$.

This property is known as geodesic stability. Bonk [**7**] proved that, in fact, geodesic stability is equivalent to hyperbolicity. There is an explicit expression for H_0, but it is very complicated. However, we have the following particular result which gives a simple bound.

Theorem 2.7 (Bonk [7, Proposition 3.1]). *Let X be a δ-hyperbolic geodesic metric space and let g be an a-chord–arc curve joining x and y. Then $g \subset N_{M/2}(\gamma)$ for every geodesic γ joining x and y, with*

$$M = M(\delta,a) := 2(1+8\delta a)(8\delta a^2 + 12\delta a + 2a) + 8\delta a + 4\delta + 4.$$

Furthermore, $H(g,\gamma) \leqslant M$.

Definition 2.8. For every non-exceptional domain $\Omega \subset \mathbb{C}$ and for every $z \in \Omega$, define $\delta_\Omega(z) := \inf\{|z-a| : a \in \partial\Omega\}$ and $\beta_\Omega(z)$ as the function

$$\beta_\Omega(z) := \inf\left\{\left|\log\left|\frac{z-a}{b-a}\right|\right| : a,b \in \partial\Omega,\ |z-a| = \delta_\Omega(z)\right\}.$$

It is clear that the infimum in $\delta_\Omega(z)$ and in $\beta_\Omega(z)$ is attained.

The function β_Ω was introduced by Beardon and Pommerenke [**6**], who showed that it provides the connection between the densities of the hyperbolic and the quasi-hyperbolic metrics. Combining the argument of Beardon and Pommerenke [**6**, Theorem 1] with the inequality of Minda [**26**, Theorem 5], we obtain the following result (note that the definition of the Poincaré metric used in this paper differs from that in [**6**, **26**] by a factor of 2).

Theorem 2.9. *For every non-exceptional domain $\Omega \subset \mathbb{C}$ and for every $z \in \Omega$, we have that*

$$1 \leqslant \lambda_\Omega(z)\delta_\Omega(z)(k_0 + \beta_\Omega(z)) \leqslant 2k_0 + \tfrac{1}{2}\pi,$$

where $k_0 = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796\dots$. Furthermore, for every $z \in \Omega$ and $a, b \in \partial\Omega$, we have

$$\lambda_\Omega(z) \geqslant \biggl(|z-a|\biggl(k_0 + \biggl|\log\biggl|\frac{z-a}{b-a}\biggr|\biggr|\biggr)\biggr)^{-1}.$$

3. Technical lemmas

In this section some technical lemmas are collected. All of them have been used in the next section in order to simplify the proof of Theorem 4.2.

Definition 3.1. If k_0 is the constant in Theorem 2.9, for $c \geqslant 1$, let us define the function

$$k(c) := c(2k_0 + \tfrac{1}{2}\pi)\biggl(1 + \frac{1}{k_0}\log c\biggr).$$

Lemma 3.2. *Let us consider any non-exceptional domain Ω, $z \in \Omega$, $a \in \partial\Omega$ with $|z-a| = \delta_\Omega(z)$, and $c \geqslant 1$. For every $w \in \Omega$ with $|w-a| \leqslant c|z-a|$, we have*

$$\lambda_\Omega(z) \leqslant k(c)\lambda_\Omega(w).$$

Remark 3.3. This result trivially holds for the quasi-hyperbolic metric, replacing $k(c)$ by c. This remark is applicable to every lemma in this section.

Proof. Let us assume first that there exists $b \in \partial\Omega$ with

$$\beta_\Omega(z) = \biggl|\log\biggl|\frac{z-a}{b-a}\biggr|\biggr|.$$

(Although the infimum in $\beta_\Omega(z)$ is attained, perhaps $\beta_\Omega(z) = |\log|z-a'|/|b'-a'||$ with $a' \neq a$ and $|z-a'| = |z-a| = \delta_\Omega(z)$.)

Assume now that $|w-a| \leqslant |z-a|$. Since the function $f(x) := x(k_0 + |\log(x/s)|)$ is increasing in $x \in (0,\infty)$ for any fixed positive constant s,

$$\begin{aligned}\lambda_\Omega(z) &\leqslant \frac{2k_0 + \frac{1}{2}\pi}{|z-a|(k_0 + |\log|(z-a)/(b-a)||)} \\ &\leqslant \frac{2k_0 + \frac{1}{2}\pi}{|w-a|(k_0 + |\log|(w-a)/(b-a)||)} \\ &\leqslant (2k_0 + \tfrac{1}{2}\pi)\lambda_\Omega(w).\end{aligned}$$

Assume now that $|z-a| < |w-a| \leqslant c|z-a|$. Note that, for any $u, v \in \mathbb{R}$, we have

$$\frac{k_0 + |u|}{k_0 + |u-v|} \leqslant 1 + \frac{|v|}{k_0}.$$

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Hence,

$$\begin{aligned}
\lambda_\Omega(z) &\leqslant \frac{2k_0+\frac{1}{2}\pi}{|z-a|(k_0+|\log|(z-a)/(b-a)||)}\\
&\leqslant \frac{2k_0+\frac{1}{2}\pi}{(1/c)|w-a|(k_0+|\log(1/c)|(w-a)/(b-a)||)}\\
&\leqslant \frac{k_0+|\log|(w-a)/(b-a)||}{k_0+|\log|(w-a)/(b-a)|-\log c|}\,\frac{c(2k_0+\frac{1}{2}\pi)}{|w-a|(k_0+|\log|(w-a)/(b-a)||)}\\
&\leqslant \left(1+\frac{1}{k_0}\log c\right)\frac{c(2k_0+\frac{1}{2}\pi)}{|w-a|(k_0+|\log|(w-a)/(b-a)||)}\\
&= \frac{k(c)}{|w-a|(k_0+|\log|(w-a)/(b-a)||)}\\
&\leqslant k(c)\lambda_\Omega(w).
\end{aligned}$$

Let us assume now that there is no $b \in \partial\Omega$ with

$$\beta_\Omega(z) = \left|\log\left|\frac{z-a}{b-a}\right|\right|.$$

Without loss of generality we can assume that $a = 0$ and $z > 0$. For $0 < \varepsilon < z$, we have $\delta_\Omega(z-\varepsilon) = z-\varepsilon$ and $|z-\varepsilon-\zeta| > \delta_\Omega(z-\varepsilon)$ for every $\zeta \in \partial\Omega \setminus \{0\}$. Hence, there exists $b \in \partial\Omega$ with

$$\beta_\Omega(z-\varepsilon) = \left|\log\left|\frac{z-\varepsilon}{b}\right|\right|.$$

Therefore, $|w| \leqslant c_\varepsilon|z-\varepsilon|$, for some constant c_ε with $c_\varepsilon \to c$ as $\varepsilon \to 0$. Then the theorem follows by the previous case, since λ_Ω and $k(c)$ are continuous functions. $\square$

Lemma 3.4. *Let us consider any non-exceptional domain Ω and two curves σ, η in Ω with the same Euclidean length and parametrized with Euclidean arc length. Assume that there exists a constant $c \geqslant 1$ with the following property: for each fixed t, there exists $a_t \in \partial\Omega$ with $|\sigma(t)-a_t| = \delta_\Omega(\sigma(t))$ and $|\eta(t)-a_t| \leqslant c|\sigma(t)-a_t|$. Then $L_\Omega(\sigma) \leqslant k(c)L_\Omega(\eta)$.*

Proof. Lemma 3.2 gives that $\lambda_\Omega(\sigma(t)) \leqslant k(c)\lambda_\Omega(\eta(t))$ for every t. Since $\eta(t)$ and $\sigma(t)$ are parametrized with Euclidean arc length, this inequality gives $L_\Omega(\sigma) \leqslant k(c)L_\Omega(\eta)$. $\square$

Using Lemma 3.4 (with $\sigma(t) = z_0 + \mathrm{i}t$, $t \in [0, r]$ and $c = \sqrt{2}$), we obtain the following result.

