COMPOSITIO MATHEMATICA

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Compositio Math. **152** (2016), 2603–2625.

doi:10.1112/S0010437X16007685

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EST. 1865

Compositio Math. 152 (2016) 2603–2625
doi:10.1112/S0010437X16007685

Chern–Schwartz–MacPherson classes for Schubert cells in flag manifolds

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Abstract

We obtain an algorithm computing the Chern–Schwartz–MacPherson (CSM) classes of Schubert cells in a generalized flag manifold G/B. In analogy to how the ordinary divided difference operators act on Schubert classes, each CSM class of a Schubert class is obtained by applying certain Demazure–Lusztig-type operators to the CSM class of a cell of dimension one less. These operators define a representation of the Weyl group on the homology of G/B. By functoriality, we deduce algorithmic expressions for CSM classes of Schubert cells in any flag manifold G/P. We conjecture that the CSM classes of Schubert cells are an effective combination of (homology) Schubert classes, and prove that this is the case in several classes of examples. We also extend our results and conjecture to the torus equivariant setting.

1. Introduction

A classical problem in algebraic geometry is to define characteristic classes of singular algebraic varieties generalizing the notion of the total Chern class of the tangent bundle of a non-singular variety. The existence of a functorial theory of Chern classes for possibly singular varieties was conjectured by Grothendieck and Deligne, and established by MacPherson [Mac74]. This theory associates a class $c_*(\varphi) \in H_*(X)$ with every constructible function φ on X, such that $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ if X is a smooth compact complex variety. (The theory was later extended to arbitrary algebraically closed fields of characteristic 0, with values in the Chow group [Ken90, Alu06b].) The strong functoriality properties satisfied by these classes determine them uniquely; we refer to § 3 below for details. If X is a compact complex variety, then the class $c_*(\mathbb{1}_X)$ coincides with a class defined earlier by Schwartz [Sch65a, Sch65b] independently of the work mentioned above. This class is commonly known as the *Chern–Schwartz–MacPherson* (CSM) class of X. In general, we denote by $c_{\mathrm{SM}}(W)$ the class $c_*(\mathbb{1}_W) \in H_*(X)$ for any constructible (e.g. locally closed) subset $W \subseteq X$.

Let G be a complex simple Lie group and let B be a Borel subgroup. Denote by W the Weyl group of G. The goal of this note is to provide an algorithm calculating the CSM classes of the Schubert cells $X(w)^\circ := BwB/B$ in the generalized flag manifold G/B, as w varies in W. To describe the answer, we need to recall two well-known families of operators on the homology group $H_*(G/B)$; we refer to § 2 below for details.

Received 3 January 2016, accepted in final form 22 April 2016, published online 14 November 2016.
2010 Mathematics Subject Classification 14M15 (primary), 14C17, 14N15 (secondary).
Keywords: Chern–Schwartz–MacPherson class, homogeneous space, Schubert variety, Demazure–Lusztig operator.
P. A. was supported in part by a Simons Collaboration Grant and by NSA Award H98230-15-1-0027; L. C. M. was supported in part by a Simons Collaboration Grant and by NSA Young Investigator Award H98230-13-1-0208.

Let $s_1, \ldots, s_r \in W$ be the simple reflections corresponding respectively to the simple roots $\alpha_1, \ldots, \alpha_r$ of G, and let $\ell : W \to \mathbb{N}$ be the length function. Denote by $X(w) := \overline{BwB/B}$ the Schubert variety corresponding to w; it is a subvariety of G/B of complex dimension $\ell(w)$. For each $1 \leqslant k \leqslant r$, the classical BGG operator [BGG73] is an operator $\partial_k : H_*(G/B) \to H_{*+2}(G/B)$ which sends the Schubert class $[X(w)]$ to $[X(ws_k)]$ if $\ell(ws_k) > \ell(w)$ and to 0 otherwise. The Weyl group admits a right action on $H_*(G/B)$, which was originally used to *define* the BGG operator ∂_k. For $1 \leqslant k \leqslant r$, define the non-homogeneous operator

$$\mathcal{T}_k := \partial_k - s_k : H_*(G/B) \to H_*(G/B),$$

where s_k denotes the (right) action of the simple reflection s_k. The main result of this note is the following.

Theorem 1.1. *Let $w \in W$ be a Weyl group element and $1 \leqslant k \leqslant r$. Then*

$$\mathcal{T}_k(c_{\mathrm{SM}}(X(w)^\circ)) = c_{\mathrm{SM}}(X(ws_k)^\circ).$$

In the case $w = \mathrm{id}$, the Schubert cell $X(\mathrm{id})^\circ$ is a point, and $c_{\mathrm{SM}}([pt]) = [pt]$. More generally, if $w = s_{i_1} \ldots s_{i_k}$, then the theorem implies that the CSM class $c_{\mathrm{SM}}(X(w)^\circ)$ is obtained by composing the operators $\mathcal{T}_{i_k} \cdots \mathcal{T}_{i_1}$. This is reminiscent of the classical situation in Schubert calculus, where one generates all the Schubert classes by applying successively the BGG operators ∂_k. To further the analogy, $\partial_k^2 = 0$ and the BGG operators satisfy the braid relations. An easy calculation shows that $\mathcal{T}_k^2 = 1$, and then Theorem 1.1 can be used to show that the operators $\mathcal{T}_k$ also satisfy the braid relations (Proposition 4.1). In particular, there is a well-defined operator $\mathcal{T}_w$ associated with any $w \in W$, and this yields a representation of the Weyl group on $H_*(G/B)$. The CSM classes of the Schubert cells $X(w)^\circ$ are the values obtained by applying to the class of a point $[pt]$ the operators $T_{w^{-1}}$ in this representation. Further, the action of each simple reflection $\mathcal{T}_k$ on Schubert classes can be written explicitly using the Chevalley formula (Proposition 4.3). This gives an *explicit* algorithm to calculate the CSM class of any Schubert cell.

We also note that $\mathcal{T}_k$ is related to a specialization of the *Demazure–Lusztig operator* defined in [Gin98, § 12], in relation to degenerate Hecke algebras. We plan to investigate this connection further in a future paper.

Perhaps the most surprising feature of the CSM classes (and of the operators $\mathcal{T}_k$) is a positivity property. It follows from the definition of CSM classes that

$$c_{\mathrm{SM}}(X(w)^\circ) = \sum_{v \leqslant w} \mathbf{c}(v; w)[X(v)],$$

where $\leqslant$ denotes the Bruhat ordering and $\mathbf{c}(w; w) = \mathbf{c}(\mathrm{id}; w) = 1$. Despite the fact that $\mathcal{T}_k$ does not preserve the positivity of a combination of Schubert classes, we conjecture that $\mathbf{c}(v; w) > 0$ for all $v \leqslant w$. We have checked this by explicit computations for all Schubert cells $X(w)$ in type A flag manifolds $\mathrm{Fl}(n)$ for $n \leqslant 7$. We were also able to prove that this positivity holds for several families of Schubert cells across all Lie types; see § 5 below.

