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RANDOM FOURIER SERIES ON COMPACT NONCOMMUTATIVE GROUPS

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1. Let G be a compact group, let I be a subset of its dual object Γ, which, without loss of generality, will be assumed to be a countable subset. Let D_i, $i \in I$, be irreducible representations of G of degree d_i. The Fourier series of a function f in $L^1(G)$ is defined by

$$f(x) \sim \sum_{\gamma \in \Gamma} d_\gamma \operatorname{tr}(A_\gamma D_\gamma(x))$$

where

$$A_\gamma = \int_G f(x) D_\gamma(x^{-1}) dx.$$

The following definition was given in [**7**, Section 37] and [**9**].

Definition. $I \subseteq \Gamma$ is called a Sidon set if every continuous function

$$f(x) \backsim \sum_{n \in I} d_n \operatorname{tr}(A_n D_n(x))$$

satisfies:

$$\sum_{n \in I} d_n \operatorname{tr}(|A_n|) < \infty.$$

Analogously, I is said to be a $\Lambda(p)$ set, $p > 1$, if every function in $L^1(G)$ whose spectral support is contained in I belongs to $L^p(G)$. I is said to be local Sidon (respectively, local $\Lambda(p)$) if

$$\operatorname{tr}(|A_n|) \leqq K||\operatorname{tr}(A_n D_n(x))||_\infty$$
$$(\text{respectively, } ||\operatorname{tr}(A_n D_n(x))||_p \leqq K'||\operatorname{tr}(A_n D_n(x))||_1)$$

for each $n \in I$ and for some positive K (respectively, K') depending on I but not on n.

Replacing, in the previous definitions, each function algebra by its center, the corresponding definitions of central lacunary sets are obtained.

We shall use the fact that a Sidon set I is a $\Lambda(p)$ set for each $p < \infty$, and for each $f \in L^2(G)$ the inequality

$$||f||_p \leqq B(p)||f||_2$$

holds, where $B(p)$ is a positive number depending only on p and I.

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Let S_i be a closed subgroup of the unitary group $U(d_i)$ and $\mathfrak{G} = \prod_{i\in I} S_i$. Let J_i be the projection of $\mathfrak{G}$ onto its ith factor. The formal series

$$(*) \quad f_V(x) \sim \sum_{i\in I} d_i \operatorname{tr}(A_i V_i D_i(x)),$$

where $\{V_i\} = V \in \mathfrak{G}$, may be considered as a random Fourier series in the independent random variables J_i defined on $\mathfrak{G}$, regarded as a probability space with its Haar measure.

Our purpose is to investigate which conditions on the symmetry groups S_i and the representations D_i are necessary or sufficient in order that f_V be almost surely in $L^p(G)$ for all $p < \infty$ whenever it is in $L^1(G)$ with positive probability.

It has been proved in [**5**; **6**], that the assumption that the symmetry groups be large enough, for instance $S_i = U(d_i)$, is a sufficient condition, without any requirement about the representations D_i. In [**2**] it has been proved that even the weaker condition of random symmetry with respect to the orthogonal group $O(d_i)$ is sufficient, no matter what the D_i are like. It is reasonable, however, to expect that the weaker random condition is assumed, the stronger spectral condition will be needed. For instance, (*) will represent a function in $L^p(G)$, without any random condition (i.e., when S_i is trivial for all $i \in I$) if and only if I is $\Lambda(p)$. We shall investigate the random symmetries for which the assumption that the spectral support I is a local $\Lambda(p)$ set for all $p < \infty$ will be a necessary or sufficient condition. In particular we'll prove that, if $S_i = D_i(G)$, or $S_i = \mathbf{Z}_2 \times D_i(G)$ (or, more generally, $S_i = H_i \times D_i(G)$, where the H_i's are any sequence of subgroups of the one-dimensional torus $\mathbf{T}$), then *every random series* (*) *which is the Fourier series of a function in* $L^1(G)$ *with positive probability is almost surely the Fourier series of a function in every* $L^p(G)$ $(p < \infty)$ *if and only if* I *is a local* $\Lambda(p)$ *set for each* $p < \infty$.