Lemma 3.5. *Let us consider a Denjoy domain Ω, $z_0 \in \Omega \cap \bar{\mathbb{H}}$, a curve η with Euclidean length r starting at z_0 and $\sigma := [z_0, z_0 + \mathrm{i}r]$. Then $L_\Omega(\sigma) \leqslant k(\sqrt{2})L_\Omega(\eta)$.*

Lemma 3.6. *Let us consider a Denjoy domain Ω, $z_0 \in \Omega$ with $\operatorname{Im} z_0 \geqslant T > 0$, a curve η with Euclidean length T starting at z_0 and $\sigma := [z_0, z_0 + T]$. Then $L_\Omega(\sigma) \leqslant k(3)L_\Omega(\eta)$.*

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Proof. Consider the curve η parametrized with Euclidean arc length starting at z_0. For each fixed $t \in [0, T]$, let us define $\sigma(t) := z_0 + t$, and consider $a_t \in \partial\Omega$ with $|\sigma(t) - a_t| = \delta_\Omega(\sigma(t))$. We have

$$|\eta(t) - a_t| \leqslant |\eta(t) - z_0| + |z_0 - \sigma(t)| + |\sigma(t) - a_t| \leqslant 2t + |\sigma(t) - a_t|.$$

Since $t \leqslant T \leqslant \operatorname{Im} z_0 \leqslant |\sigma(t) - a_t|$, we deduce that

$$|\eta(t) - a_t| \leqslant 3|\sigma(t) - a_t|.$$

Lemma 3.4 with $c = 3$ gives the result. □

Definition 3.7. Let us define the function $F\colon \mathbb{C} \to \mathbb{C}$ as

$$F(re^{\mathrm{i}t}) := \begin{cases} r + \mathrm{i}r\tan t & \text{if } r \geqslant 0, \quad 0 \leqslant t \leqslant \frac{1}{4}\pi, \\ r\operatorname{cotan} t + \mathrm{i}r & \text{if } r \geqslant 0, \ \frac{1}{4}\pi \leqslant t \leqslant \frac{1}{2}\pi. \end{cases}$$

$F(-z) = -F(z)$ and $F(\bar{z}) = \overline{F(z)}$ for every $z \in \mathbb{C}$.

Note that the transformation F has a simple geometric meaning: the image by F of the circle $\{|z| = r\}$ is the boundary of the square $[-r, r] \times [-r, r]$ (i.e. F applies C-lines on B-lines; see Definition 4.1). This function will allow us to obtain information about C-lines from results about B-lines (see the proof of Theorem 4.2).

It is not difficult to check the following inequalities.

Lemma 3.8. *This function F satisfies*

$$\frac{1}{\sqrt{2}}|z - x| \leqslant |F(z) - x| \leqslant \sqrt{3}|z - x|$$

for every $z \in \mathbb{C}$ and every $x \in \mathbb{R}$.

Lemma 3.9. *The following inequalities hold for the function F and every Denjoy domain Ω.*

(i) *For every $z \in \Omega$,*

$$\frac{1}{k(\sqrt{2})}\lambda_\Omega(F(z)) \leqslant \lambda_\Omega(z) \leqslant k(\sqrt{3})\lambda_\Omega(F(z)).$$

(ii) *For every curve γ contained in any circle $\{|z| = r\} \cap \Omega$,*

$$L_\Omega(\gamma) \leqslant k(\sqrt{3})L_\Omega(F(\gamma)).$$

(iii) *For every curve g contained in Ω,*

$$L_\Omega(F(g)) \leqslant 2\sqrt{2}k(\sqrt{2})L_\Omega(g).$$

(iv) *For every $z_1, z_2 \in \Omega$,*

$$d_\Omega(F(z_1), F(z_2)) \leqslant 2\sqrt{2}k(\sqrt{2})d_\Omega(z_1, z_2).$$

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4. Chord–arc curves in every Denjoy domain

The following curves will play a key role in our results.

Definition 4.1. We denote by *A-lines* the set of curves which can be written as

$$\{z \in \bar{\mathbb{H}} \cap \Omega\colon\ \operatorname{Im} z = a\}$$

for some constant $a \in \mathbb{R}$.

We denote by *B-lines* the set of curves which can be written as

$$([a, a + \mathrm{i}r] \cup [a + \mathrm{i}r, a + 2r + \mathrm{i}r] \cup [a + 2r + \mathrm{i}r, a + 2r]) \cap \Omega$$

for some constants $a \in \mathbb{R}$, $r > 0$.

Half-circles of the type

$$\{z \in \bar{\mathbb{H}} \cap \Omega\colon |z - x_0| = r\}, \quad x_0 \in \mathbb{R},\ r > 0,$$

are called *C-lines*.

Note that A-lines and C-lines are the geodesics for the Poincaré metric in $\mathbb{H}$ (and also for the quasi-hyperbolic metric, since both metrics are the same in $\mathbb{H}$). It is useful to consider B-lines, since in practical cases the computations with B-lines are easier than with the C-lines.

The following surprising result shows that the geodesics for $\mathbb{H}$ are chord–arc curves in every Denjoy domain (with universal constants), whether or not some of the endpoints of the curves belong to $\partial\Omega$.

Theorem 4.2. *Let Ω be any Denjoy domain. Then the following result holds for the Poincaré metric.*

(i) *Every A-line is $k(\sqrt{2})$-chord–arc.*

(ii) *Every B-line is k_1-chord–arc, with $k_1 := k(\sqrt{2}) + k(3)$.*

(iii) *Every C-line is k_2-chord–arc, with $k_2 := 2\sqrt{2}k(\sqrt{2})k(\sqrt{3})k_1$.*

Remark 4.3. By symmetry, a similar result holds for $\{z \in \Omega\colon\ \operatorname{Im} z \leqslant 0\}$.

Proof. Consider σ, which is either an A-line, a B-line or a C-line parametrized with Poincaré arc length, and $s < t$ in the domain of σ.

Assume first that σ is an A-line $\sigma = \{z \in \bar{\mathbb{H}} \cap \Omega\colon\ \operatorname{Im} z = a\}$. Let us consider a hyperbolic geodesic η joining $\sigma(s)$ and $\sigma(t)$. Without loss of generality we can assume that $\operatorname{Im}\sigma(s) < \operatorname{Im}\sigma(t)$. Since the graph of σ is a straight line, we obtain $L_{\mathrm{Eucl}}(\sigma|_{[s,t]}) \leqslant L_{\mathrm{Eucl}}(\eta)$, and we can denote by η_0 the subcurve of η starting at $\sigma(s)$ with $L_{\mathrm{Eucl}}(\eta_0) = L_{\mathrm{Eucl}}(\sigma|_{[s,t]})$. Applying Lemma 3.5, we deduce

$$t - s = L_\Omega(\sigma|_{[s,t]}) \leqslant k(\sqrt{2})L_\Omega(\eta_0) \leqslant k(\sqrt{2})L_\Omega(\eta) = k(\sqrt{2})d_\Omega(\sigma(s), \sigma(t)).$$

Consider now a B-line

$$\sigma := ([a, a + \mathrm{i}r] \cup [a + \mathrm{i}r, a + 2r + \mathrm{i}r] \cup [a + 2r + \mathrm{i}r, a + 2r]) \cap \Omega.$$

If $\sigma(s)$ and $\sigma(t)$ are both either in $[a, a+\mathrm{i}r]$ or in $[a+2r+\mathrm{i}r, a+2r]$, it suffices to apply the previous argument. Hence, without loss of generality we can assume that $\sigma(s) \in [a, a+\mathrm{i}r]$ and $\sigma(t) \in [a + 2r + \mathrm{i}r, a + 2r]$, since the other cases are easier.

Let us consider a hyperbolic geodesic η joining $\sigma(s)$ and $\sigma(t)$. Denote by η_1 the subcurve of η starting at $\sigma(s)$ with $L_{\mathrm{Eucl}}(\eta_1) = r - \operatorname{Im}\sigma(s)$, and by η_2 the subcurve of η finishing in $\sigma(t)$ with $L_{\mathrm{Eucl}}(\eta_2) = r - \operatorname{Im}\sigma(t)$. Since the Euclidean length of η is at least $2r$, η_1 and η_2 are disjoint. Applying Lemma 3.5 twice, we deduce

$$\begin{aligned} L_\Omega([\sigma(s), a + \mathrm{i}r]) + L_\Omega([a + 2r + \mathrm{i}r, \sigma(t)]) &\leqslant k(\sqrt{2})L_\Omega(\eta_1) + k(\sqrt{2})L_\Omega(\eta_2) \\ &\leqslant k(\sqrt{2})L_\Omega(\eta) \\ &= k(\sqrt{2})d_\Omega(\sigma(s), \sigma(t)). \end{aligned}$$

We now bound $L_\Omega([a + \mathrm{i}r, a + 2r + \mathrm{i}r])$.