Let $P \subset G$ be a parabolic subgroup containing B and let $p : G/B \to G/P$ be the natural projection. Then $p(X(w)^\circ) = X(wW_P)^\circ$, where W_P is the subgroup of W generated by the reflections in P. The functoriality of CSM classes can be used to prove that

$$p_*(c_{\mathrm{SM}}(X(w)^\circ)) = c_{\mathrm{SM}}(X(wW_P)^\circ)$$

https://doi.org/10.1112/S0010437X16007685 Published online by Cambridge University Press

(see Proposition 3.5 below). In particular, the (conjectured) positivity of CSM classes of Schubert cells in G/B implies the positivity of classes in any G/P. In the case when G/P is a Grassmann manifold this was proved in several cases by the authors of this note [AM09, Mih15], Jones [Jon10], Stryker [Str11], and it was settled for all cases by Huh [Huh16]. Huh was able to realize the homogeneous components of CSM classes as a class of an effective cycle, but unfortunately his main technical requirements do not seem to hold for arbitrary flag manifolds G/B.

Our calculation of CSM classes is based on a construction of these classes in terms of bundles of logarithmic tangent fields. For every $W \subseteq X$, the class $c_{\mathrm{SM}}(\mathbb{1}_W) \in H_*(X)$ may be obtained by pushing forward to X the total Chern class of $T_{\widetilde{W}}(-\log D)$, where $\widetilde{W}$ is a resolution of $\overline{W}$ such that $D := \widetilde{W} \smallsetminus W$ is a simple normal crossing (SNC) divisor [Alu99, Alu06a]. We refer to § 3 for details. This approach was used successfully in previous work by the authors [AM09] to compute CSM classes of Schubert cells in the Grassmann manifold and prove partial positivity results. In that case one can use a resolution of a Schubert variety which is a (smooth) Schubert variety, but in a partial flag manifold. Jones [Jon10] gave an alternative computation of the classes, by means of a different (small) resolution, and also obtained partial positivity results. The resolution used in [AM09] has finitely many orbits of the Borel subgroup B, and this was a key fact in Huh's full proof of the positivity conjecture for CSM classes in that case [Huh16].

For generalized flag manifolds G/B a resolution is given by *Bott–Samelson varieties*. These are iterated $\mathbb{P}^1$-bundles, and they can be constructed from any (possibly non-reduced) word consisting of simple reflections. Section 2.3 is devoted to the definition and cohomological properties of Bott–Samelson varieties. The key technical result in the paper is Theorem 3.3, establishing the necessary cohomological formulas calculating the push-forward of the Chern class of the logarithmic tangent bundle. The operators ∂_k and s_k, and $\mathcal{T}_k = \partial_k - s_k$, appear naturally in these push-forward formulas. Properties of the operators $\mathcal{T}_k$ are discussed in § 4, and in § 5 we formulate the positivity conjecture and a related identity and provide partial evidence for these statements. In § 6 we consider the torus-equivariant setting, and prove (Theorem 6.4) that a direct analogue of Theorem 1.1 holds for the *equivariant* CSM classes of Schubert cells. We also propose an equivariant generalization of the conjecture presented in § 5.

Equivariant Chern–Schwartz–MacPherson classes were defined by Ohmoto [Ohm06] and have recently been the object of further study: Weber [Web12] proved localization formulas for these classes, and in [RV15] Rimányi and Varchenko used Weber's results to prove that equivariant CSM classes of Schubert cells agree with the κ classes they studied in earlier work [RTV14]. The κ classes are related to the stable envelopes of Maulik and Okounkov [MO12], which are classes in the equivariant cohomology of the cotangent bundle of flag manifolds. In future work we plan to investigate further the connection between the CSM classes and stable envelopes, both from the localization point of view (cf. [RV15, Su15]) and from the point of view of Ginzburg's theory of bivariant Chern classes [Gin86].

2. Preliminaries

The goal of this section is to recall some basic facts on the cohomology of flag manifolds and on Bott–Samelson resolutions. We refer to [BK05, §§ 2.1 and 2.2] for more details and references to the standard literature.

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2.1 Flag manifolds and Schubert varieties

Let G be a complex simple Lie group and let $T \subseteq B \subseteq G$ be a maximal torus included in a Borel subgroup of G. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the Lie algebras of T and G, and let $R \subseteq \mathfrak{h}^*$ be the associated root system with the set of positive roots R^+ determined by B. Denote by $\Delta := \{\alpha_1, \ldots, \alpha_r\} \subseteq R^+$ the set of simple roots. Let $R^\vee$ denote the set of coroots $\alpha^\vee \in \mathfrak{h}$ and $\langle \cdot, \cdot \rangle : R \otimes R^\vee \to \mathbb{Z}$ the evaluation pairing.

To each root $\alpha \in R$ one associates a reflection s_α in the Weyl group $W = N_G(T)/T$. The set of simple reflections $s_i := s_{\alpha_i}$ generates W; thus, each $w \in W$ can be written as $w = s_{i_1} \cdots s_{i_k}$. The minimal such integer k is denoted by $\ell(w)$, the length of w. If $k = \ell(w)$, then the decomposition $w = s_{i_1} \cdots s_{i_k}$ is said to be *reduced*. There is a partial order on W called the *Bruhat ordering*, defined as follows: $u < v$ if there exists a chain

$$u_0 := u \to u_1 := us_{\beta_1} \to u_2 := u_1 s_{\beta_2} \to \cdots \to u_n := v = u_{n-1} s_{\beta_n},$$

where the β_i are roots in R such that $\ell(u_i) > \ell(u_{i-1})$. Let G/B be the generalized flag manifold. This is a projective homogeneous space for G (hence a non-singular variety) and it is stratified by *Schubert cells* $X(w)^\circ := BwB/B$, where $w \in W$, and each such cell $X(w)^\circ$ is isomorphic to $\mathbb{C}^{\ell(w)}$. The closure $X(w) := \overline{BwB/B}$ is called a *Schubert variety*. Each Schubert variety $X(w)$ has a fundamental class $[X(w)] \in H_{2\ell(w)}(G/B)$, and these classes form a $\mathbb{Z}$-basis for the (co)homology. It may be verified that $X(v) \subseteq X(w)$ if and only if $v \leqslant w$ in the Bruhat order. In fact, $X(w) = \bigcup_{v \leqslant w} X(v)^\circ$. It follows [Ful98, Example 1.9.1] that any class in $H_*(X(w))$ may be written uniquely as an integer linear combination $\sum_{v \leqslant w} c_v [X(v)]$.

