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On the Maximum Curvature of Closed Curves in Negatively Curved Manifolds

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Abstract. Motivated by Almgren's work on the isoperimetric inequality, we prove a sharp inequality relating the length and maximum curvature of a closed curve in a complete, simply connected manifold of sectional curvature at most −1. Moreover, if equality holds, then the norm of the geodesic curvature is constant and the torsion vanishes. The proof involves an application of the maximum principle to a function defined on pairs of points.

1 Introduction

One of the most important results in geometric analysis is the classical isoperimetric inequality in Euclidean space. There are a number of interesting generalizations of this classical inequality. For example, in 1980, Gromov [9] proved a sharp isoperimetric inequality for manifolds with Ricci curvature bounded below by a positive constant. In 1992, Kleiner [12] obtained a sharp isoperimetric inequality for complete, simply connected three-manifolds with sectional curvature $K \leq -1$. To state this result, suppose that Ω is a domain in a complete, simply connected three-manifold with sectional curvature at most -1, and B is a geodesic ball in hyperbolic space $\mathbb{H}^n$ such that $\mathrm{vol}(\Omega) = \mathrm{vol}(B)$. Then $\mathrm{area}(\partial\Omega) \geq \mathrm{area}(\partial B)$ (*cf.* [12, Theorem 2]). We note that Schulze [14] later found an alternative proof using geometric flow techniques. Moreover, Croke [6] obtained a similar result in dimension 4. Finally, in dimension 3, Bray [2] established a sharp volume comparison theorem involving scalar curvature. In addition, Bray analyzed the isoperimetric profile of certain asymptotically flat manifolds arising in general relativity (see also [7, 8]).

In a different direction, Almgren [1] proved an isoperimetric inequality for least area surfaces in Euclidean space. More precisely, suppose that V is a closed m-dimensional surface in $\mathbb{R}^n$ such that $\mathrm{vol}_m(V) = \mathrm{vol}_m(\partial B^{m+1})$, where B^{m+1} denotes the unit ball in $\mathbb{R}^{m+1}$. Then there exists an $(m+1)$-dimensional surface Σ in $\mathbb{R}^n$ with boundary V such that $\mathrm{vol}_{m+1}(\Sigma) \leq \mathrm{vol}_{m+1}(B^{m+1})$. Almgren's inequality is closely related to the Sobolev inequality of Michael and Simon [13], and to Gromov's filling radius inequality (*cf.* [10]).

The isoperimetric inequalities in [1] and [12] both rely on a sharp lower bound for the supremum of the mean curvature.

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Theorem 1.1 (F. J. Almgren, Jr. [1]) *Suppose that V is a closed m-dimensional surface in $\mathbb{R}^n$ such that $\operatorname{vol}_m(V) = \operatorname{vol}_m(\partial B^{m+1})$, where B^{m+1} denotes the unit ball in $\mathbb{R}^{m+1}$. Then $\sup|H| \geq m$, where H denotes the mean curvature vector of V.*

Theorem 1.2 (B. Kleiner [12]) *Let V be a closed surface in a complete, simply connected three-manifold with sectional curvature at most -1, and let B be a geodesic ball in hyperbolic space $\mathbb{H}^n$ such that $\operatorname{area}(V) = \operatorname{area}(\partial B)$. Then the supremum of the mean curvature of V is bounded from below by the mean curvature of ∂B.*

We note that Simon [15] has obtained a remarkable estimate for the diameter of a surface in terms of the L^2-norm of its mean curvature.

In this paper, we prove a sharp estimate for the maximum curvature of a closed curve in a complete, simply connected manifold of sectional curvature at most -1.

Theorem 1.3 *Let M be a complete, simply connected manifold of dimension n with sectional curvature $K \leq -1$. Then any closed curve in M satisfies the inequality*

$$L^2 \sup(|\kappa|^2 - 1) \geq 4\pi^2,$$

where L denotes the length of the curve and κ denotes its geodesic curvature. Finally, if equality holds, then the geodesic curvature κ is a parallel section of the normal bundle, and the two-plane spanned by the tangent vector τ and the geodesic curvature κ has sectional curvature equal to -1.