Let us consider a connected component η_* of $\eta \cap \{z \in \mathbb{C} \colon \operatorname{Im} z \leqslant r\}$. Then η_* joins $z_1 := x_1 + \mathrm{i}y_1$ and $z_2 := x_2 + \mathrm{i}y_2$, with $0 \leqslant y_1, y_2 \leqslant r$, and we define

$$\sigma_* := [x_1 + \mathrm{i}r, x_2 + \mathrm{i}r].$$

Since Ω is a Denjoy domain, we conclude that $b \mapsto \lambda_\Omega(a + \mathrm{i}b)$ is decreasing for $b > 0$ (see [**26**, Theorem 4.1 (i)]); hence, $L_\Omega(\sigma_*) \leqslant L_\Omega(\eta_*)$.

Let us consider now the closure η^* of a connected component of $\eta \cap \{z \in \mathbb{C} \colon \operatorname{Im} z > r\}$; hence, η^* joins $z_3 := x_3 + \mathrm{i}r$ with $z_4 := x_4 + \mathrm{i}r$, and we define $\sigma^* := [z_3, z_4]$. If $T := \frac{1}{2}(z_4 - z_3)$, then we define $\sigma_1^* := [z_3, z_3 + T]$ and $\sigma_2^* := [z_3 + T, z_4]$.

Denote by η_1^* the subcurve of η^* starting at z_3 with $L_{\mathrm{Eucl}}(\eta_1^*) = T$, and by η_2^* the subcurve of η^* finishing at z_4 with $L_{\mathrm{Eucl}}(\eta_2^*) = T$. Since the Euclidean length of η^* is at least $2T$, η_1^* and η_2^* are disjoint. Since σ is a B-line, we deduce that $\operatorname{Im} z_3 = \operatorname{Im} z_4 = r \geqslant T$. Therefore, applying Lemma 3.6 twice, we deduce

$$L_\Omega(\sigma^*) = L_\Omega(\sigma_1^*) + L_\Omega(\sigma_2^*) \leqslant k(3)L_\Omega(\eta_1^*) + k(3)L_\Omega(\eta_2^*) \leqslant k(3)L_\Omega(\eta^*).$$

Hence,

$$L_\Omega([a + \mathrm{i}r, a + 2r + \mathrm{i}r]) \leqslant k(3)L_\Omega(\eta),$$

and consequently,

$$t - s = L_\Omega(\sigma|_{[s,t]}) \leqslant k(\sqrt{2})L_\Omega(\eta) + k(3)L_\Omega(\eta) = (k(\sqrt{2}) + k(3))d_\Omega(\sigma(s), \sigma(t)).$$

This completes the proof of part (ii).

Finally, let us consider a C-line σ. Applying a transformation $Tz = z + c$ if necessary, without loss of generality we can assume that the image of σ is $\{x^2 + y^2 = r^2\}$ for some $r > 0$. Using part (ii) of Lemma 3.9, we obtain

$$t - s = L_\Omega(\sigma|_{[s,t]}) \leqslant k(\sqrt{3})L_\Omega(F(\sigma)|_{[s,t]}).$$

Since we have proved that $F(\sigma)$ is k_1-chord–arc, we have

$$L_\Omega(F(\sigma)|_{[s,t]}) \leqslant k_1 d_\Omega(F(\sigma(t)), F(\sigma(s))).$$

This inequality and part (iv) of Lemma 3.9 give

$$L_\Omega(\sigma|_{[s,t]}) \leqslant k(\sqrt{3})k_1 d_\Omega(F(\sigma(t)), F(\sigma(s))) \leqslant 2\sqrt{2}k(\sqrt{2})k(\sqrt{3})k_1 d_\Omega(\sigma(t), \sigma(s)).$$

This completes the proof of the theorem. □

Using the same argument as in the proof of Theorem 4.2, and always replacing $k(c)$ by c (see Remark 3.3), we obtain a similar result for the quasi-hyperbolic metric.

Theorem 4.4. *Let Ω be any Denjoy domain. Then the following result holds for the quasi-hyperbolic metric.*

(i) *Every A-line is $\sqrt{2}$-chord–arc.*

(ii) *Every B-line is k_1'-chord–arc, with $k_1' := \sqrt{2} + 3$.*

(iii) *Every C-line is k_2'-chord–arc, with $k_2' := 4\sqrt{3}k_1'$.*

Now we prove that chord–arc curves are uniformly close to geodesics in every Denjoy domain.

Theorem 4.5. *For every Denjoy domain Ω with its Poincaré metric, and for every $z, w \in \Omega \cap \bar{\mathbb{H}}$, let γ be the geodesic joining z and w in $\Omega \cap \bar{\mathbb{H}}$ and let g be the subarc of either an A-line, a B-line or a C-line joining z and w. Then $H(\gamma, g) \leqslant M(\delta_0, k_2)$, where $\delta_0 := \log(1 + \sqrt{2})$, k_2 is the constant in Theorem 4.2, and $M(\delta, a)$ is the function in Theorem 2.7.*

Proof. Let us consider the bordered Riemann surface $\Omega^+ = \Omega \cap \bar{\mathbb{H}}$. By [**4**, p. 130], we know that the unit disc and the upper half-plane are δ_0-hyperbolic. Since Ω is symmetric about the real axis, we have that the Poincaré metric in Ω is also symmetric about the real axis, i.e. $\lambda_\Omega(\bar{z}) = \lambda_\Omega(z)$ for every $z \in \Omega$. This implies that each connected component of $\Omega \cap \mathbb{R}$ is a geodesic. Ω^+ is isometric to a geodesically convex subset of the unit disc, since it is a simply connected set bounded by disjoint geodesics; therefore, it is also δ_0-hyperbolic.

By Theorem 4.2, g is k_2-chord–arc, with k_2 the constant in Theorem 4.2, and Theorem 2.7 completes the proof. □

We also have a similar result for the quasi-hyperbolic metric, but without a beautiful expression for the constant.

Theorem 4.6. *For every Denjoy domain Ω with its quasi-hyperbolic metric, and for every $z, w \in \Omega \cap \bar{\mathbb{H}}$, let γ be a geodesic joining z and w in $\Omega \cap \bar{\mathbb{H}}$ and let g be the subarc of either an A-line, a B-line or a C-line joining z and w. Then $H(\gamma, g) \leqslant H_0$, for some universal constant H_0.*

Proof. Let us consider the bordered Riemann surface $\Omega^+ = \Omega \cap \bar{\mathbb{H}}$. This set Ω^+ with its quasi-hyperbolic metric is c-hyperbolic for a universal constant c [**20**, Lemma 3.1].

By Theorem 4.4, g is a k_2'-chord–arc (is a $(k_2', 0)$-quasi-geodesic), with $k_2' := 4\sqrt{3}(\sqrt{2}+3)$.

By Theorem 2.6, we have $H(\gamma, g) \leqslant H_0$, for some universal constant H_0 (depending only on c and k_2', which are universal constants). □

5. Distance estimates and lengths of simple closed geodesics

Using the results in the previous sections, here we obtain good estimates for the Poincaré distance of

(i) any pair of points $z, w \in \Omega \cap \mathbb{R}$ (see Theorem 5.2),

(ii) any pair of connected components of $\Omega \cap \mathbb{R}$ (see Theorem 5.3),

(iii) any point $z \in \Omega \cap \mathbb{R}$ and any connected component of $\Omega \cap \mathbb{R}$ (see Theorem 5.5).

In this section we consider only the Poincaré metric, since there exists a simple function comparable to the quasi-hyperbolic distance for every Denjoy domain (see, for example, [**22**, Lemma 5.1]), which allows us to solve these three problems for this latter metric.

We obtain these estimates up to multiplicative constants, which are the best possible results for the Poincaré metric, since the sharpest known estimates for the density of the Poincaré metric in Theorem 2.9 also have this property.

Note that (ii) is equivalent to estimating the length of simple closed geodesics, a very interesting and difficult problem for the Poincaré metric. These geodesics are a key concept of Riemannian geometry. The closed geodesics are the periodic orbits of the dynamical system associated to a manifold on its unit tangent bundle, and they provide tools to study the geodesic flow, just as the fixed points of an automorphism help to study it. Lastly, closed geodesics are becoming increasingly important in the study of heat and wave equations, and the study of the spectrum of the manifold. The lengths of all closed geodesics largely determine the spectrum. Conversely, the spectrum completely determines the lengths of the closed geodesics [**11**, **13**, **18**].