Under this respect, the behaviour of these random series is particularly interesting because it contrasts with the property of the same series expressed by Billard's theorem: if (*) is the Fourier series of a function in $L^\infty(G)$ with positive probability (with respect to the group $\prod_{i\in I} \mathbf{Z}_2 \times D_i(G)$) then it is almost surely the Fourier series of a continuous function, without any requirement on the spectral support I ([**3**, Theorem 1]). Let's notice, however, that this result is false, in general, when the symmetry group is chosen to be $\prod_{i\in I} D_i(G)$, at least if I is not local $\Lambda(p)$ for some p ([**3**, Section 3]; see also Remark 3 below).

We remark that, by an easy application of Fubini's theorem, our results can be extended to any random Fourier series

$$\sum_{i\in I} d_i \operatorname{tr}(A_i X_i D_i(x)),$$

where the X_i's are a sequence of independent random variables symmetric with respect to $\mathfrak{G} = \prod_{i\in I} S_i$, and with values on the algebra of the d_i-dimensional square matrices.

Furthermore, we shall obtain similar results in the case of much weaker random conditions (for instance, with respect to the Cantor group).

In Section 3 we shall investigate the existence of local $\Lambda(p)$ sets for compact groups, and give partial results which are consistent with the following conjecture: Γ *is local* $\Lambda(p)$ *for some* $p > 1$ *if and only if it is bounded dimensional,* i.e., $\sup \{d_\gamma, \gamma \in \Gamma\} < \infty$. By a result of C. C. Moore [**8**], this would amount to identifying the groups whose dual objects are local $\Lambda(p)$ sets with the finite extensions of abelian groups.

The results of this paper partially appeared in our thesis of "laurea" at the University of Genova.

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2. Let S_i be an irreducible subgroup of $U(d_i)$. The set $\mathfrak{J} = \{J_i, i \in I\}$ of all projections of $\mathfrak{G}$ onto its factors is then a subset of the dual object of $\mathfrak{G}$. The following lemma can be proved in the same way as Lemma 1 of [**6**], provided one observes that, since $\mathfrak{J}$ consists of independent representations, it is local $\Lambda(4)$ if and only if it is a $\Lambda(4)$ set (or one can apply Theorem 3 of [**9**], since $\mathfrak{J}$ is central Sidon).

LEMMA 1. *Assume* $\mathfrak{J}$ *is a local* $\Lambda(4)$ *set for* $\mathfrak{G}$. *Then, for each measurable subset* M *of* $\mathfrak{G}$ *of positive Haar measure, and for each* $\epsilon > 0$, *there exists a finite set* $\mathfrak{J}_0 \subseteq \mathfrak{J}$, *which only depends on* M *and* ϵ, *such that, if* $F \in L^2(G)$, *and*

$$F(V) = \sum_{i \in \mathfrak{J} - \mathfrak{J}_0} d_i \operatorname{tr} (A_i J_i(V)) = \sum_{i \in \mathfrak{J} - \mathfrak{J}_0} d_i \operatorname{tr} (A_i V_i)$$

$(\{V_i\} = V \in \mathfrak{G})$, *the following inequality holds:*

$$(1 - \epsilon) m(M) \sum_{i \in \mathfrak{J} - \mathfrak{J}_0} d_i \operatorname{tr} (A_i A_i^*) \leqq \int_M |F(V)|^2 dV.$$

On the basis of this lemma, by the same argument of Lemma II.4.2 of [**4**], or as in [**5**, Theorem 4; **6**, Theorem 3], one can prove the following result:

THEOREM 1. *Suppose that the projection set* $\mathfrak{J}$ *is a local* $\Lambda(p)$ *set for* $\mathfrak{G}$, $p \geqq 4$, *and that there exists a subset* M *of* $\mathfrak{G}$, *of positive Haar measure, such that*

$$\sum_{j \in I} d_j \operatorname{tr} (A_j V_j D_j(x))$$

represents a function $f_V \in L^1(G)$ *for each* $\{V_i\} = V \in M$. *Then* f_V *belongs to* $L^p(G)$ *for almost all* $V \in \mathfrak{G}$.