In particular, if M is the n-dimensional hyperbolic space, then equality holds if and only if the curve is a circle.

The proof of Theorem 1.3 is quite different from Theorems 1.1 and 1.2. While the latter results rely on tube formulae and the Gauss–Bonnet theorem, the proof of Theorem 1.3 involves an application of the maximum principle to a suitable two-point function. This method goes back to the work of Huisken [11] on the curve shortening flow and plays a key role in the proof of the Lawson conjecture (*cf.* [3]). We refer to [4] and [5] for some recent surveys on this topic.

We now sketch the main idea of the proof. Let $F: \mathbb{R}/(L\mathbb{Z}) \to M$ be a closed curve of length L whose geodesic curvature satisfies $\sup|\kappa| \leq \sqrt{1 + 4\pi^2/L^2}$. In Section 2, we use the maximum principle to show that

$$2 \sinh \frac{\rho(F(x), F(y))}{2} \leq \frac{L}{\pi} \sin \frac{\pi d(x, y)}{L},$$

where ρ denotes the Riemannian distance in M and $d(x, y)$ denotes the distance of two points $x, y \in \mathbb{R}/(L\mathbb{Z})$. This estimate is the opposite of the one established in [11]. While Huisken bounds the extrinsic distance of two points from below by their intrinsic distance, our estimate gives an upper bound for the extrinsic distance in terms of the intrinsic distance. In Section 3, we fix an arbitrary point y and consider the function

$$v(x) := 4 \sinh^2 \frac{\rho(F(x), F(y))}{2} - \frac{L^2}{\pi^2} \sin^2 \frac{\pi d(x, y)}{L} \leq 0.$$

A calculation shows that the derivatives of the function v at the point y vanish up to third order and the fourth derivative is nonnegative. Since $v \leq 0$, it follows that the fourth derivative of v at the point y must be equal to 0. This implies that $|\kappa| = \sqrt{1+4\pi^2/L^2}$ at each point on the curve. Having established that fact, we next show that the fifth derivative of v at the point y vanishes, and the sixth derivative is nonnegative. Again, this implies that the sixth derivative of v at the point y is equal to 0. From this we deduce that the torsion is equal to 0.

2 Application of the Two-Point Maximum Principle

Proposition 2.1 *Let M be a complete, simply connected manifold of dimension n with sectional curvature $K \le -1$. Let $\rho(p,q)$ denote the distance between two points $p,q \in M$, and let $F\colon \mathbb{R}/(L\mathbb{Z}) \to M$ be a closed curve of length L whose geodesic curvature satisfies* $\sup|\kappa| \le \sqrt{1+4\pi^2/L^2}$. *Then*

$$2\sinh\frac{\rho(F(x),F(y))}{2} \le \frac{L}{\pi}\sin\frac{\pi d(x,y)}{L},$$

where $d(x,y)$ denotes the distance of two points $x,y \in \mathbb{R}/(L\mathbb{Z})$.

Proof Suppose that the assertion is false. Then

$$\alpha := \inf_{x\neq y}\frac{L}{2\pi}\frac{\sin\frac{\pi d(x,y)}{L}}{\sinh\frac{\rho(F(x),F(y))}{2}} < 1.$$

We define

$$Z(x,y) := 2\alpha\sinh\frac{\rho(F(x),F(y))}{2} - \frac{L}{\pi}\sin\frac{\pi d(x,y)}{L}$$

for all points $x,y \in \mathbb{R}/(L\mathbb{Z})$. Clearly, $Z(x,y) \le 0$ for all points $x,y \in \mathbb{R}/(L\mathbb{Z})$. Moreover, since $\alpha < 1$, we can find a pair of points $\bar{x} \neq \bar{y}$ such that $Z(\bar{x},\bar{y}) = 0$. For abbreviation, we put $a := \frac{\rho(F(\bar{x}),F(\bar{y}))}{2}$. Moreover, let $\gamma\colon[-a,a] \to M$ be a unit-speed geodesic such that $\gamma(-a) = F(\bar{x})$ and $\gamma(a) = F(\bar{y})$.