Let $\mathfrak{h}^*_\mathbb{Z}$ be the integral weight lattice and let $\lambda \in \mathfrak{h}^*_\mathbb{Z}$ be an integral weight. Then one constructs the G-equivariant line bundle over G/B

$$\mathcal{L}_\lambda := G \times^B \mathbb{C}_{-\lambda} = (G \times \mathbb{C})/B,$$

where B acts on $G \times \mathbb{C}$ by $b \cdot (g, u) = (gb^{-1}, \lambda(b)^{-1}u)$ (and the action of $B = UT$ on $\mathbb{C}$ extends the action of T so that it is trivial over the unipotent group U).

The *Chevalley formula* states that

$$c_1(\mathcal{L}_\lambda) \cdot [X(w)] = \sum \langle \lambda, \beta^\vee \rangle [X(ws_\beta)], \tag{1}$$

where the sum is over all positive roots β such that $\ell(ws_\beta) = \ell(w) - 1$. See e.g. [FW04, Lemma 8.1].

2.2 Two operators acting on $H^*(G/B)$

For each simple root $\alpha_k \in \Delta$ one can construct the BGG operator $\partial_k : H^*(G/B) \to H^*(G/B)$ defined in [BGG73] as follows. Let $P_k \subseteq G$ be the minimal parabolic subgroup corresponding to α_k. Then the natural projection $\pi : G/B \to G/P_k$ is a $\mathbb{P}^1$-bundle and there is a fiber square

$$\begin{array}{ccc} G/B \times_{G/P_k} G/B & \xrightarrow{pr_1} & G/B \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p_k} \\ G/B & \xrightarrow{p_k} & G/P_k. \end{array}$$

The BGG operator is defined to be $\partial_k = p_k^*(p_k)_*$. We record next two properties of this operator; see e.g. [Knu03, Proposition 2] or [Tym08] for simple proofs.

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PROPOSITION 2.1. *The operator ∂_k satisfies the following properties.*

(a) *For all Weyl group elements $w \in W$,*

$$\partial_k([X(w)]) = \begin{cases} 0 & \text{if } \ell(ws_k) < \ell(w), \\ [X(ws_k)] & \text{if } \ell(ws_k) > \ell(w). \end{cases} \tag{2}$$

In particular, $\partial_k^2 = 0$ and the BGG operators satisfy the same braid relations as the elements of the Weyl group.

(b) *For all $\gamma_1, \gamma_2 \in H^*(G/B)$,*

$$\partial_k(\gamma_1\gamma_2) = \partial_k(\gamma_1)\gamma_2 + \gamma_1\partial_k(\gamma_2) - c_1(\mathcal{L}_{\alpha_k})\partial_k(\gamma_1)\partial_k(\gamma_2),$$

where $c_1(\mathcal{L}_{\alpha_k})$ denotes the Chern class of $\mathcal{L}_{\alpha_k}$.

For each $w \in W$, there is a well-defined map $r_w : G/T \to G/T$ obtained by multiplying on the right with any lift of w in $N_G(T)$. This induces an algebra isomorphism $r_w^* : H^*(G/T) \to H^*(G/T)$. Note that the projection $G/T \to G/B$ is a $U \simeq B/T$-bundle and because U is contractible this implies that the cohomology rings $H^*(G/B)$ and $H^*(G/T)$ are isomorphic. This defines a right action of W on $H^*(G/B)$, denoted again by w; it will be clear from the context whether we refer to the Weyl group element or to its action on $H^*(G/B)$. It is well known (see e.g. [BGG73, § 1]) that for $w = s_k$ this operator satisfies

$$s_k = \mathrm{id} - c_1(\mathcal{L}_{\alpha_k})\partial_k, \tag{3}$$

where the Chern class $c_1(\mathcal{L}_{\alpha_k})$ acts on $H^*(G/B)$ by multiplication. Combining this with the Chevalley formula, we obtain an identity

$$s_k[X(w)] = \begin{cases} [X(w)] & \text{if } \ell(ws_k) < \ell(w), \\ -[X(w)] - \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)] & \text{if } \ell(ws_k) > \ell(w), \end{cases} \tag{4}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_k s_\beta)$.

For use in § 4, we also record the following commutation relation of the operators ∂_k and s_k.

LEMMA 2.2. *With notation as above,*

$$s_k\partial_k = \partial_k, \quad \partial_k s_k = -\partial_k.$$

In particular, $\partial_k s_k + s_k\partial_k = 0$.

Proof. The first equality follows immediately from the definition of s_k and the fact that $\partial_k^2 = 0$. The second equality is a consequence of Proposition 2.1(b):

$$\begin{aligned} \partial_k s_k &= \partial_k - \partial_k(c_1(\mathcal{L}_{\alpha_k})\partial_k) \\ &= \partial_k - \partial_k(c_1(\mathcal{L}_{\alpha_k}))\partial_k - c_1(\mathcal{L}_{\alpha_k})\partial_k^2 + c_1(\mathcal{L}_{\alpha_k})\partial_k(c_1(\mathcal{L}_{\alpha_k}))\partial_k^2 \\ &= \partial_k - \partial_k(c_1(\mathcal{L}_{\alpha_k}))\partial_k \\ &= -\partial_k, \end{aligned}$$

where we used the fact that $\partial_k(c_1(\mathcal{L}_{\alpha_k})) = \langle \alpha_k, \alpha_k^\vee \rangle = 2$ as may be checked using Proposition 2.1(a) and the Chevalley formula. □

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2.3 Bott–Samelson varieties

For each word $s_{i_1}s_{i_2}\cdots_{i_k}$ for an element $w \in W$ one can construct a tower of $\mathbb{P}^1$ bundles, the *Bott–Samelson variety* $Z := Z_{i_1,\dots,i_k}$, endowed with a map $\theta := \theta_{i_1,\dots,i_k} : Z \to G/B$. If the word is reduced, then this map is birational onto $X(w)$, giving a resolution of singularities for $X(w)$ depending on the choice of the word for w. There are several ways to do this, but for our purpose we present an inductive construction, which can be found e.g. in [BK05, § 2.2].