COROLLARY 1. *Let* $\mathfrak{G} = \prod_{i \in I} D_i(G)$ (*or also, more generally,* $\mathfrak{G} = \prod_{i \in I} (H_i \times D_i(G))$, H_i *any closed subgroup of* $\mathbf{T}$). *Suppose there exists a subset* M *of* $\mathfrak{G}$, *of positive measure, such that*

$$\sum_{j \in I} d_j \operatorname{tr} (A_j V_j D_j(x))$$

represents a function $f_V \in L^1(G)$ *for each* $V = \{V_i\} \in M$. *If* I *is a local* $\Lambda(p)$ *set for* $\mathfrak{G}$ $(p \geqq 4)$, *then* f_V *belongs to* $L^p(G)$ *for almost all* $V \in \mathfrak{G}$.

Proof. It is enough to observe that $\mathfrak{J}$ is a local $\Lambda(p)$ set, since

$$||\text{tr}\,(A_n S_n(x))||_{L^p(G)} = ||\text{tr}\,(A_n J_n(V))||_{L^p(\mathfrak{G})},$$

and apply Theorem 1.

We do not know whether Lemma 1 and Theorem 1 hold with a weaker assumption than $p \geqq 4$. We also notice that, if I_i, $\mathfrak{G}_j$, $\mathfrak{J}_j$ $(j = 1, 2)$ satisfy the hypothesis of Lemma 1, and I_1, I_2 are disjoint sets, then the random symmetry condition with respect to the group $\mathfrak{G}_1 \times \mathfrak{G}_2$ is sufficient for Theorem 1 to hold, since $\mathfrak{J}_1 \cup \mathfrak{J}_2$ is local $\Lambda(p)$. Similarly, we can extend the random condition in the following way: suppose $\mathfrak{J}$ is a local $\Lambda(p)$ set for $\mathfrak{G}$, for some $p \geqq 4$, and define

$$\mathfrak{G}^* = \mathfrak{G} \times \prod_{\Gamma - I} U(d_\gamma),$$

i.e., $\mathfrak{G}^* = \prod_{\gamma\in\Gamma} R_\gamma$, where $R_\gamma = S_\gamma$ if $\gamma \in I$, $R_\gamma = U(d_\gamma)$ if $\gamma \in \Gamma - I$. Then the projection set $\mathfrak{J}^*$ of $\mathfrak{G}^*$ is local $\Lambda(p)$, and Theorem 1 holds with $\mathfrak{G}$ and $\mathfrak{J}$ replaced by $\mathfrak{G}^*$ and $\mathfrak{J}^*$. Similarly, Corollary 1 holds with $\mathfrak{G} = \prod_{i\in I} (H_i \times D_i(G))$ replaced by $\mathfrak{G}^* = \prod_{\gamma\in\Gamma} R_\gamma$, where $R_\gamma = U(d_\gamma)$ for $\gamma \in \Gamma - I$, $R_j = H_j \times D_j$ for $j \in I$.

Let's now drop the assumption of irreducibility of the groups S_i, in order to test if even weaker random conditions yield results similar to that of Corollary 1. For the sake of simplicity, we shall restrict to the case $\mathfrak{G} = \prod_{i\in I} H_i$, where I is a local $\Lambda(2s)$ set for G, and, for each $i \in I$, H_i is a compact subgroup of $\mathbf{T}$; if I is infinite and $H_i = \mathbf{Z}_2$ for all $i \in I$ then $\mathfrak{G}$ is the Cantor group, and the random variables J_i may be regarded as Rademacher functions on $\mathbf{T}$.

THEOREM 2. *Let* s *be a positive integer larger than* 1, I *a local* $\Lambda(2s)$ *set for* G, H_i *a closed subgroup of* $\mathbf{T}$ *for each* $i \in I$ *and* $\mathfrak{G} = \prod_{i\in I} H_i$. *If*

$$f(x, \epsilon) \sim \sum_{n\in I} d_n \epsilon_n \,\text{tr}\,(A_n D_n(x))$$

(*where* $\epsilon = \{\epsilon_n\} \in \mathfrak{G}$) *is a function in* $L^1(G)$ *for all* ϵ *in a subset* M *of* $\mathfrak{G}$ *of positive measure, then* $f(x, \epsilon)$ *belongs to* $L^{2s}(G)$ *for almost all* ϵ *in* $\mathfrak{G}$.