In the sequel, we identify $\bar{x}$ and $\bar{y}$ with points in the interval $[0,L]$. Without loss of generality, we may assume that $0 \le \bar{x} \le \bar{y} \le \frac{1}{2}L$. Since the function $Z(x,y)$ attains its global maximum at the point $(\bar{x},\bar{y})$, we have

$$\begin{aligned} 0 &= \frac{d}{ds}Z(\bar{x}+s,\bar{y})\Big|_{s=0} = -\alpha\cosh a\langle\gamma'(-a),F'(\bar{x})\rangle + \cos\frac{\pi d(\bar{x},\bar{y})}{L}, \\ 0 &= \frac{d}{ds}Z(\bar{x},\bar{y}-s)\Big|_{s=0} = -\alpha\cosh a\langle\gamma'(a),F'(\bar{y})\rangle + \cos\frac{\pi d(\bar{x},\bar{y})}{L}. \end{aligned}$$

In particular, we can write

$$F'(\bar{x}) = \frac{\cos\frac{\pi d(\bar{x},\bar{y})}{L}}{\alpha\cosh a}\gamma'(-a) + \xi \quad\text{and}\quad F'(\bar{y}) = \frac{\cos\frac{\pi d(\bar{x},\bar{y})}{L}}{\alpha\cosh a}\gamma'(a) + \eta,$$

where $\langle\xi,\gamma'(-a)\rangle = \langle\eta,\gamma'(a)\rangle = 0$.

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We next compute

$$\begin{aligned}
0 &\geq \frac{d^2}{ds^2} Z(\bar{x}+s, \bar{y}-s)\Big|_{s=0} \\
&= -\alpha \cosh a \big\langle \gamma'(-a), \kappa(\bar{x}) \big\rangle + \alpha \cosh a \big\langle \gamma'(a), \kappa(\bar{y}) \big\rangle \\
&\quad + \frac{1}{2} \alpha \sinh a \Big(\langle \gamma'(-a), F'(\bar{x}) \rangle + \langle \gamma'(a), F'(\bar{y}) \rangle \Big)^2 \\
&\quad + \alpha \cosh a \, I_\gamma(V, V) + \frac{4\pi}{L} \sin \frac{\pi d(\bar{x}, \bar{y})}{L}.
\end{aligned}$$

Here, V is the unique Jacobi field along γ satisfying $V(-a) = \xi$ and $V(a) = \eta$. Moreover, I_γ denotes the index form; that is,

$$I_\gamma(V, V) = \int_{-a}^{a} \Big(|D_t V(t)|^2 - R\big(\gamma'(t), V(t), \gamma'(t), V(t)\big) \Big) \, dt.$$

Note that

$$|\xi| = |\eta| = \sqrt{1 - \frac{\cos^2 \frac{\pi d(\bar{x}, \bar{y})}{L}}{\alpha^2 \cosh^2 a}}.$$

Since M has sectional curvature at most -1, standard Jacobi field estimates imply that

$$|V(t)| \leq \frac{\cosh t}{\cosh a} \sqrt{1 - \frac{\cos^2 \frac{\pi d(\bar{x}, \bar{y})}{L}}{\alpha^2 \cosh^2 a}}$$

for all $t \in [-a, a]$. Moreover, equality holds for $t = -a$ and $t = a$. Consequently,

$$\begin{aligned}
-\frac{d}{dt} |V(t)|^2 \Big|_{t=-a} &\geq 2 \frac{\sinh a}{\cosh a} \Big(1 - \frac{\cos^2 \frac{\pi d(\bar{x}, \bar{y})}{L}}{\alpha^2 \cosh^2 a} \Big), \\
\frac{d}{dt} |V(t)|^2 \Big|_{t=a} &\geq 2 \frac{\sinh a}{\cosh a} \Big(1 - \frac{\cos^2 \frac{\pi d(\bar{x}, \bar{y})}{L}}{\alpha^2 \cosh^2 a} \Big).
\end{aligned}$$

This gives

$$I_\gamma(V, V) = \frac{1}{2} \frac{d}{dt} |V(t)|^2 \Big|_{t=a} - \frac{1}{2} \frac{d}{dt} |V(t)|^2 \Big|_{t=-a} \geq 2 \frac{\sinh a}{\cosh a} \Big(1 - \frac{\cos^2 \frac{\pi d(\bar{x}, \bar{y})}{L}}{\alpha^2 \cosh^2 a} \Big).$$