If the word is empty, then define $Z := pt$. In general, assume that we have constructed $Z' := Z_{i_1,\dots,i_{k-1}}$ and the map $\theta' : Z' \to X(w')$ for $w' = s_{i_1}\cdots s_{i_{k-1}}$. Define $Z = Z_{i_1,\dots,i_k}$ so that the left-hand square in the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\theta_1} & G/B \times_{G/P_{i_k}} G/B & \xrightarrow{pr_1} & G/B \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p_{i_k}} \\ Z' & \xrightarrow{\theta'} & G/B & \xrightarrow{p_{i_k}} & G/P_{i_k} \end{array} \tag{5}$$

is a fiber square; the morphisms pr_1, pr_2, p_{i_k} are the natural projections. From this construction it follows that $Z_{i_1,\dots,i_k}$ is a smooth, projective variety of dimension k.

The Bott–Samelson variety Z is equipped with a SNC divisor D_Z. We recall next an explicit inductive construction of this divisor, which will be needed later. If $Z = pt$, then $D_Z = \emptyset$. In general, G/B is the projectivization $\mathbb{P}(E)$ of a homogeneous rank-2 vector bundle $E \to G/P_k$, defined up to tensoring with a line bundle. Define $\mathcal{E} := E \otimes \mathcal{O}_E(1)$, a vector bundle over $G/B = \mathbb{P}(E)$. Then we have the Euler sequence of the projective bundle $\mathbb{P}(E)$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{6}$$

where $\mathcal{Q}$ is the relative tangent bundle $T_{p_{i_k}}$. Note that $\mathcal{E}$ is independent of the specific choice of E, and $pr_2 : G/B \times_{G/P_{i_k}} G/B \to G/B$, that is, the pull-back of $\mathbb{P}(E)$ via p_{i_k} may be identified with $\mathbb{P}(\mathcal{E})$. Let $\mathcal{E}' := (\theta')^*\mathcal{E}$ and $\mathcal{Q}' := (\theta')^*\mathcal{Q}$, and pull back the previous sequence via θ' to get an exact sequence

$$0 \longrightarrow \mathcal{O}_{Z'} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{Q}' \longrightarrow 0\,. \tag{7}$$

The inclusion $\mathcal{O}_{Z'} \hookrightarrow \mathcal{E}'$ gives a section $\sigma : Z' \to Z$ of π and therefore a divisor $D_k := \sigma(Z') = \mathbb{P}(\mathcal{O}_{Z'})$ in $\mathbb{P}(\mathcal{E}') = Z$. The SNC divisor on Z is defined by

$$D_Z = \pi^{-1}(D_{Z'}) \cup D_k,$$

where $D_{Z'}$ is the inductively constructed SNC divisor on Z'. The following result is well known; see e.g. [BK05, § 2.2].

Proposition 2.3. *If $s_{i_1}\dots s_{i_k}$ is a reduced word for w, then the image of the composition $\theta = pr_1 \circ \theta_1 : Z_{i_1,\dots,i_k} \to G/B$ is the Schubert variety $X(w)$. Moreover, $\theta^{-1}(X(w) \smallsetminus X(w)^\circ) = D_{Z_{i_1,\dots,i_k}}$ and the restriction map*

$$\theta : Z_{i_1,\dots,i_k} \smallsetminus D_{Z_{i_1,\dots,i_k}} \to X(w)^\circ$$

is an isomorphism.

Let $h_k := c_1(\mathcal{O}_{\mathcal{E}'}(1)) \in H^2(Z)$. For later use we record next the class of the divisor D_k in Z, and the Chern classes of the relative tangent bundle $T_\pi = T_{\mathbb{P}(\mathcal{E}')|Z'}$ and of T_Z.

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Proposition 2.4. *The following identities hold in* $H^*(Z)$*:*

(a) $D_k = c_1(\pi^*(\mathcal{Q}') \otimes \mathcal{O}_{\mathcal{E}'}(1)) \in H^*(Z)$;
(b) $h_k \cdot D_k = 0$;
(c) $c(T_\pi) = (1 + D_k)(1 + h_k)$ *and therefore*

$$c(T_Z) = \pi^*(c(T_{Z'}))(1 + h_k)(1 + D_k).$$

Proof. (a) follows from the definition of D_k and [Ful98, Example 3.2.17], since $\mathcal{Q}' = \mathcal{E}'/\mathcal{O}_{Z'}$.

(b) holds, since $h_k|_{D_k} = c_1(\mathcal{O}_{\mathcal{O}_{Z'}}(1)) = 0$.

To prove (c), note that by (7) the Chern roots of $\mathcal{E}'$ are 0 and $c_1(\mathcal{Q}')$; it follows from (a) that the Chern roots of $\pi^*\mathcal{E}' \otimes \mathcal{O}_{\mathcal{E}'}(1)$ are h_k and D_k. The Euler sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \pi^*\mathcal{E}' \otimes \mathcal{O}_{\mathcal{E}'}(1) \longrightarrow T_\pi \longrightarrow 0$$

then implies that $c(T_\pi) = c(\pi^*\mathcal{E}' \otimes \mathcal{O}_{\mathcal{E}'}(1)) = (1 + h_k)(1 + D_k)$. The last statement follows from $c(T_Z) = \pi^*(c(T_{Z'}))c(T_\pi)$. □

3. Chern–Schwartz–MacPherson classes of Schubert cells in G/B

3.1 CSM classes

Let Y be an algebraic variety over $\mathbb{C}$. Denote by $\mathcal{F}(Y)$ the group of constructible functions on Y: the elements of $\mathcal{F}(Y)$ are finite sums $\sum c_i \mathbb{1}_{W_i}$, where $c_i \in \mathbb{Z}$, $W_i \subseteq Y$ are locally closed subvarieties and $\mathbb{1}_W$ denotes the characteristic function taking value 1 on $p \in W$ and 0 otherwise. If $f : Y \to X$ is a proper morphism of varieties, one can define a push-forward $f_* : \mathcal{F}(Y) \to \mathcal{F}(X)$ by setting

$$f_*(\mathbb{1}_W)(p) = \chi(f^{-1}(p) \cap W)$$

for $W \subseteq Y$ a subvariety and $p \in X$, and extending by linearity to every $\varphi \in \mathcal{F}(Y)$; this makes $\mathcal{F}$ into a covariant functor. Here χ denotes the topological Euler characteristic. MacPherson [Mac74] proved a conjecture of Deligne and Grothendieck stating that there exists a natural transformation $c_* : \mathcal{F} \to H_*$ such that if Y is non-singular, then $c_*(\mathbb{1}_Y) = c(T_Y) \cap [Y]$. The naturality of c_* means that if $f : Y \to X$ is a proper morphism, then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(Y) & \xrightarrow{c_*} & H_*(Y) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{F}(X) & \xrightarrow{c_*} & H_*(X) \end{array}$$

That is,

$$f_*(c_*(\varphi)) = c_*(f_*(\varphi)) \tag{8}$$

in $H_*(X)$ for all constructible functions φ. Resolution of singularities and the normalization requirement easily imply that c_* is unique.