Lemma 5.1. *Let Ω be any Denjoy domain, with $a \in \mathbb{R}$ and $r > 0$. Then we have*

$$L_\Omega([a + \mathrm{i}r, a \pm r + \mathrm{i}r]) \leqslant k(2) L_\Omega([a, a + \mathrm{i}r]).$$

Proof. We shall prove that

$$\lambda_\Omega(a + t + \mathrm{i}r) \leqslant k(2)\lambda_\Omega(a + \mathrm{i}r) \tag{5.1}$$

for every real t with $|t| \leqslant r$. Since Ω is a Denjoy domain, we conclude that $b \mapsto \lambda_\Omega(a + \mathrm{i}b)$ is decreasing for $b > 0$ (see [**26**, Theorem 4.1 (i)]), and then

$$\lambda_\Omega(a + t + \mathrm{i}r) \leqslant k(2)\lambda_\Omega(a + \mathrm{i}(r - |t|)) \quad \text{for every } t \in [-r, r].$$

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This inequality proves the lemma, since the three intervals involved have the same Euclidean length.

Now let us prove (5.1). Choose $a_t \in \partial\Omega$ with $\delta_\Omega(a+t+\mathrm{i}r) = |a+t+\mathrm{i}r-a_t|$. Let us note that

$$\begin{aligned}|a+\mathrm{i}r-a_t| &\leqslant |a+\mathrm{i}r-(a+t+\mathrm{i}r)| + |a+t+\mathrm{i}r-a_t|\\ &= |t| + |a+t+\mathrm{i}r-a_t|\\ &\leqslant r + |a+t+\mathrm{i}r-a_t|\\ &\leqslant 2|a+t+\mathrm{i}r-a_t|.\end{aligned}$$

Therefore, Lemma 3.2 gives $\lambda_\Omega(a+t+\mathrm{i}r) \leqslant k(2)\lambda_\Omega(a+\mathrm{i}r)$. □

The next result allows us to estimate the distance of any pair of points of $\Omega \cap \mathbb{R}$ in Ω.

Theorem 5.2. *Let Ω be any Denjoy domain and let g be any B-line. Then we have*

$$\frac{1}{k(2)+1} L_\Omega(g) \leqslant L_\Omega([a, a+\mathrm{i}r] \cup [a+2r, a+2r+\mathrm{i}r]) < L_\Omega(g).$$

Furthermore,

$$\frac{1}{k(2)+1} d_\Omega(a, a+2r) \leqslant L_\Omega([a, a+\mathrm{i}r] \cup [a+2r, a+2r+\mathrm{i}r]) < k_1 d_\Omega(a, a+2r)$$

for every $a, a+2r \in \mathbb{R}$, with $k_1 = k(\sqrt{2}) + k(3)$.

Proof. Applying Lemma 5.1 twice, we obtain, for every B-line g,

$$L_\Omega([a+\mathrm{i}r, a+2r+\mathrm{i}r]) \leqslant k(2) L_\Omega([a, a+\mathrm{i}r] \cup [a+2r, a+2r+\mathrm{i}r]),$$

$$\begin{aligned}L_\Omega(g) &= L_\Omega([a, a+\mathrm{i}r]) + L_\Omega([a+\mathrm{i}r, a+2r+\mathrm{i}r])\\ &\quad + L_\Omega([a+2r, a+2r+\mathrm{i}r])\\ &\leqslant (k(2)+1) L_\Omega([a, a+\mathrm{i}r] \cup [a+2r, a+2r+\mathrm{i}r]),\end{aligned}$$

which is the first inequality in the second display. The first one is trivial.

In order to finish the proof we just need to note that

$$d_\Omega(a, a+2r) \leqslant L_\Omega(g) \leqslant k_1 d_\Omega(a, a+2r)$$

by Theorem 4.2. □

The next result allows us to estimate the distance of any pair of connected components of $\Omega \cap \mathbb{R}$ or, equivalently, the length of simple closed geodesics in Ω.

Theorem 5.3. *Let Ω be any Denjoy domain with $\Omega \cap \mathbb{R} = \bigcup_n (a_n, b_n)$. Denote by x_n the midpoint of (a_n, b_n) and by γ_{mn} the shortest geodesic joining (a_m, b_m) and (a_n, b_n) with $a_m < a_n$. There exist universal constants c_1, c_2 and c_3 verifying the following.*

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(i) *If* $b_m - a_m \leqslant a_n - b_m$ *and* $b_n - a_n \leqslant a_n - b_m$, *then*

$$\begin{aligned} c_1 L_\Omega(\gamma_{mn}) &\leqslant L_\Omega([x_m, x_m + \tfrac{1}{2}\mathrm{i}|x_n - x_m|]) + L_\Omega([x_n, x_n + \tfrac{1}{2}\mathrm{i}|x_n - x_m|]) \\ &\leqslant c_2 L_\Omega(\gamma_{mn}). \end{aligned}$$

(ii) *If* $b_n - a_n \leqslant b_m - a_m$, $a_n - b_m \leqslant b_m - a_m$ *and*

$$r(a_m, b_m, a_n, b_n) := \frac{(b_m - a_m)(b_n - a_n)}{(a_n - b_m)(b_n - a_m)} \leqslant r_0$$

for some positive constant r_0, *then*

$$c_3 L_\Omega(\gamma_{mn}) \leqslant L_\Omega([x_n, x_n + \mathrm{i}(x_n - b_m)]) \leqslant c_2(3r_0 + 2) L_\Omega(\gamma_{mn}).$$

In fact, we can choose

$$c_1 = \frac{1}{k(2)+1}, \qquad c_2 = 2k(1)(k(\sqrt{2}) + k(3)), \qquad c_3 = \frac{1}{(k(2)+1)(k(3\sqrt{2})+1)}.$$

(iii) *If* $r(a_m, b_m, a_n, b_n) \geqslant r_0$ *for some* $r_0 > 1$, *then there exist constants* c_4, c_5, *which just depend on* r_0, *such that*

$$c_4 L_\Omega(\gamma_{mn}) \leqslant \frac{1}{\log r(a_m, b_m, a_n, b_n)} \leqslant c_5 L_\Omega(\gamma_{mn}).$$

Remark 5.4.

(i) By symmetry, we can always assume $a_m < a_n$ and $b_n - a_n \leqslant b_m - a_m$; therefore, these hypotheses are just technical, and Theorem 5.3 covers all possible cases.

(ii) We also allow $a_m = -\infty$. The case $a_m = -\infty$ and $b_n = \infty$ is direct, since then ∞ is a puncture and $L_\Omega(\gamma_{mn}) = 0$.

(iii) Although $b_n - a_m > 0$, it is possible to have $a_n - b_m = 0$, and then

$$r(a_m, b_m, a_n, b_n) = \infty$$

(therefore $a_n = b_m$ and $L_\Omega(\gamma_{mn}) = 0$).

Proof. Recall that the first part of Theorem 5.2 states that, for every B-line g,

$$L_\Omega(g) \leqslant (k(2)+1) L_\Omega([a, a + \mathrm{i}r] \cup [a + 2r + \mathrm{i}r, a + 2r]).$$

Note that, since the map $b \mapsto \lambda_\Omega(a + \mathrm{i}b)$ is decreasing for $b > 0$ (see [**26**, Theorem 4.1 (i)]), we have, for every constant $Q \geqslant 1$,

$$L_\Omega([x, x + \mathrm{i}Qy]) \leqslant Q L_\Omega([x, x + \mathrm{i}y]).$$

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If $\eta := [x, x + \mathrm{i}y]$, we denote by $Q\eta$ the segment $Q\eta := [x, x + \mathrm{i}Qy]$. Then $L_\Omega(Q\eta) \leqslant QL_\Omega(\eta)$.

Let us consider the B-line B joining x_m and x_n. If y_m and y_n are the endpoints of γ_{mn}, consider the B-line B' joining y_m and y_n.

Let us denote by σ_j (respectively, σ'_j) the vertical segment of B (respectively, B') starting in (a_j, b_j), for $j = m, n$. We define $\tilde{\sigma}_n := [x_n, x_n + \mathrm{i}(x_n - b_m)]$.

We denote by h (respectively, h', $\tilde{h}$) the maximum of the imaginary part of the points in σ_n (respectively, σ'_n, $\tilde{\sigma}_n$).