Moreover, using the inequality $\sup |\kappa| \leq \sqrt{1 + 4\pi^2/L^2}$, we obtain

$$\begin{aligned}
\langle \gamma'(-a), \kappa(\bar{x}) \rangle &\leq \sqrt{1 - \langle \gamma'(-a), F'(\bar{x}) \rangle^2} |\kappa(\bar{x})| \leq \sqrt{1 - \frac{\cos^2 \frac{\pi d(\bar{x}, \bar{y})}{L}}{\alpha^2 \cosh^2 a}} \sqrt{1 + \frac{4\pi^2}{L^2}}, \\
-\langle \gamma'(a), \kappa(\bar{y}) \rangle &\leq \sqrt{1 - \langle \gamma'(a), F'(\bar{y}) \rangle^2} |\kappa(\bar{y})| \leq \sqrt{1 - \frac{\cos^2 \frac{\pi d(\bar{x}, \bar{y})}{L}}{\alpha^2 \cosh^2 a}} \sqrt{1 + \frac{4\pi^2}{L^2}}.
\end{aligned}$$

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Putting these facts together, we obtain

$$
\begin{aligned}
0 &\geq \frac{d^2}{ds^2} Z(\overline{x}+s,\overline{y}-s)\Big|_{s=0} \\
&\geq -2\alpha\cosh a\sqrt{1-\frac{\cos^2\frac{\pi d(\overline{x},\overline{y})}{L}}{\alpha^2\cosh^2 a}}\sqrt{1+\frac{4\pi^2}{L^2}} \\
&\quad + 2\alpha\sinh a\frac{\cos^2\frac{\pi d(\overline{x},\overline{y})}{L}}{\alpha^2\cosh^2 a} + 2\alpha\sinh a\Big(1-\frac{\cos^2\frac{\pi d(\overline{x},\overline{y})}{L}}{\alpha^2\cosh^2 a}\Big) + \frac{4\pi}{L}\sin\frac{\pi d(\overline{x},\overline{y})}{L} \\
&= -2\sqrt{\alpha^2\cosh^2 a-\cos^2\frac{\pi d(\overline{x},\overline{y})}{L}}\sqrt{1+\frac{4\pi^2}{L^2}} \\
&\quad + 2\alpha\sinh a + \frac{4\pi}{L}\sin\frac{\pi d(\overline{x},\overline{y})}{L} \\
&= -2\sqrt{\alpha^2-1+\alpha^2\sinh^2 a+\sin^2\frac{\pi d(\overline{x},\overline{y})}{L}}\sqrt{1+\frac{4\pi^2}{L^2}} \\
&\quad + 2\alpha\sinh a + \frac{4\pi}{L}\sin\frac{\pi d(\overline{x},\overline{y})}{L} \\
&= -2\sqrt{\alpha^2-1+\alpha^2\Big(1+\frac{4\pi^2}{L^2}\Big)\sinh^2 a}\sqrt{1+\frac{4\pi^2}{L^2}} + 2\alpha\Big(1+\frac{4\pi^2}{L^2}\Big)\sinh a.
\end{aligned}
$$

This contradicts the assumption that $\alpha < 1$. ∎

3 Proof of Theorem 1.3

In this section, we describe how Theorem 1.3 follows from Proposition 2.1. We first collect several auxiliary results. In this section, we fix a point $q \in M$, and denote by ρ_q the Riemannian distance from q.

Lemma 3.1 *We have $\nabla\rho_q^2 = 0$, $D^2\rho_q^2 = 2g$, and $D^3\rho_q^2 = 0$ at the point q. Moreover, at the point q, we have*

$$
\begin{aligned}
(D^4\rho_q^2)_q(V,W,V,W) = (D^4\rho_q^2)_q(W,V,V,W) &= \frac{2}{3}R(V,W,V,W), \\
(D^4\rho_q^2)_q(V,V,W,W) &= -\frac{4}{3}R(V,W,V,W).
\end{aligned}
$$

Proof The first statement is standard, so we focus on the second statement. We define a vector field S by $S = \rho_q\nabla\rho_q$. Clearly, $D_S S = S$ and $D^2_{S,S}S = 0$. We next assume that W is an arbitrary smooth vector field defined in a neighborhood of q. We define a $(1,1)$-tensor Q by $Q(W) := D_W S$.