If Y is a compact complex variety, the class $c_*(\mathbb{1}_Y)$ coincides with a class defined earlier by Schwartz [Sch65a, Sch65b]; this class is the CSM class of Y. Taking f to be a constant map, the commutativity of the above diagram implies that $\int c_*(\mathbb{1}_Y) = \chi(Y)$, so this class provides a natural generalization of the Poincaré–Hopf theorem to possibly singular varieties.

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Abusing language a little, we denote by $c_{\mathrm{SM}}(W) := c_*(1_W) \in H_*Y$ the CSM class of any constructible set W in a variety Y; by additivity of Euler characteristics, $\int c_{\mathrm{SM}}(W) = \chi(W)$.

Our main tool will be the observation that if Z is a non-singular variety and $W \subseteq Z$ is an open subvariety such that $Z \smallsetminus W$ is a SNC divisor with components D_i, then

$$c_{\mathrm{SM}}(W) = \frac{c(T_Z)}{\prod_i (1+D_i)} \cap [Z] \in H_*Z \tag{9}$$

(cf. [GP02, Proposition 15.3] and [Alu99, Theorem 1]).

In fact, this observation may be used to extend the scope of the natural transformation c_* to arbitrary algebraically closed fields of characteristic 0, with values in the Chow group A_*. In this generality, c_* may be constructed as follows. Every constructible function on Y can be written as a linear combination of characteristic functions $\mathbb{1}_W$ for W locally closed and non-singular in Y, so it suffices to describe $c_{\mathrm{SM}}(W) = c_*(\mathbb{1}_W)$ for such W. By resolution of singularities, there exists a desingularization $\pi : Z \to \overline{W}$ of the closure $\overline{W}$ of W in Y such that $D := \pi^{-1}(\overline{W} \smallsetminus W)$ is a SNC divisor in Z. Then one may take the push-forward of (9) to Y as the *definition* of $c_*(\mathbb{1}_W)$: one can show that over algebraically closed fields of characteristic 0 the resulting c_* is independent of the choices and satisfies the Deligne–Grothendieck axioms mentioned above [Alu06a, Alu06b].

3.2 A recursive formula for CSM classes of Schubert cells

We will now apply the identity (9) to calculate the CSM class of a Schubert cell $X(w)^\circ \subseteq G/B$. This class may be viewed as an element of $H_*(G/B)$, and in fact of $H_*(X(w))$, and hence it can be written as an integer linear combination of classes $[X(v)]$ for $v \leqslant w$ in the Bruhat order, as we observed in § 2.1. We will give an algorithm which yields this linear combination. All the necessary ingredients were developed in § 2.3 and we keep the notation of that section. In particular, we recall the fiber diagram (5):

$$\begin{array}{ccccc} Z & \xrightarrow{\theta_1} & G/B \times_{G/P_{i_k}} G/B & \xrightarrow{pr_1} & G/B \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p_{i_k}} \\ Z' & \xrightarrow{\theta'} & G/B & \xrightarrow{p_{i_k}} & G/P_{i_k}. \end{array}$$

Let $s_{i_1} \cdots s_{i_k}$ be any word (reduced or otherwise) and let $Z := Z_{i_1,\ldots,i_k}$ be the corresponding Bott–Samelson variety. Recall from § 2.3 that $G/B \times_{G/P_{i_k}} G/B = \mathbb{P}(\mathcal{E})$ for a canonically defined rank-2 vector bundle $\mathcal{E}$ on G/B; thus, $Z = \mathbb{P}(\mathcal{E}')$ is the projectivization of the pull-back $\mathcal{E}' = \theta'^*(\mathcal{E})$. In $H^2(Z_{i_1,\ldots,i_k})$, we have the tautological class $h_k = c_1(\mathcal{O}_{\mathcal{E}'}(1))$, as well as the pull-backs of h_j from $Z_{i_1,\ldots,i_j}$ for $j < k$; we will omit the pull-back notation.

Let D_Z be the SNC divisor defined in § 2.3 and denote by $Z^\circ = Z^\circ_{i_1,\ldots,i_k}$ the complement $Z \smallsetminus D_Z$. By (9),

$$c_{\mathrm{SM}}(Z^\circ) = \frac{c(T_Z)}{(1+D_1)\cdots(1+D_k)} \cap [Z], \tag{10}$$

where $D_1, \ldots, D_k$ are the components of D_Z.

Lemma 3.1. *With notation as above, the following holds in $H_*(Z)$:*

$$c_{\mathrm{SM}}(Z^\circ) = (1+h_k) \cdot \pi^*(c_{\mathrm{SM}}(Z'^\circ)) = \prod_{j=1}^{k} (1+h_j) \cdot [Z].$$

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Proof. The first formula follows from (10) and Proposition 2.4(c), and the second formula is an immediate consequence. □

Now let $X(w)$ be the Schubert variety determined by $w \in W$, that is, the closure of $X(w)^\circ$ in G/B, and fix a reduced decomposition $s_{i_1} \cdots s_{i_k}$ of w and the corresponding Bott–Samelson variety Z. As recalled in Proposition 2.3, the composition $pr_1 \circ \theta_1$ gives a proper birational morphism (hence a desingularization) $\theta : Z \to X(w)$, restricting to an isomorphism on $\theta^{-1}(X(w)^\circ) = Z^\circ$.

LEMMA 3.2. *The CSM class of the Schubert cell $X(w)^\circ$ is given by*

$$c_{\mathrm{SM}}(X(w)^\circ) = \theta_*((1 + h_k) \cdot \pi^*(c_{\mathrm{SM}}(Z'^\circ))).$$

Proof. By construction, $\theta_*(\mathbb{1}_{Z^\circ}) = \mathbb{1}_{X(w)^\circ}$; therefore, the functoriality of CSM classes (8) implies that $c_{\mathrm{SM}}(X(w)^\circ) = \theta_*(c_{\mathrm{SM}}(Z^\circ))$, and the stated formula then follows from Lemma 3.1. □

Lemma 3.2 motivates the study of the quantity $\theta_*((1 + h_k) \cdot \pi^*(\gamma))$ for $\gamma \in H_*(Z')$. The next theorem gives the key formulas needed for explicit calculations, in terms of the operators introduced in § 2.2.