First we prove part (i). Then $b_m - a_m \leqslant a_n - b_m$ and $b_n - a_n \leqslant a_n - b_m$ imply

$$2h' = y_n - y_m \geqslant a_n - b_m \geqslant \tfrac{1}{2}(x_n - x_m) = h.$$

If ζ_j is a point in $\{a_j, b_j\}$ $(j = m, n)$ with $\delta_\Omega(y_j) = |y_j - \zeta_j|$, then we also have $\delta_\Omega(x_j) = |x_j - \zeta_j|$, since $|x_j - a_j| = |x_j - b_j|$. Hence,

$$|y_j + \mathrm{i}t - \zeta_j| \leqslant |x_j + \mathrm{i}t - \zeta_j|.$$

Since $h \leqslant 2h'$, we have $\sigma_j \subseteq 2\sigma'_j$, and Lemma 3.4 gives

$$L_\Omega(\sigma_j) \leqslant k(1)L_\Omega(2\sigma'_j) \leqslant 2k(1)L_\Omega(\sigma'_j).$$

Therefore, using Theorem 4.2,

$$\begin{aligned}
\frac{1}{k(2)+1}L_\Omega(\gamma_{mn}) &\leqslant \frac{1}{k(2)+1}L_\Omega(B) \\
&\leqslant L_\Omega(\sigma_m) + L_\Omega(\sigma_n) \\
&\leqslant 2k(1)L_\Omega(\sigma'_m) + 2k(1)L_\Omega(\sigma'_n) \\
&\leqslant 2k(1)L_\Omega(B') \\
&\leqslant 2k(1)(k(\sqrt{2}) + k(3))L_\Omega(\gamma_{mn}).
\end{aligned}$$

We now prove (ii). Since $b_n - a_n \leqslant b_m - a_m$ and $a_n - b_m \leqslant b_m - a_m$, we have $b_n - a_m \leqslant 3(b_m - a_m)$ and, consequently, $b_n - a_n \leqslant 3r_0(a_n - b_m)$.

We distinguish two cases.

Case (a). We assume first that $\frac{1}{2}(b_m - a_m) \leqslant x_n - b_m$. We have

$$x_n - b_m = \tfrac{1}{2}(b_n - a_n) + a_n - b_m \leqslant \tfrac{1}{2}(b_m - a_m) + b_m - a_m = 3(b_m - x_m),$$

and hence

$$\begin{aligned}
|x_n + \mathrm{i}t - b_m| &\leqslant t + x_n - b_m \\
&\leqslant 3\sqrt{2}\tfrac{1}{\sqrt{2}}(t + b_m - x_m) \\
&\leqslant 3\sqrt{2}|x_m + \mathrm{i}t - b_m| \\
&= 3\sqrt{2}\delta_\Omega(x_m + \mathrm{i}t).
\end{aligned}$$

Lemma 3.4 gives $L_\Omega(\sigma_m) \leqslant k(3\sqrt{2})L_\Omega(\sigma_n)$. Therefore,

$$\begin{aligned}\frac{1}{(k(2)+1)(k(3\sqrt{2})+1)}L_\Omega(\gamma_{mn}) &\leqslant \frac{1}{(k(2)+1)(k(3\sqrt{2})+1)}L_\Omega(B)\\ &\leqslant \frac{1}{k(3\sqrt{2})+1}(L_\Omega(\sigma_m)+L_\Omega(\sigma_n))\\ &\leqslant L_\Omega(\sigma_n).\end{aligned}$$

Since we are assuming $\frac{1}{2}(b_m - a_m) \leqslant x_n - b_m$, we have

$$\begin{aligned}h &= \tfrac{1}{2}(x_n - x_m)\\ &\geqslant \tfrac{1}{2}(x_n - b_m)\\ &= \tfrac{1}{2}\tilde{h},\\ &= \tfrac{1}{2}(x_n - x_m)\\ &= \tfrac{1}{2}(x_n - b_m) + \tfrac{1}{2}(b_m - x_m)\\ &= \tfrac{1}{2}(x_n - b_m) + \tfrac{1}{4}(b_m - a_m)\\ &\leqslant \tfrac{1}{2}(x_n - b_m) + \tfrac{1}{2}(x_n - b_m)\\ &= \tilde{h},\end{aligned}$$

and then $h \leqslant \tilde{h} \leqslant 2h$. Therefore, $\sigma_n \subset \tilde{\sigma}_n \subset 2\sigma_n$ and

$$L_\Omega(\sigma_n) \leqslant L_\Omega(\tilde{\sigma}_n) \leqslant L_\Omega(2\sigma_n) \leqslant 2L_\Omega(\sigma_n).$$

We also have

$$\begin{aligned}h &\leqslant \tilde{h}\\ &= x_n - b_m\\ &= x_n - a_n + a_n - b_m\\ &= \tfrac{1}{2}(b_n - a_n + 2(a_n - b_m))\\ &\leqslant \tfrac{1}{2}(3r_0(a_n - b_m) + 2(a_n - b_m))\\ &\leqslant (3r_0 + 2)h',\end{aligned}$$

and then $\tilde{h} \leqslant 2h \leqslant 2(3r_0+2)h'$. A similar argument to that used in the proof of (i), using Lemma 3.4, gives

$$L_\Omega(\tilde{\sigma}_n) \leqslant k(1)L_\Omega(2(3r_0+2)\sigma'_n) \leqslant 2(3r_0+2)k(1)L_\Omega(\sigma'_n).$$

Hence, using Theorem 4.2,

$$\begin{aligned}\frac{1}{(k(2)+1)(k(3\sqrt{2})+1)}L_\Omega(\gamma_{mn}) &\leqslant L_\Omega(\sigma_n) \leqslant L_\Omega(\tilde{\sigma}_n)\\ &\leqslant 2(3r_0+2)k(1)L_\Omega(\sigma'_n)\\ &\leqslant 2(3r_0+2)k(1)L_\Omega(B')\\ &\leqslant 2(3r_0+2)k(1)k_1L_\Omega(\gamma_{mn})\\ &= c_2(3r_0+2)L_\Omega(\gamma_{mn}).\end{aligned}$$

Case (b). We now consider the case when $x_n - b_m < \frac{1}{2}(b_m - a_m)$. Note that in this case it is possible that x_m is not well defined, since the case $a_m = -\infty$ is allowed, and then $x_m = -\infty$. We define B in this case as the B-line joining $x_m^* := 2b_m - x_n$ and x_n. Note that, by our hypothesis,

$$x_m^* = 2b_m - x_n = b_m - (x_n - b_m) > b_m - \tfrac{1}{2}(b_m - a_m) = x_m,$$

and then x_m^* is nearer to b_m than x_m; hence, $\delta(x_m^* + \mathrm{i}t) = |x_m^* + \mathrm{i}t - b_m|$. We also have $\tilde{h} = h = x_n - b_m$ and $\tilde{\sigma}_n = \sigma_n$. Then

$$h = x_n - b_m = \tfrac{1}{2}(b_n - a_n) + 2\tfrac{1}{2}(a_n - b_m) \leqslant (3r_0 + 2)\tfrac{1}{2}(a_n - b_m) \leqslant (3r_0 + 2)h'.$$

A similar argument to that used in the proof of (i), using Lemma 3.4, gives

$$L_\Omega(\sigma_n) \leqslant k(1)L_\Omega((3r_0 + 2)\sigma_n') \leqslant (3r_0 + 2)k(1)L_\Omega(\sigma_n').$$

We also have

$$|x_n + \mathrm{i}t - b_m| = |x_m^* + \mathrm{i}t - b_m| = \delta_\Omega(x_m^* + \mathrm{i}t),$$

and then Lemma 3.4 gives $L_\Omega(\sigma_m) \leqslant k(1)L_\Omega(\sigma_n)$. Therefore, using Theorem 4.2,

$$\begin{aligned}
\frac{1}{(k(2)+1)(k(1)+1)} L_\Omega(\gamma_{mn}) &\leqslant \frac{1}{(k(2)+1)(k(1)+1)} L_\Omega(B) \\
&\leqslant \frac{1}{k(1)+1}(L_\Omega(\sigma_m) + L_\Omega(\sigma_n)) \\
&\leqslant L_\Omega(\sigma_n) \\
&\leqslant (3r_0 + 2)k(1)L_\Omega(\sigma_n') \\
&\leqslant (3r_0 + 2)k(1)L_\Omega(B') \\
&\leqslant (3r_0 + 2)k(1)k_1 L_\Omega(\gamma_{mn}) \\
&\leqslant c_2(3r_0 + 2)L_\Omega(\gamma_{mn}).
\end{aligned}$$

This completes the proof of (ii).