Differentiating the identity $D_S S = S$ gives

$$
\begin{aligned}
D^2_{W,S}S &= D_W D_S S - D_{D_W S}S = Q(W) - Q\big(Q(W)\big), \\
D^2_{S,W}S &= D^2_{W,S}S - \sum_i R(S,W,S,e_i)e_i \\
&= Q(W) - Q\big(Q(W)\big) - \sum_i R(S,W,S,e_i)e_i.
\end{aligned}
$$

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Consequently, we have

$$(D_S Q)(W) = Q(W) - Q\big(Q(W)\big) - \sum_i R(S, W, S, e_i) e_i$$

in a neighborhood of the point q. Integrating this identity along radial geodesics, we conclude that

$$Q(W) = W - \frac{1}{3} \sum_i R(S, W, S, e_i) e_i + O(\rho_q^3)$$

in a neighborhood of the point q.

We next differentiate the identity $D^2_{S,S} S = 0$. This gives

$$\begin{aligned} D^3_{W,S,S} S &= D_W(D^2_{S,S} S) - D^2_{D_W S, S} S - D^2_{S, D_W S} S \\ &= -2Q\big(Q(W)\big) + 2Q\big(Q(Q(W))\big) + \sum_i R\big(S, Q(W), S, e_i\big) e_i \\ &= \frac{1}{3} \sum_i R(S, W, S, e_i) e_i + O(\rho_q^3) \end{aligned}$$

in a neighborhood of q. In particular, we have

$$\langle D^3_{W,S,S} S, W \rangle = \frac{1}{3} R(S, W, S, W) + O(\rho_q^3).$$

Since $\nabla \rho_q^2 = 2S$, it follows that

$$(D^4 \rho_q^2)(W, S, W, S) = (D^4 \rho_q^2)(W, S, S, W) = \frac{2}{3} R(S, W, S, W) + O(\rho_q^3)$$

in a neighborhood of q. Consequently, at the point q, we have

$$(D^4 \rho_q^2)(W, V, W, V) = (D^4 \rho_q^2)(W, V, V, W) = \frac{2}{3} R(V, W, V, W)$$

for all tangent vectors V, W. Finally, a standard commutator identity gives

$$(D^3 \rho_q^2)(V, W, V) - (D^3 \rho_q^2)(W, V, V) = \sum_i R(V, W, V, e_i) D\rho_q^2(e_i),$$

hence

$$\begin{aligned} &(D^4 \rho_q^2)(W, V, W, V) - (D^4 \rho_q^2)(W, W, V, V) = \\ &\qquad \sum_i R(V, W, V, e_i)(D^2 \rho_q^2)(W, e_i) + \sum_i (D_W R)(V, W, V, e_i) D\rho_q^2(e_i). \end{aligned}$$

In particular, we have

$$(D^4 \rho_q^2)(W, V, W, V) - (D^4 \rho_q^2)(W, W, V, V) = 2R(V, W, V, W)$$

at the point q. Thus, we conclude that

$$(D^4 \rho_q^2)(W, W, V, V) = -\frac{4}{3} R(V, W, V, W).$$

This proves the assertion. ∎

Lemma 3.2 *At the point q, we have $(D^{l+1} \rho_q^2)(V, \dots, V, W) = 0$ for all $l \geq 2$ and all vectors $V, W \in T_q M$.*

Proof Recall that the vector field $S = \rho_q \nabla \rho_q$ satisfies $D^2_{S,S} S = 0$. By induction on l, we can show that $D^l_{S,\dots,S} S = 0$ for all $l \geq 2$. In particular, at the point q, we have $D^l_{V,\dots,V} S = 0$ for all $l \geq 2$ and all vectors $V \in T_q M$. The assertion follows. ■

Lemma 3.3 *At the point q, we have*

$$\begin{aligned}(D^5\rho_q^2)(V,V,V,V,W) &= (D^5\rho_q^2)(V,V,V,W,V) = (D^5\rho_q^2)(V,V,W,V,V)\\ &= (D^5\rho_q^2)(V,W,V,V,V) = (D^5\rho_q^2)(W,V,V,V,V) = 0\end{aligned}$$

for all vectors $V, W \in T_q M$.