THEOREM 3.3. *Let $\gamma \in H_*(Z')$. Then the following holds in $H_*(G/B)$:*

(a) $\theta_*(\pi^*(\gamma)) = \partial_{i_k}(\theta'_*(\gamma))$;
(b) $\theta_*(h_k \cdot \pi^*(\gamma)) = -s_{i_k}(\theta'_*(\gamma))$.

Therefore,

$$\theta_*((1 + h_k) \cdot \pi^*(\gamma)) = \mathcal{T}_{i_k}(\theta'_*(\gamma)),$$

where $\mathcal{T}_i : H^*(G/B) \to H^*(G/B)$ is the operator given by $\mathcal{T}_i = \partial_i - s_i$.

Before proving the theorem, we note that if $\gamma = c_{\mathrm{SM}}(Z'^\circ)$, then $\theta'_*(\gamma) = c_{\mathrm{SM}}(X(w')^\circ)$, where $w' = s_{i_1} \cdots s_{i_{k-1}} = w s_{i_k}$. Therefore, Theorem 3.3 gives a recursive formula to calculate the CSM classes.

COROLLARY 3.4. *Let $w \in W$ be a non-identity element and let s_k be a simple reflection such that $\ell(ws_k) < \ell(w)$. Then the following recursive identity holds:*

$$c_{\mathrm{SM}}(X(w)^\circ) = \mathcal{T}_k(c_{\mathrm{SM}}(X(ws_k)^\circ)),$$

with the initial condition that $c_{\mathrm{SM}}(X(\mathrm{id})^\circ) = c_{\mathrm{SM}}(pt) = [pt]$.

The explicit action of the operator $\mathcal{T}_k$ on Schubert classes $[X(u)]$ is obtained by combining identities (2) and (4) above. The resulting formula together with other properties of the operator $\mathcal{T}_k$ will be presented in § 4 below.

Proof of Theorem 3.3. Both the left- and right-hand squares in (5) are fiber squares, and p_{i_k} is flat and $\theta = pr_1\theta_1$ is proper, so

$$\theta_*\pi^*(\gamma) = p_{i_k}^*(p_{i_k})_*\theta'_*(\gamma) = \partial_{i_k}(\theta'_*(\gamma))$$

by [Ful98, Proposition 1.7] and the definition of ∂_{i_k} given in § 2.2. This proves (a).

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For (b), let $\underline{\gamma} := \theta'_*(\gamma) \in H^*(G/B)$ and

$$\tilde{h}_k = c_1(\mathcal{O}_{\mathcal{E}}(1)) \in H^2(G/B \times_{G/P_{i_k}} G/B),$$

so that $h_k = \theta_1^*(\tilde{h}_k)$. Then

$$\begin{aligned}\theta_*(h_k \cdot \pi^*(\gamma)) &= (pr_1)_*(\theta_1)_*(\theta_1^*(\tilde{h}_k) \cdot \pi^*(\gamma)) \\ &= (pr_1)_*(\tilde{h}_k \cdot (\theta_1)_*\pi^*(\gamma)) \\ &= (pr_1)_*(\tilde{h}_k \cdot pr_2^*(\theta')_*(\gamma)) \\ &= (pr_1)_*(\tilde{h}_k \cdot pr_2^*(\underline{\gamma})).\end{aligned}$$

In the second equality we used the projection formula, and in the third we used the fact that the left-hand square in (5) is a fiber square and that pr_2 is flat and θ' is proper. Now recall that G/B is the projectivization $\mathbb{P}(E)$ of a vector bundle E over G/P_{i_k}, and $\mathcal{E}$ is $p_{i_k}^*(E) \otimes \mathcal{O}_E(1)$ as a bundle over G/B. We can compute the tautological subbundle $\mathcal{O}_{\mathcal{E}}(-1)$ of $pr_2^*(\mathcal{E})$, a bundle over $G/B \times_{G/P_{i_k}} G/B$, by using [Ful98, Appendix B.5.5]:

$$\mathcal{O}_{\mathcal{E}}(-1) = \mathcal{O}_{p_{i_k}^*(E)\otimes\mathcal{O}_E(1)}(-1) = pr_2^*\mathcal{O}_{p_{i_k}^*(E)}(-1) \otimes pr_2^*\mathcal{O}_E(1) = pr_1^*\mathcal{O}_E(-1) \otimes pr_2^*\mathcal{O}_E(1).$$

Letting $\eta = c_1(\mathcal{O}_E(1))$, this implies that

$$\tilde{h}_k = c_1(pr_1^*\mathcal{O}_E(1)) + c_1(pr_2^*\mathcal{O}_E(-1)) = pr_1^*(\eta) - pr_2^*(\eta),$$

and the projection formula gives

$$\begin{aligned}(pr_1)_*(\tilde{h}_k \cdot pr_2^*(\underline{\gamma})) &= (pr_1)_*((pr_1^*(\eta) - pr_2^*(\eta)) \cdot pr_2^*(\underline{\gamma})) \\ &= \eta \cdot (pr_1)_* pr_2^*(\underline{\gamma}) - (pr_1)_* pr_2^*(\eta \cdot \underline{\gamma}) \\ &= \eta \cdot p_{i_k}^*(p_{i_k})_*(\underline{\gamma}) - p_{i_k}^*(p_{i_k})_*(\eta \cdot \underline{\gamma}),\end{aligned}$$

where the last equality follows since the second square in (5) is also a fiber square and p_{i_k} is both flat and proper. By definition, $\partial_{i_k} = p_{i_k}^*(p_{i_k})_*$. Putting all together, we have shown that

$$\theta_*(h_k \cdot \pi^*\gamma) = \eta \cdot \partial_{i_k}(\underline{\gamma}) - \partial_{i_k}(\eta \cdot \underline{\gamma}).$$

Since $\mathcal{O}_E(1)$ has degree 1 on the fibers of p_{i_k}, and p_{i_k} has relative dimension 1, we have

$$\partial_{i_k}(\eta) = p_{i_k}^*(p_{i_k})_*(\eta) = [G/B].$$

We use this and part (b) of Proposition 2.1 to get

$$\begin{aligned}\partial_{i_k}(\eta \cdot \underline{\gamma}) &= \partial_{i_k}(\eta) \cdot \underline{\gamma} + \eta \cdot \partial_{i_k}(\underline{\gamma}) - c_1(\mathcal{L}_{\alpha_{i_k}}) \cdot \partial_{i_k}(\eta) \cdot \partial_{i_k}(\underline{\gamma}) \\ &= \underline{\gamma} + \eta \cdot \partial_{i_k}(\underline{\gamma}) - c_1(\mathcal{L}_{\alpha_{i_k}}) \cdot \partial_{i_k}(\underline{\gamma})\end{aligned}$$

and finally

$$\theta_*(h_k \cdot \pi^*\gamma) = -\underline{\gamma} + c_1(\mathcal{L}_{\alpha_{i_k}}) \cdot \partial_{i_k}(\underline{\gamma}) = (-\mathrm{id} + c_1(\mathcal{L}_{\alpha_{i_k}}))(\underline{\gamma}) = -s_{i_k}(\underline{\gamma})$$

by (3), concluding the proof of (b). □

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3.3 Chern classes of Schubert cells in G/P