Finally, we prove (iii). Assume that $r := r(a_m, b_m, a_n, b_n) \geqslant r_0$ for some $r_0 > 1$. Let us consider the Möbius map

$$T(z) := \frac{(b_m - a_m)(z - a_n)}{(a_n - b_m)(z - a_m)}.$$

It is clear that $T(a_m) = \infty$, $T(b_m) = -1$, $T(a_n) = 0$ and $T(b_n) = r$. If we define

$$S_r := \mathbb{C} \setminus \{-1, 0, r\} \quad \text{and} \quad T_r := \mathbb{C} \setminus \{[-1, 0] \cup [r, \infty)\},$$

then $T_r \subset T(\Omega) \subset S_r$. It is easy to check that

$$\sigma_r := \{z \in \mathbb{C} \colon |z + 1| = \sqrt{1 + r}\}$$

is the simple closed geodesic in S_r (and in T_r) that surrounds $\{-1, 0\}$ and does not surround $\{r\}$. Since $T_r \subset T(\Omega) \subset S_r$, we have

$$L_{S_r}(\sigma_r) \leqslant L_{T(\Omega)}(T(\gamma_{mn})) = L_\Omega(\gamma_{mn}) \leqslant L_{T_r}(\sigma_r).$$

Then we just need to apply [**2**, Lemma 4.5]. This completes the proof. □

The next result allows us to estimate the distance from any connected component of $\Omega \cap \mathbb{R}$ to a point of $\Omega \cap \mathbb{R}$.

Theorem 5.5. *Let Ω be any Denjoy domain with $\Omega \cap \mathbb{R} = \bigcup_n (a_n, b_n)$. Given $x \in (a_n, b_n)$, denote by γ_m^x the shortest geodesic joining x and (a_m, b_m). There exist universal constants c_1, c_2, C_1 and C_2 verifying the following.*

(i) *If $b_m - a_m \leqslant 2d_{\mathrm{Eucl}}(x, (a_m, b_m))$, then*

$$c_1 L_\Omega(\gamma_m^x) \leqslant L_\Omega([x_m, x_m + \tfrac{1}{2}\mathrm{i}|x - x_m|]) + L_\Omega([x, x + \tfrac{1}{2}\mathrm{i}|x - x_m|]) \leqslant c_2 L_\Omega(\gamma_m^x),$$

where x_m is the midpoint of (a_m, b_m).

(ii) *If $b_m - a_m > 2d_{\mathrm{Eucl}}(x, (a_m, b_m))$, then*

$$C_1 L_\Omega(\gamma_m^x) \leqslant L_\Omega([x, x + \mathrm{i}(x - b_m)]) \leqslant C_2 L_\Omega(\gamma_m^x).$$

In fact, we can choose c_1, c_2 as in Theorem 5.3, and

$$C_1 = \frac{1}{(k(2)+1)(k(1)+1)}, \qquad C_2 = 2(k(\sqrt{2}) + k(3)).$$

Proof. By symmetry, without loss of generality we can assume that $a_m < a_n$.

Recall that the first part of Theorem 5.2 states that, for every B-line g,

$$L_\Omega(g) \leqslant (k(2)+1) L_\Omega([a, a + \mathrm{i}r] \cup [a + 2r, a + 2r + \mathrm{i}r]).$$

As we saw at the beginning of the proof of Theorem 5.3, we also have

$$L_\Omega([x_0, x_0 + \mathrm{i}Qy_0]) \leqslant Q L_\Omega([x_0, x_0 + \mathrm{i}y_0]) \quad \text{for every } x_0 \in \mathbb{R},\ y_0 > 0,\ Q \geqslant 1.$$

If $b_m - a_m \leqslant 2(x - b_m)$, we have defined x_m as the midpoint of (a_m, b_m). If $b_m - a_m > 2(x - b_m)$, let us define x_m as $x_m := 2b_m - x$.

Let us consider the B-line B joining x_m and x. If $y_m := \gamma_m^x \cap (a_m, b_m)$, consider the B-line B' joining y_m and x.

Let us denote by σ_j (respectively, σ_j') the vertical segment of B (respectively, B') starting in (a_j, b_j), for $j = m, n$.

We denote by h (respectively, h') the maximum of the imaginary part of the points in σ_n (respectively, σ_n').

First we prove (i). Then $b_m - a_m \leqslant 2(x - b_m)$ and this implies

$$2h' = x - y_m \geqslant x - b_m \geqslant \tfrac{1}{2}(x - b_m + \tfrac{1}{2}(b_m - a_m)) = h.$$

If ζ is a point in $\{a_m, b_m\}$ with $\delta_\Omega(y_m) = |y_m - \zeta|$, then we also have $\delta_\Omega(x_m) = |x_m - \zeta|$, since $|x_m - a_m| = |x_m - b_m|$. Hence,

$$|y_m + \mathrm{i}t - \zeta| \leqslant |x_m + \mathrm{i}t - \zeta|.$$

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Since $h \leqslant 2h'$, we have $\sigma_j \subseteq 2\sigma_j'$. Then

$$L_\Omega(\sigma_n) \leqslant L_\Omega(2\sigma_n') \leqslant 2L_\Omega(\sigma_n'),$$

and Lemma 3.4 gives

$$L_\Omega(\sigma_m) \leqslant k(1)L_\Omega(2\sigma_m') \leqslant 2k(1)L_\Omega(\sigma_m').$$

Therefore, using Theorem 4.2,

$$\begin{aligned}\frac{1}{k(2)+1}L_\Omega(\gamma_m^x) &\leqslant \frac{1}{k(2)+1}L_\Omega(B)\\ &\leqslant L_\Omega(\sigma_m)+L_\Omega(\sigma_n)\\ &\leqslant 2k(1)L_\Omega(\sigma_m')+2L_\Omega(\sigma_n')\\ &\leqslant 2k(1)L_\Omega(B') \leqslant 2k(1)(k(\sqrt{2})+k(3))L_\Omega(\gamma_m^x).\end{aligned}$$

We now prove (i). Since $2(x-b_m) < b_m - a_m$ and $x_m = 2b_m - x$, we have

$$x_m > \tfrac{1}{2}(a_m+b_m), \qquad b_m - x_m = \delta_\Omega(x_m), \qquad x - b_m = b_m - x_m.$$

Hence,

$$|x - b_m + \mathrm{i}t| = |x_m + \mathrm{i}t - b_m| = \delta_\Omega(x_m+\mathrm{i}t).$$

Lemma 3.4 gives $L_\Omega(\sigma_m) \leqslant k(1)L_\Omega(\sigma_n)$. Therefore,

$$\begin{aligned}\frac{1}{(k(2)+1)(k(1)+1)}L_\Omega(\gamma_m^x) &\leqslant \frac{1}{(k(2)+1)(k(1)+1)}L_\Omega(B)\\ &\leqslant \frac{1}{k(1)+1}(L_\Omega(\sigma_m)+L_\Omega(\sigma_n))\\ &\leqslant L_\Omega(\sigma_n).\end{aligned}$$

Note that $2h' = x - y_m \geqslant x - b_m = h$. Therefore, $\sigma_n \subseteq 2\sigma_n'$ and, using Theorem 4.2,

$$L_\Omega(\sigma_n) \leqslant L_\Omega(2\sigma_n') \leqslant 2L_\Omega(\sigma_n') \leqslant 2L_\Omega(B') \leqslant 2(k(\sqrt{2})+k(3))L_\Omega(\gamma_m^x).$$

□

Lastly, we need a technical lemma.

Lemma 5.6. *Let us consider a Denjoy domain Ω, $x \in \Omega \cap \mathbb{R}$ and $0 \leqslant u < v$. Then*

$$\begin{aligned}&\frac{v-u}{k(\sqrt{1+v^2}/\sqrt{1+u^2})\sqrt{1+u^2}(k_0+\beta_\Omega(x+\mathrm{i}u\delta_\Omega(x)))}\\ &\qquad\leqslant L_\Omega([x+\mathrm{i}u\delta_\Omega(x), x+\mathrm{i}v\delta_\Omega(x)])\\ &\qquad\leqslant \frac{(2k_0+\frac{1}{2}\pi)k(1)(v-u)}{\sqrt{1+u^2}(k_0+\beta_\Omega(x+\mathrm{i}u\delta_\Omega(x)))}.\end{aligned}$$

Proof. Let us consider $0 \leqslant u \leqslant y \leqslant v$, and $a \in \partial\Omega$ with $\delta_\Omega(x) = |x-a|$. We obviously have

$$\delta_\Omega(x + \mathrm{i}y\delta_\Omega(x)) = |x + \mathrm{i}y\delta_\Omega(x) - a|.$$