Proof. By the zero-one law, we can assume that M has measure one. By the same argument as in Lemma II.4.2 of [**4**] or Theorem 3 of [**6**], for each $\delta > 0$ one can find a subset P of $G \times \mathfrak{G}$ such that $m(P) \geqq 1 - \delta^2$ and

$$\sup_P |f(x, \epsilon)| = C_\delta < \infty.$$

Now, if $P_x = \{\epsilon\colon (x, \epsilon) \in P\}$ and $E = \{x \in G\colon m(P_x) > 1 - \delta\}$, it follows:

$$1 - \delta^2 \leqq \int_G m(P_x)dx \leqq \int_E dx + \int_{E^c} (1 - \delta)dx = 1 - \delta m(E^c),$$

hence $m(E) \geqq 1 - \delta$. Since the projection set $\mathfrak{J}$ is a Sidon set for $\mathfrak{G}$, for every

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square-summable sequence $\{a_n\}_{n\in I}$ the following inequalities hold:

$$\int_{P_x^c} \left|\sum_{n\in I} \epsilon_n a_n\right|^2 d\epsilon \leqq \left(\int_{\mathfrak{G}} \left|\sum_{I} \epsilon_n a_n\right|^4 d\epsilon\right)^{1/2} \cdot \left(\int_{P_x^c} d\epsilon\right)^{1/2}$$
$$\leqq B(4)^2 (m(P_x^c))^{1/2} \sum_I |a_n|^2.$$

Therefore, if $x \in E$ and $\delta < 1/4B(4)^2$,

$$\int_{P_x} \left|\sum_{n\in I} \epsilon_n a_n\right|^2 d\epsilon = \int_{\mathfrak{G}} \left|\sum_{n\in I} \epsilon_n a_n\right|^2 d\epsilon - \int_{P_x^c} \left|\sum_{n\in I} \epsilon_n a_n\right|^2 d\epsilon$$
$$\geqq 1/2 \sum_{n\in I} |a_n|^2$$

Hence, for $x \in E$:

$$(**) \quad 1/2 \sum_I d_n^2 |\operatorname{tr}(A_n D_n(x))|^2 \leqq \int_{P_x} |f(x,\epsilon)|^2 d\epsilon \leqq C_\delta^2 m(P_x) \leqq C_\delta^2.$$

Similarly,

$$\int_{E^c} |\operatorname{tr}(A_n D_n(x))|^2 dx \leqq \left(\int_G |\operatorname{tr}(A_n D_n(x))|^{2s} dx\right)^{1/s} \cdot \left(\int_{E^c} dx\right)^{(s-1)/s}$$
$$\leqq k^2 \delta^{(s-1)/s} \int_G |\operatorname{tr}(A_n D_n(x))|^2 dx,$$

where k is the $\Lambda(2s)$ constant of I. Therefore, if δ is small enough:

$$\int_E |\operatorname{tr}(A_n D_n(x))|^2 dx = \int_G |\operatorname{tr}(A_n D_n(x))|^2 dx - \int_{E^c} |\operatorname{tr}(A_n D_n(x))|^2 dx$$
$$\geqq 1/2 \int_G |\operatorname{tr}(A_n D_n(x))|^2 dx.$$