Fix a parabolic subgroup $P \subset G$ containing the Borel subgroup B. Let $W_P \subseteq W$ be the subgroup generated by the simple reflections in P. It is known (see e.g. [Hum90, § 1.10]) that each coset in W/W_P has a unique minimal length representative; we denote by W^P the set of these representatives. If $w \in W$, then one can define a length function $\ell : W/W_P \to \mathbb{N}$ by $\ell(wW_P) := \ell(w')$, where $w' \in W^P$ is in the coset of w.

The space G/P is a projective manifold of dimension $\ell(w_0 W_P)$, where w_0 is the longest element in W. For each $w \in W^P$, there are a Schubert cell $X(wW_P)^\circ := BwP/P$ of dimension $\ell(wW_P)$ and the corresponding Schubert variety $X(wW_P) := \overline{BwP/P}$; see e.g. [BL00, § 2.6]. The fundamental classes $[X(wW_P)] \in H_{2\ell(wW_P)}(G/P)$ $(w \in W^P)$ form a $\mathbb{Z}$-basis for the homology $H_*(G/P)$. The natural projection $p : G/B \to G/P$ satisfies $p(X(w)) = X(wW_P)$ and the induced map in homology is given by

$$p_*[X(w)] = \begin{cases} [X(wW_P)] & \text{if } \ell(w) = \ell(wW_P), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

PROPOSITION 3.5. *With notation as above,*

$$c_{\mathrm{SM}}(X(wW_P)^\circ) = p_*(c_{\mathrm{SM}}(X(w)^\circ)) \in H_*(G/P) \tag{12}$$

for all $w \in W$. Further, if $u \leqslant w$ and $\ell(u) = \ell(uW_P)$, then the coefficient of $[X(u)]$ in $c_{\mathrm{SM}}(X(w)^\circ)$ equals the coefficient of $[X(uW_P)]$ in $c_{\mathrm{SM}}(X(wW_P)^\circ)$.

Proof. The topological Euler characteristic χ of the fibers of the restriction of p to $X(w)^\circ$ is constant; hence, the push-forward $p_*(\mathbb{1}_{X(w)^\circ})$ equals $\chi \cdot \mathbb{1}_{X(wW_P)^\circ}$. By functoriality of CSM classes (8), this implies that $p_*(c_{\mathrm{SM}}(X(w)^\circ)) = \chi \cdot c_{\mathrm{SM}}(X(wW_P)^\circ)$. Since the coefficient of $[pt]$ in both CSM classes equals 1, it follows that $\chi = 1$. The last claim follows from (11). □

Thus, the CSM classes of Schubert cells in G/P are determined by the corresponding classes in G/B. For example, the CSM classes of Schubert cells in the ordinary Grassmannian, determined explicitly in [AM09], can also be computed in principle using the recursive formula obtained in Corollary 3.4; see Example 4.4 for a concrete example. Further, the push-forward formula (11) implies that if the positivity conjecture discussed in § 5 is true for the CSM classes of Schubert cells in G/B, then the analogous conjecture must be true for CSM classes of Schubert cells in G/P for any parabolic P containing B.

4. The operators $\mathcal{T}_k$ and a Weyl group representation on $H^*(G/B)$

In this section we analyze the operator $\mathcal{T}_k = \partial_k - s_k : H_*(G/B) \to H_*(G/B)$, which gives the recursion for CSM classes of Schubert cells as proven in Corollary 3.4. We start by recording the main algebraic properties of the operators $\mathcal{T}_k$.

PROPOSITION 4.1. *The following identities hold.*

(a) $\mathcal{T}_k^2 = 1$.

(b) *The operators $\mathcal{T}_k$ satisfy the braid relations, i.e. $(\mathcal{T}_i \mathcal{T}_j)^{m_{i,j}} = 1$, where $m_{i,j}$ is the order of the element $s_i s_j \in W$. Also, if $w = s_{i_1} \cdots s_{i_k}$ is a representation of an element $w \in W$ as a word in simple reflections, then the operator $\mathcal{T}_w := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k}$ is independent of the choice of the word representing w.*

(c) *For any $u, v \in W$, $\mathcal{T}_u \cdot \mathcal{T}_v = \mathcal{T}_{uv}$.*

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Proof. First we note that

$$\mathcal{T}_k^2 = (\partial_k - s_k)^2 = s_k^2 = 1,$$

since $\partial_k^2 = 0$ and $\partial_k s_k + s_k \partial_k = 0$ by Lemma 2.2. This proves (a). To prove the first part of (b), it suffices to show that the relations hold after applying the operators to the classes $c_{\mathrm{SM}}(X(w)^\circ)$, since these form a basis for $H_*(G/B)$. These relations then follow immediately from the fact that for all $w \in W$ and all simple reflections s_k,

$$\mathcal{T}_k(c_{\mathrm{SM}}(X(w)^\circ)) = c_{\mathrm{SM}}(X(ws_k)^\circ) \tag{13}$$

as a consequence of Corollary 3.4 and (a). The independence of $\mathcal{T}_w$ on the specific word for w is also an immediate consequence of (13). Finally, (c) follows from the independence of $\mathcal{T}_w$ on the word representing w. □

The proposition implies that the operators $\mathcal{T}_w$ define a representation of the Weyl group W on $H_*(G/B)$. We record an immediate consequence of the identity (13) from the proof of Proposition 4.1.

Corollary 4.2. *Let u, w be two Weyl group elements. Then the identity*

$$\mathcal{T}_u(c_{\mathrm{SM}}(X(w)^\circ)) = c_{\mathrm{SM}}(X(wu^{-1})^\circ)$$

holds in $H_*(G/B)$. In particular, $c_{\mathrm{SM}}(X(w)^\circ) = \mathcal{T}_{w^{-1}}([pt])$.

Combining the actions of ∂_k and s_k on Schubert classes found in the identities (2) and (4) from § 2.2, we obtain the following explicit formula for $\mathcal{T}_k$.