It is easy to check that

$$1 \leqslant \frac{|x + \mathrm{i}y\delta_\Omega(x) - a|}{|x + \mathrm{i}u\delta_\Omega(x) - a|} = \frac{\delta_\Omega(x)\sqrt{1+y^2}}{\delta_\Omega(x)\sqrt{1+u^2}} \leqslant \frac{\sqrt{1+v^2}}{\sqrt{1+u^2}},$$

and applying Lemma 3.2 we obtain

$$\frac{1}{k(1)}\lambda_\Omega(x + \mathrm{i}y\delta_\Omega(x)) \leqslant \lambda_\Omega(x + \mathrm{i}u\delta_\Omega(x)) \leqslant k\left(\frac{\sqrt{1+v^2}}{\sqrt{1+u^2}}\right)\lambda_\Omega(x + \mathrm{i}y\delta_\Omega(x)).$$

Consequently, using Theorem 2.9,

$$\begin{aligned}
L_\Omega([x + \mathrm{i}u\delta_\Omega(x), x + \mathrm{i}v\delta_\Omega(x)]) &= \int_u^v \lambda_\Omega(x + \mathrm{i}y\delta_\Omega(x))\delta_\Omega(x)\,\mathrm{d}y \\
&\leqslant (v-u)\delta_\Omega(x)k(1)\lambda_\Omega(x + \mathrm{i}u\delta_\Omega(x)) \\
&\leqslant \frac{(2k_0 + \frac{1}{2}\pi)k(1)(v-u)}{\sqrt{1+u^2}(k_0 + \beta_\Omega(x + \mathrm{i}u\delta_\Omega(x)))}, \\
L_\Omega([x + \mathrm{i}u\delta_\Omega(x), x + \mathrm{i}v\delta_\Omega(x)]) &\geqslant \frac{(v-u)\delta_\Omega(x)}{k(\sqrt{1+v^2}/\sqrt{1+u^2})}\lambda_\Omega(x + \mathrm{i}u\delta_\Omega(x)) \\
&\geqslant \frac{v-u}{k(\sqrt{1+v^2}/\sqrt{1+u^2})\sqrt{1+u^2}(k_0 + \beta_\Omega(x + \mathrm{i}u\delta_\Omega(x)))}.
\end{aligned}$$

□

Theorems 5.2, 5.3 and 5.5 estimate distances (which are very difficult to compute) in terms of lengths of vertical segments. The following result gives a practical criterion for estimating $L_\Omega([a, a + \mathrm{i}r])$ in a simple way, by using a comparable quantity (which is easy to compute).

We define, as usual, the integer part of $x \in \mathbb{R}$ as $[x] := n$ if $x \in [n, n+1)$.

Theorem 5.7. *Let us consider a Denjoy domain Ω, $a \in \Omega \cap \mathbb{R}$, $r > 0$ and $m := [\log_2(r/\delta_\Omega(a))]$. Then*

(i) *if $r \geqslant \delta_\Omega(a)$ $(m \geqslant 0)$,*

$$\begin{aligned}
\frac{1}{\sqrt{2}k(2)}&\left(\frac{1}{k_0 + \beta_\Omega(a)} + \sum_{n=0}^{m-1}\frac{1}{k_0 + \beta_\Omega(a + \mathrm{i}2^n\delta_\Omega(a))}\right) \\
&\leqslant L_\Omega([a, a + \mathrm{i}r]) \\
&\leqslant (4k_0 + \pi)k(1)\left(\frac{1}{k_0 + \beta_\Omega(a)} + \sum_{n=0}^{m-1}\frac{1}{k_0 + \beta_\Omega(a + \mathrm{i}2^n\delta_\Omega(a))}\right);
\end{aligned}$$

(ii) *if* $r < \delta_\Omega(a)$,

$$\frac{1}{k(\sqrt{2})}\frac{r}{\delta_\Omega(a)(k_0+\beta_\Omega(a))} \leqslant L_\Omega([a, a+\mathrm{i}r]) \leqslant (2k_0 + \tfrac{1}{2}\pi)k(1)\frac{r}{\delta_\Omega(a)(k_0+\beta_\Omega(a))}.$$

Remark 5.8. As usual, we define $\sum_{n=0}^{-1} := 0$.

Proof. In order to prove (i), note that $m \leqslant \log_2(r/\delta_\Omega(a)) < m+1$, and therefore $2^m\delta_\Omega(a) \leqslant r < 2^{m+1}\delta_\Omega(a)$. Recall that, as we have seen in the beginning of the proof of Theorem 5.3, we have $L_\Omega(Q\eta) \leqslant QL_\Omega(\eta)$ for every constant $Q \geqslant 1$. Therefore,

$$\begin{aligned} L_\Omega([a, a+\mathrm{i}2^m\delta_\Omega(a)]) &\leqslant L_\Omega([a, a+\mathrm{i}r]) \\ &< L_\Omega([a, a+\mathrm{i}2\cdot 2^m\delta_\Omega(a)]) \\ &\leqslant 2L_\Omega([a, a+\mathrm{i}2^m\delta_\Omega(a)]), \end{aligned}$$

and

$$L_\Omega([a, a+\mathrm{i}2^m\delta_\Omega(a)]) = L_\Omega([a, a+\mathrm{i}\delta_\Omega(a)]) + \sum_{n=0}^{m-1} L_\Omega([a+\mathrm{i}2^n\delta_\Omega(a), a+\mathrm{i}2^{n+1}\delta_\Omega(a)]).$$

If $u = 0$ and $v = 1$, then

$$\frac{v-u}{\sqrt{1+u^2}} = 1, \qquad \frac{\sqrt{1+v^2}}{\sqrt{1+u^2}} = \sqrt{2};$$

if $u = 2^n$ and $v = 2^{n+1}$, then

$$\frac{v-u}{\sqrt{1+u^2}} = \frac{2^n}{\sqrt{1+2^{2n}}} \in [1/\sqrt{2}, 1], \qquad \frac{\sqrt{1+v^2}}{\sqrt{1+u^2}} = \frac{\sqrt{1+2^{2n+2}}}{\sqrt{1+2^{2n}}} < 2.$$

These facts and Lemma 5.6 give (i).

Now let us prove (ii). If $u = 0$ and $v = r/\delta_\Omega(a)$, then

$$\frac{v-u}{\sqrt{1+u^2}} = \frac{r}{\delta_\Omega(a)}, \qquad \frac{\sqrt{1+v^2}}{\sqrt{1+u^2}} < \sqrt{2}.$$

These facts and Lemma 5.6 give (ii). □

6. Isoperimetric inequalities

Let us consider a non-exceptional Riemann surface S with its Poincaré metric. We say that S satisfies the *linear isoperimetric inequality* (LII) if there exists a constant $h > 0$ such that, for every relatively compact domain (open and connected set) G with smooth boundary, we have that

$$A_S(G) \leqslant hL_S(\partial G). \tag{6.1}$$

We denote by $h(S)$ the best constant in (6.1).

There are a number of natural questions concerning the LII property of Riemann surfaces. Particularly interesting are the stability under appropriate maps, its relation with other conformal invariants and its characterization for plane domains.

Concerning the study of the stability of the LII, in [**14**, Theorem 1] it was proved that the LII is invariant by quasi-conformal maps.

One of the conformal invariants related with the LII property is the bottom of the spectrum of the Laplace–Beltrami operator, $b(S)$, defined in terms of Rayleigh's quotient. The number $b(S)$ belongs to $[0, \frac{1}{4}]$ and a celebrated theorem of Elstrodt *et al*. [**36**, p. 333] relates it to another important conformal invariant of S, its exponent of convergence $\delta(S)$ (see, for example, [**27**, p. 21] for basic background). It is a well-known fact that $0 \leqslant \delta(S) \leqslant 1$ (see, for example, [**27**, p. 21]).

It is also well known (see, for example, [**10**, p. 95], [**12**], [**14**, Theorem 2]) that

$$\tfrac{1}{4} \leqslant b(S)h(S)^2 \quad \text{and} \quad b(S)h(S) \leqslant \tfrac{3}{2}.$$

Therefore, S has the LII property if and only if $b(S) > 0$ or, equivalently, $\delta(S) < 1$.

It is also known that $\delta(S)$ coincides with the Hausdorff dimension of the conical limit set of the covering group of S (see, for example, [**27**, p. 154]). This tells us that the LII property must also be related to the size of the 'boundary' of S.

Although the characterization of LII for plane domains is a very difficult problem, there exists such a characterization of LII for Denjoy domains in [**2**]. We need some definitions in order to explain this result.

Definition 6.1. A subset I of a non-exceptional Riemann surface S is strongly uniformly separated in S if there exists a positive constant ρ such that the hyperbolic balls $B_S(p, \rho)$, where $p \in I$, are simply connected and pairwise disjoint.