Proof By Lemma 3.2, we have

$$(D^5\rho_q^2)(V,V,V,V,W) = (D^5\rho_q^2)(V,V,V,W,V) = 0$$

at the point q. Using standard commutator identities, we obtain

$$\begin{aligned}&(D^5\rho_q^2)(V,V,V,W,V) - (D^5\rho_q^2)(V,V,W,V,V)\\ &\quad= \sum_i R(V,W,V,e_i)(D^3\rho_q^2)(V,V,e_i) + 2\sum_i (D_V R)(V,W,V,e_i)(D^2\rho_q^2)(V,e_i)\\ &\qquad+ \sum_i (D^2_{V,V} R)(V,W,V,e_i) D\rho_q^2(e_i)\\ &\quad= 4(D_V R)(V,W,V,V) = 0\end{aligned}$$

and

$$\begin{aligned}&(D^5\rho_q^2)(V,V,W,V,V) - (D^5\rho_q^2)(V,W,V,V,V)\\ &\quad= 2\sum_i R(V,W,V,e_i)(D^3\rho_q^2)(V,V,e_i) + 2\sum_i (D_V R)(V,W,V,e_i)(D^2\rho_q^2)(V,e_i)\\ &\quad= 4(D_V R)(V,W,V,V) = 0\end{aligned}$$

at the point q. Consequently, $(D^5\rho_q^2)(V,V,W,V,V) = (D^5\rho_q^2)(V,W,V,V,V) = 0$ at the point q. Finally, since $D^3\rho_q^2 = 0$ at the point q, we have

$$(D^5\rho_q^2)(V,W,V,V,V) - (D^5\rho_q^2)(W,V,V,V,V) = 0$$

at the point q. Therefore, we have $(D^5\rho_q^2)(W,V,V,V,V) = 0$ at the point q, as claimed. ■

We now describe the proof of Theorem 1.3. Let M be a complete, simply connected manifold of dimension n with sectional curvature $K \leq -1$. Suppose that $F\colon \mathbb{R}/(L\mathbb{Z}) \to M$ is a closed curve of length L whose geodesic curvature satisfies $\sup|\kappa| \leq \sqrt{1 + 4\pi^2/L^2}$. Moreover, we denote by τ the unit tangent vector field to the curve. We claim that $|\kappa| = \sqrt{1 + 4\pi^2/L^2}$, $D^\perp_\tau \kappa = 0$, and $R(\tau,\kappa,\tau,\kappa) = -|\kappa|^2$ at each point on the curve. In order to prove this, we fix y, and let $q := F(y)$. Moreover, we define

$$u(x) := \rho_q^2(F(x)) \quad \text{and} \quad v(x) := 4\sinh^2\frac{\rho_q(F(x))}{2} - \frac{L^2}{\pi^2}\sin^2\frac{\pi d(x,y)}{L},$$

where, as usual, ρ_q denotes the distance function from q. The function u satisfies