Now, by (**):

$$1/4 \sum_I d_n^2 \int_G |\operatorname{tr}(A_n D_n(x))|^2 dx \leqq C_\delta^2 m(E) \leqq C_\delta^2,$$

so that

$$||f(x,\epsilon)||_{L^2(G)} \leqq 2C_\delta < \infty.$$

Let's now prove that $f(x, \epsilon) \in L^2(G)$ implies $f(x,\epsilon) \in L^{2s}(G)$ for almost all ϵ in $\mathfrak{G}$:

$$\int_{\mathfrak{G}} \int_G |f(x,\epsilon)|^{2s} dx d\epsilon \leqq (B(2s))^{2s} \int_G \left(\int_{\mathfrak{G}} |f(x,\epsilon)|^2 d\epsilon\right)^s dx$$
$$= (B(2s))^{2s} \int_G \left(\sum_{n\in I} d_n^2 |\operatorname{tr}(A_n D_n(x))|^2\right)^s dx$$
$$\leqq (B(2s))^{2s} \sum_{n_1 \ldots n_s \in I} d_{n_1}^2 \ldots d_{n_s}^2 ||\operatorname{tr}(A_{n_1} D_{n_1}(x))||_{2s}^2 \ldots$$
$$\times ||\operatorname{tr}(A_{n_s} D_{n_s}(x))||_{2s}^2$$
$$\leqq k^{2s} (B(2s))^{2s} \sum_{n_1 \ldots n_s \in I} d_{n_1} \ldots d_{n_s} \operatorname{tr}(A_{n_1} A_{n_1}^*) \ldots \operatorname{tr}(A_{n_s} A_{n_s}^*)$$
$$= k^{2s} (B(2s))^{2s} ||f(x,\epsilon)||_2^{2s} < \infty.$$

Hence $f(x, \epsilon) \in L^{2s}(G)$ for almost all ϵ in $\mathfrak{G}$.

Remark 1. In the proof of Theorem 2 we did not need the inequality of Lemma 1. We ignore if such inequality still holds.

Let us now investigate the random properties of Fourier series whose spectral support is not a local $\Lambda(p)$ set.

THEOREM 3. *Let $I \subseteq \Gamma$ be not a local $\Lambda(p)$ set for some p, $1 < p < \infty$, let $\{H_i, i \in I\}$ be a sequence of closed subgroups of* $\mathbf{T}$ *and $\mathfrak{G} = \prod_{i \in I} H_i$ or $\mathfrak{G} = \prod_{i \in I}(H_i \times D_i(G))$. Then there exists a random Fourier series*

$$\sum_{i \in I} d_i \operatorname{tr}(A_i V_i D_i(x))$$

($\{V_i\} = V \in \mathfrak{G}$), *which represents for all $V \in \mathfrak{G}$ a function in $L^1(G)$, and for no $V \in \mathfrak{G}$ a function in $L^p(G)$.*

Proof. Let $f_i(x) = d_i \operatorname{tr}(A_i D_i(x))$, $i \in I$, be a sequence of functions such that

$$\lim_i ||f_i||_p / ||f_i||_1 = \infty.$$

Passing to a suitably normalized subsequence, one can assume:

$$||f_i||_1 \leqq 2^{-i}$$
$$||f_i||_p \geqq 3||f_{i-1}||_p.$$

Let E be the index set of such a subsequence, and denote by $f_i(x, V)$ the function $d_i \operatorname{tr}(A_i V_i D_i(x))$. Since $||f_i(x, V)||_q = ||f_i(x)||_q$ for all $q \geqq 1$, the series

$$f(x, V) = \sum_{i \in E} f_i(x, V)$$

surely converges in $L^1(G)$; on the other hand, for each $V \in \mathfrak{G}$:

$$\left|\left| \sum_{i=1}^{r} f_i(x, V) \right|\right|_p \geqq ||f_r(x)||_p - \sum_{i=1}^{r-1} ||f_i(x)||_p \geqq 1/2||f_r(x)||_p \underset{r}{\rightarrow} \infty.$$

so that the event

$$\left\{ \max_r \lim \left|\left| \sum_{i=1}^{r} f_i(x, V) \right|\right|_p < \infty \right\}$$

is surely not verified. This proves the theorem.