Definition 6.2. Given a Denjoy domain Ω we denote by $I = I(\Omega)$ the isolated points of $\partial\Omega$, and we define $\Omega_0 := \Omega \cup I$. Then Ω_0 is also a Denjoy domain and $\Omega = \Omega_0 \setminus I$.

Definition 6.3. We say that a finite subset $A = \{\alpha_1, \dots, \alpha_{2n}\}$, $n \geqslant 2$, of points of $\partial\Omega \cup \{\infty\}$ is a *border set* of $\partial\Omega$ if A verifies the following two conditions.

(i) A is 'ordered' in $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, i.e. there exists $j \in \mathbb{Z}_{2n} = \mathbb{Z}/(2n\mathbb{Z})$ such that $\alpha_{j+1} < \dots < \alpha_{j+2n}$, where the subscripts belong to $\mathbb{Z}_{2n}$.

(ii) The set $\bigcup_{k=1}^{n} (\alpha_{2k-1}, \alpha_{2k})$ is contained in Ω.

Obviously, every subset $A = \{\alpha_1, \dots, \alpha_{2n}\}$ of $\bar{\mathbb{R}}$ can be 'ordered' in such a way that condition (i) is satisfied. So (ii) is the significant condition in the definition above.

Example 6.4. Let us consider the Denjoy domain $\Omega := \mathbb{C} \setminus \bigcup_{n=1}^{\infty} [2n-1, 2n]$. It is clear that the ordered sets $\{2, 3, 6, 7, 10, 11\}$ and $\{4, 5, \infty, 1\}$ are border sets of $\partial\Omega$, but $\{1, 4, 5, \infty\}$ is not.

Definition 6.5. Given a border set of $\partial\Omega$ with four points, $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, we denote by $\gamma(A)$ the unique simple closed geodesic in Ω which separates $[\alpha_2, \alpha_3]$ from $[\alpha_4, \alpha_1]$ ($\gamma(A)$ meets $\mathbb{R}$ only in (α_1, α_2) and (α_3, α_4)).

The characterization of the LII in [**2**] is as follows.

Theorem 6.6 (Alvarez *et al.* [2, Theorems 4 and 5]). *Let Ω be a Denjoy domain. Then, Ω satisfies the LII if and only if I is strongly uniformly separated in Ω_0 and there exists a positive constant c such that, for any border set of $\partial\Omega_0$, $A = \{\alpha_1, \dots, \alpha_{2n}\}$ with $n \geqslant 3$, we have that*

$$\frac{1}{n}\sum_{j=1}^{n} L_{\Omega_0}(\gamma(\{\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}\})) > c.$$

Remark 6.7. At the sight of this characterization of LII, it is clear that we just need to estimate the lengths of simple closed geodesics up to multiplicative constants; however, an additive constant in the estimate would be a 'large error'. For this reason, we need A-lines and B-lines to be chord–arc instead of (a, b)-quasi-geodesics (with $b > 0$).

Furthermore, [**2**, Theorem 4] provides an estimate of

$$L_{\Omega_0}(\gamma(\{\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}\})).$$

Unfortunately, this estimate involves a different Möbius map $U = U_{\{\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}\}}$ for each border set, the expression of which is not nice [**2**, p. 378], and there is no explicit expression for the constants in the estimates. In addition, there are no criteria that guarantee that the set I is strongly uniformly separated; rather than having a topological condition like '$B_{\Omega_0}(x, \rho)$ is simply connected', we would prefer to have a metric condition (especially having good results at our disposal which allow us to estimate the metric easily).

Using the results of this paper we obtain an improvement of Theorem 6.6, which removes the inconveniences of the results in [**2**, Theorem 4]. We have a direct estimate of

$$L_{\Omega_0}(\gamma(\{\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}\}))$$

(without any Möbius map), by Theorems 5.3 and 5.7.

Let us define first a function D_Ω, if $\Omega \cap \mathbb{R} = \bigcup_n (a_n, b_n)$, as follows.

If $a, b \in \Omega \cap \mathbb{R}$, we define $D_\Omega(a, b)$ as the function comparable to $d_\Omega(a, b)$ appearing in Theorem 5.2, i.e.

$$D_\Omega(a, b) := L_\Omega([a, a + \tfrac{1}{2}\mathrm{i}|b - a|] \cup [b, b + \tfrac{1}{2}\mathrm{i}|b - a|]).$$

If $a \in \Omega \cap \mathbb{R}$, we define $D_\Omega(a, (a_m, b_m))$ as the function comparable to $d_\Omega(a, (a_m, b_m))$ appearing in Theorem 5.5:

$$D_\Omega(a, (a_m, b_m)) := L_\Omega([x_m, x_m + \tfrac{1}{2}\mathrm{i}|a - x_m|]) + L_\Omega([a, a + \tfrac{1}{2}\mathrm{i}|a - x_m|])$$

if $b_m - a_m \leqslant 2d_{\mathrm{Eucl}}(a, (a_m, b_m))$, and

$$D_\Omega(a, (a_m, b_m)) := L_\Omega([a, a + \mathrm{i}(a - b_m)])$$

if $b_m - a_m > 2d_{\mathrm{Eucl}}(a, (a_m, b_m))$.

We define $D_\Omega((a_m,b_m),(a_n,b_n))$ as the function comparable to $d_\Omega((a_m,b_m),(a_n,b_n))$ appearing in Theorem 5.3:

$$D_\Omega((a_m,b_m),(a_n,b_n)) := L_\Omega([x_m, x_m + \tfrac{1}{2}\mathrm{i}|x_n - x_m|]) + L_\Omega([x_n, x_n + \tfrac{1}{2}\mathrm{i}|x_n - x_m|])$$

if $b_m - a_m \leqslant a_n - b_m$ and $b_n - a_n \leqslant a_n - b_m$;

$$D_\Omega((a_m,b_m),(a_n,b_n)) := L_\Omega([x_n, x_n + \mathrm{i}(x_n - b_m)])$$

if $b_n - a_n \leqslant b_m - a_m$, $a_n - b_m \leqslant b_m - a_m$ and $r(a_m,b_m,a_n,b_n) < 2$; and

$$D_\Omega((a_m,b_m),(a_n,b_n)) := 1/\log r(a_m,b_m,a_n,b_n)$$

if $r(a_m,b_m,a_n,b_n) \geqslant 2$.

If $a \in (a_m,b_m)$, we also define $D_\Omega(a)$ as $D_\Omega(a) := \inf_{n\neq m} D_\Omega(a,(a_n,b_n))$.

Therefore, D_Ω can easily be estimated by Theorem 5.7.

Now we can state our characterization of LII.

Theorem 6.8. *Let Ω be a Denjoy domain. Then, Ω satisfies the LII if and only if there exists a positive constant c such that*

(i) *for any border set of $\partial\Omega_0$, $A = \{\alpha_1,\dots,\alpha_{2n}\}$ with $n \geqslant 3$, we have that*

$$\frac{1}{n}\sum_{j=1}^{n} D_{\Omega_0}((\alpha_{2j-1},\alpha_{2j}),(\alpha_{2j+1},\alpha_{2j+2})) > c\,;$$

(ii) *$D_{\Omega_0}(x_1,x_2) > c$ for any $x_1, x_2 \in I$;*

(iii) *$D_{\Omega_0}(x) > c$ for any $x \in I$.*

Proof. Theorems 5.2, 5.3 and 5.5 allow us to use the simple function D_Ω instead of d_Ω.

By Theorem 6.6, it is sufficient to show that the condition '$B_\Omega(x,\rho)$ is simply connected for every $x \in I$' is equivalent to (iii). This equivalence is a consequence of the following two facts:

$$\begin{aligned}&\sup\{t > 0\colon B_{\Omega_0}(x,t) \text{ is simply connected}\}\\ &\qquad\qquad = \tfrac{1}{2}\min\{L_{\Omega_0}(\gamma) : \gamma \text{ is a geodesic loop with base point } x\},\end{aligned}$$

and a geodesic loop in Ω_0 is not homotopically trivial in Ω_0.

We prove only the second fact since the first one is well known. Let us consider a geodesic loop γ with base point x, a universal covering map $\pi\colon \mathbb{D} \to \Omega_0$, and the lift $\tilde\gamma$ of γ starting in $\tilde x \in \mathbb{D}$. If γ is homotopically trivial in Ω_0, then $\tilde\gamma$ finishes in $\tilde x$ too, i.e. $\tilde\gamma$ is a geodesic loop in $\mathbb{D}$, which is a contradiction since there are no geodesic loops in $\mathbb{D}$. $\square$

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