$$\begin{aligned}
u'(x) &= (D\rho_q^2)(\tau),\\
u''(x) &= (D^2\rho_q^2)(\tau,\tau) + (D\rho_q^2)(D_\tau\tau),\\
u'''(x) &= (D^3\rho_q^2)(\tau,\tau,\tau) + 3(D^2\rho_q^2)(\tau,D_\tau\tau) + (D\rho_q^2)(D_\tau D_\tau\tau),\\
u^{(4)}(x) &= (D^4\rho_q^2)(\tau,\tau,\tau,\tau) + (D^3\rho_q^2)(D_\tau\tau,\tau,\tau) + 5(D^3\rho_q^2)(\tau,\tau,D_\tau\tau)\\
&\quad + 4(D^2\rho_q^2)(\tau,D_\tau D_\tau\tau) + 3(D^2\rho_q^2)(D_\tau\tau,D_\tau\tau) + (D\rho_q^2)(D_\tau D_\tau D_\tau\tau),\\
u^{(5)}(x) &= (D^5\rho_q^2)(\tau,\tau,\tau,\tau,\tau) + (D^4\rho_q^2)(D_\tau\tau,\tau,\tau,\tau)\\
&\quad + 2(D^4\rho_q^2)(\tau,D_\tau\tau,\tau,\tau) + 7(D^4\rho_q^2)(\tau,\tau,\tau,D_\tau\tau)\\
&\quad + (D^3\rho_q^2)(D_\tau D_\tau\tau,\tau,\tau) + 9(D^3\rho_q^2)(\tau,\tau,D_\tau D_\tau\tau)\\
&\quad + 7(D^3\rho_q^2)(D_\tau\tau,\tau,D_\tau\tau) + 8(D^3\rho_q^2)(\tau,D_\tau\tau,D_\tau\tau)\\
&\quad + 5(D^2\rho_q^2)(\tau,D_\tau D_\tau D_\tau\tau) + 10(D^2\rho_q^2)(D_\tau\tau,D_\tau D_\tau\tau)\\
&\quad + (D\rho_q^2)(D_\tau D_\tau D_\tau D_\tau\tau),\\
u^{(6)}(x) &= (D^6\rho_q^2)(\tau,\tau,\tau,\tau,\tau) + (D^5\rho_q^2)(D_\tau\tau,\tau,\tau,\tau,\tau)\\
&\quad + 2(D^5\rho_q^2)(\tau,D_\tau\tau,\tau,\tau,\tau) + 3(D^5\rho_q^2)(\tau,\tau,D_\tau\tau,\tau,\tau)\\
&\quad + 9(D^5\rho_q^2)(\tau,\tau,\tau,\tau,D_\tau\tau) + (D^4\rho_q^2)(D_\tau D_\tau\tau,\tau,\tau,\tau)\\
&\quad + 3(D^4\rho_q^2)(\tau,D_\tau D_\tau\tau,\tau,\tau) + 16(D^4\rho_q^2)(\tau,\tau,\tau,D_\tau D_\tau\tau)\\
&\quad + 3(D^4\rho_q^2)(D_\tau\tau,D_\tau\tau,\tau,\tau) + 9(D^4\rho_q^2)(D_\tau\tau,\tau,\tau,D_\tau\tau)\\
&\quad + 18(D^4\rho_q^2)(\tau,D_\tau\tau,\tau,D_\tau\tau) + 15(D^4\rho_q^2)(\tau,\tau,D_\tau\tau,D_\tau\tau)\\
&\quad + (D^3\rho_q^2)(D_\tau D_\tau D_\tau\tau,\tau,\tau) + 14(D^3\rho_q^2)(\tau,\tau,D_\tau D_\tau D_\tau\tau)\\
&\quad + 9(D^3\rho_q^2)(D_\tau D_\tau\tau,\tau,D_\tau\tau) + 16(D^3\rho_q^2)(D_\tau\tau,\tau,D_\tau D_\tau\tau)\\
&\quad + 15(D^3\rho_q^2)(D_\tau\tau,D_\tau\tau,D_\tau\tau) + 35(D^3\rho_q^2)(\tau,D_\tau\tau,D_\tau D_\tau\tau)\\
&\quad + 6(D^2\rho_q^2)(\tau,D_\tau D_\tau D_\tau D_\tau\tau) + 15(D^2\rho_q^2)(D_\tau\tau,D_\tau D_\tau D_\tau\tau)\\
&\quad + 10(D^2\rho_q^2)(D_\tau D_\tau\tau,D_\tau D_\tau\tau) + (D\rho_q^2)(D_\tau D_\tau D_\tau D_\tau D_\tau\tau).
\end{aligned}$$

In particular, for $x = y$ we obtain $u(y) = 0$, $u'(y) = 0$, $u''(y) = 2$, and $u'''(y) = 0$. This immediately implies $v(y) = 0$, $v'(y) = 0$, $v''(y) = 0$, and $v'''(y) = 0$. We next compute

$$u^{(4)}(y) = 8\langle\tau,D_\tau D_\tau\tau\rangle + 6\langle D_\tau\tau,D_\tau\tau\rangle = -2|\kappa|^2.$$