Remark 2. More generally, the proof of Theorem 3 holds if $\mathfrak{G}$ is replaced by $\mathfrak{G}' = \prod_{i \in I} S_i$, where S_i is any closed subgroup of $U(d_i)$ with the property that there exists a subset M of positive measure in $\mathfrak{G}'$ and for every $V \in M$ and for every sequence of d_i-dimensional matrices A_i there exist two positive numbers C_V, C_V' depending on V but not on i, such that

$$||\operatorname{tr}(A_i V_i D_i(x))||_1 \leqq C_V ||\operatorname{tr}(A_i D_i(x))||_1,$$
$$||\operatorname{tr}(A_i V_i D_i(x))||_p \geqq C_V' ||\operatorname{tr}(A_i D_i(x))||_p.$$

With respect to $\mathfrak{G}'$, there exists a random Fourier series which represents almost surely a function in $L^1(G)$ but not in $L^p(G)$. The first explicit example of a random Fourier series (with respect to the group $\mathfrak{G} = \prod_{i\in I}(\mathbf{Z}_2 \times D_i(G))$) which is in L^2 but not in L^4 was produced in [**4**]. Our result shows that this phenomenon is quite general.

3. In this section we consider the problem of the classification of the compact groups G whose dual objects Γ do not contain any infinite subset which is a local $\Lambda(p)$ set for every $p < \infty$.

The following result is readily proved:

PROPOSITION 1. *If I is a subset of Γ such that* $\sup\{d_\gamma, \gamma \in I\} < \infty$, *then I is a local Sidon set, hence a local $\Lambda(p)$ set for every $p < \infty$.*

Hence, one must require that G is not a finite extension of an abelian group [**8**]. On the other hand, Theorem 3 of [**1**] shows that no local $\Lambda(4)$ subset I of the dual object of a compact Lie group can satisfy the condition $\sup\{d_\gamma, \gamma \in I\} = \infty$. One notices that, if G is not totally disconnected, its dual object contains the dual object of a compact Lie group, and therefore it contains subsets which are not local $\Lambda(4)$. The following theorem shows that, at least for a wide class of totally disconnected groups, it is not difficult to find subsets of Γ which are not local $\Lambda(4)$.

PROPOSITION 2. *Let G be a direct product of noncommutative compact groups G_α. Then there exists a subset I of Γ which is not local central $\Lambda(4)$.*

Proof. Let $\{G_i, i \in I\}$ be a sequence of distinct elements in $\{G_\alpha\}$, and denote by σ_i the character of an irreducible representation D_i of G_i of dimension larger than one. Then $\|\sigma_i\|_4 \geqq 2^{1/4}$. Define

$$\chi_n = \sigma_1 \ldots \sigma_n.$$

Since χ_n is the character of the irreducible representation $D_1 \otimes \ldots \otimes D_n$, it follows $\|\chi_n\|_2 = 1$, while, by independence, $\|\chi_n\|_4 \geqq 2^{n/4}$.

Remark 3. By exhibiting a large class of groups whose dual objects contain subsets which are not local $\Lambda(p)$ sets for some $p > 1$, we have shown, on the basis of the results of Section 2, that the random Fourier series considered in Corollary 1 and Theorem 3 are in fact weaker, with respect to the above considered random property, than the unitary random Fourier series, i.e., the random Fourier series whose symmetry groups are full unitary groups. As remarked in the introduction, the random Fourier series with respect to the symmetry group $\mathfrak{G} = \prod_{i\in I}(\mathbf{Z}_2 \times D_i(G))$ (called Steinhaus random Fourier series in the terminology of [**3**]) have the following property: if $\sum_{n\in I} d_n\epsilon_n \operatorname{tr}(A_nD_n(y_nx))$, $(\{\epsilon_nD_n(y_nx)\} \in \mathfrak{G})$, represents almost surely a function in $L^\infty(G)$, it represents almost surely a continuous function [**3**, Theorem 1]. We observe that this property is also shared by unitary random Fourier series, because,

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if $\sum_{n \in I} d_n$ tr $(A_n U_n D_n(x))$, $(U_n \in U(d_n))$ represents almost surely a function in $L^\infty(G)$, then the random Fourier series $\sum_{n \in I} d_n \epsilon_n$ tr $(A_n U_n D_n(y_n x))$ represents a bounded measurable function almost surely with respect to $\mathfrak{G} \times \prod_{n \in I} U(d_n)$; now Fubini's theorem and Theorem 1 of [**3**] yield the result. This observation gives a positive answer to a question raised in [**4**, Chapter 2, Section 5].

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