This gives

$$v^{(4)}(y) = u^{(4)}(y) + \frac{1}{2}u''(y)^2 + \frac{8\pi^2}{L^2} = -2|\kappa|^2 + 2 + \frac{8\pi^2}{L^2} \geq 0.$$

On the other hand, it follows from Proposition 2.1 that $v(x) \leq 0$ for all x. Consequently, we have $v^{(4)}(y) = 0$. This implies $|\kappa|^2 = 1 + \frac{4\pi^2}{L^2}$ at the point y. Since the point y is arbitrary, we conclude that $|\kappa|^2 = 1 + \frac{4\pi^2}{L^2}$ at each point on the curve.

Since $|\kappa|^2$ is constant, we obtain

$$\langle \tau, D_\tau D_\tau D_\tau \tau \rangle = \langle D_\tau \tau, D_\tau D_\tau \tau \rangle = 0.$$

This implies

$$u^{(5)}(y) = 10\langle \tau, D_\tau D_\tau D_\tau \tau \rangle + 20\langle D_\tau \tau, D_\tau D_\tau \tau \rangle = 0,$$

hence $v^{(5)}(y) = 0$. Finally, using Lemmas 3.1–3.3, we obtain

$$\begin{aligned}
u^{(6)}(y) &= 3(D^4\rho_q^2)(D_\tau\tau, D_\tau\tau, \tau, \tau) + 9(D^4\rho_q^2)(D_\tau\tau, \tau, \tau, D_\tau\tau) \\
&\quad + 18(D^4\rho_q^2)(\tau, D_\tau\tau, \tau, D_\tau\tau) + 15(D^4\rho_q^2)(\tau, \tau, D_\tau\tau, D_\tau\tau) \\
&\quad + 6(D^2\rho_q^2)(\tau, D_\tau D_\tau D_\tau D_\tau \tau) + 15(D^2\rho_q^2)(D_\tau\tau, D_\tau D_\tau D_\tau \tau) \\
&\quad + 10(D^2\rho_q^2)(D_\tau D_\tau \tau, D_\tau D_\tau \tau) \\
&= -6R(\tau, D_\tau\tau, \tau, D_\tau\tau) + 12\langle \tau, D_\tau D_\tau D_\tau D_\tau \tau \rangle \\
&\quad + 30\langle D_\tau\tau, D_\tau D_\tau D_\tau \tau \rangle + 20\langle D_\tau D_\tau \tau, D_\tau D_\tau \tau \rangle \\
&= -6R(\tau, D_\tau\tau, \tau, D_\tau\tau) + 2\langle D_\tau D_\tau \tau, D_\tau D_\tau \tau \rangle \\
&= -6R(\tau, \kappa, \tau, \kappa) + 2|D_\tau^\perp \kappa|^2 + 2|\kappa|^4.
\end{aligned}$$

Since M has sectional curvature at most -1, we have $R(\tau, \kappa, \tau, \kappa) \le -|\kappa|^2$. Thus, we conclude that

$$\begin{aligned}
v^{(6)}(y) &= u^{(6)}(y) + \frac{5}{2}u''(y)u^{(4)}(y) + \frac{1}{4}u''(y)^3 - \frac{32\pi^4}{L^4} \\
&= -6R(\tau, \kappa, \tau, \kappa) + 2|D_\tau^\perp \kappa|^2 + 2|\kappa|^4 - 10|\kappa|^2 + 2 - \frac{32\pi^4}{L^4} \\
&= -6|\kappa|^2 - 6R(\tau, \kappa, \tau, \kappa) + 2|D_\tau^\perp \kappa|^2 \ge 0.
\end{aligned}$$

On the other hand, by Proposition 2.1, we have $v(x) \le 0$ for all x. Thus, we conclude that $v^{(6)}(y) = 0$. This gives $R(\tau, \kappa, \tau, \kappa) = -|\kappa|^2$ and $D_\tau^\perp \kappa = 0$ at the point y. Since y is arbitrary, the assertion follows. ■

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