Proposition 4.3.

$$\mathcal{T}_k([X(w)]) = \begin{cases} -[X(w)] & \text{if } \ell(ws_k) < \ell(w), \\ [X(ws_k)] + [X(w)] + \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)] & \text{if } \ell(ws_k) > \ell(w), \end{cases}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_k s_\beta)$.

Example 4.4. Using Corollary 4.2 and Proposition 4.3, it is straightforward to implement computations of CSM classes of Schubert cells in symbolic manipulation packages such as Maple. For instance, we obtain that the CSM class for the open cell in the flag manifold Fl(4) (in type A) is

$$\begin{aligned} c_{\mathrm{SM}}(X(4321)^\circ) = {} & [X(4321)] + [X(4312)] + [X(4231)] + [X(3421)] + 2[X(4213)] + 2[X(4132)] \\ & + [X(3412)] + 2[X(3241)] + 2[X(2431)] + [X(4123)] + 5[X(3214)] \\ & + 5[X(3142)] + 3[X(2413)] + [X(2341)] + 5[X(1432)] + 3[X(3124)] \\ & + 4[X(2314)] + 6[X(2143)] + 4[X(1423)] + 3[X(1342)] + 3[X(2134)] \\ & + 4[X(1324)] + 3[X(1243)] + [X(1234)], \end{aligned}$$

where we use the standard identification of the elements of W with permutations in indexing the 4! Schubert classes.

Note that the terms corresponding to the 'Grassmannian permutations' $(a_1 a_2 b_1 b_2)$ with $a_1 < a_2$ and $b_1 < b_2$ are

$$[X(3412)] + 3[X(2413)] + 4[X(1423)] + 4[X(2314)] + 4[X(1324)] + [X(1234)]$$

and push forward as prescribed by the identity (12) in § 3.3 to the CSM class for the open cell in $G(2,4)$ (cf. the row corresponding to $\square\!\square$ in [AM09, Example 1.2]).

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Remark 4.5. Even if $\ell(ws_k) > \ell(w)$, $\mathcal{T}_k([X(w)])$ is in general not a positive combination of Schubert classes. For example, let $G = \mathrm{SL}_4(\mathbb{C})$, and let $w = w_0 s_3$, where w_0 is the longest element in $W = S_4$, the symmetric group with four letters. Using again the standard identification of the elements of W with permutations, so that $w_0 = (4321)$ and $s_3 = (1243)$, we have that $w = (4312)$ and

$$\mathcal{T}_3([X(4312)]) = [X(4312)] + [X(4321)] - [X(4231)].$$

Nevertheless, substantial evidence suggests that the classes $\mathcal{T}_k(\gamma)$, and hence all classes $\mathcal{T}_w(\gamma)$, *are* positive linear combinations of Schubert classes if γ is a positive combination of CSM classes $c_{\mathrm{SM}}(X(u)^\circ)$; see § 5.

5. Positivity of CSM classes

5.1 Positivity

Fix $w \in W$ and consider the CSM class $c_{\mathrm{SM}}(X(w)^\circ)$. As we have shown, if $s_{i_1} \cdots s_{i_k}$ is a reduced decomposition for w, then

$$c_{\mathrm{SM}}(X(w)^\circ) = \theta_*(c_{\mathrm{SM}}(Z^\circ)) = \theta_*\left(\prod_{j=1}^{k}(1+h_j)\cdot[Z]\right), \tag{14}$$

where $\theta : Z := Z_{i_1,\ldots,i_k} \to X(w)$ is the Bott–Samelson resolution (Lemma 3.1). We have also shown that

$$c_{\mathrm{SM}}(X(w)^\circ) = \mathcal{T}_{i_k} \cdots \mathcal{T}_{i_1}([pt])$$

(Corollary 3.4). Since $c_{\mathrm{SM}}(X(w)^\circ) \in H_*(X(w))$, we have

$$c_{\mathrm{SM}}(X(w)^\circ) = \sum_{u \leqslant w} \mathbf{c}(u;w)[X(u)], \tag{15}$$

where $\mathbf{c}(u;w)$ are well-defined integers. In fact, $\mathbf{c}(w;w) = 1$, since the map θ is birational, and $\mathbf{c}(\mathrm{id};w) = 1$, since $X(w)^\circ \cong \mathbb{A}^{\ell(w)}$ and $\chi(\mathbb{A}^{\ell(w)}) = 1$.

The operator $\mathcal{T}_k$ does not preserve positivity: $\mathcal{T}_k([X(s_k)^\circ]) = -[X(s_k)^\circ]$ by Proposition 4.3, and in fact $\mathcal{T}_k([X(w)^\circ])$ may have negative contributions from Schubert classes even if $\ell(ws_k) > \ell(w)$ (Remark 4.5). Examples also show that $c_{\mathrm{SM}}(Z^\circ)$ is not necessarily a positive combination of strata of the normal crossing divisor $D_Z := Z \smallsetminus Z^\circ$ (Z°_{12321} is the smallest such example). So one should not expect any positivity properties of the CSM class *a priori*. Nevertheless, we conjecture that these classes *are* positive.

CONJECTURE 1. For all $u \leqslant w$, the coefficient $\mathbf{c}(u;w)$ from the expansion (15) is strictly positive.

Note that with notation as above, the class of the Schubert *variety* $X(w)$ is given by

$$c_{\mathrm{SM}}(X(w)) = \sum_{u \leqslant w}\left(\sum_{u \leqslant v \leqslant w} \mathbf{c}(u;v)\right)[X(u)]; \tag{16}$$

indeed, $\mathbb{1}_{X(w)} = \sum_{v \leqslant w} \mathbb{1}_{X(v)^\circ}$. So Conjecture 1 would imply that these classes are also necessarily effective.

A positivity result analogous to Conjecture 1 was conjectured by the authors in [AM09] for Schubert cells in the Grassmannian $\mathrm{Gr}(p,n)$ of subspaces of dimension p in $\mathbb{C}^n$. This conjecture was proved in [AM09] in the case $p = 2$, in [Mih15] for $p = 3$, and several classes of coefficients

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were proved to be positive by Jones [Jon10] and Stryker [Str11]. The full conjecture has recently been proven by Huh [Huh16]. By Proposition 3.5, the CSM classes of Schubert cells in any homogeneous space G/P are in fact push-forwards of CSM classes of Schubert cells in G/B; therefore, Conjecture 1 would simultaneously imply the positivity of all CSM classes of Schubert cells in all G/P, and in particular it would yield an alternative proof of Huh's theorem.

By the same token, Huh's theorem provides some evidence for Conjecture 1, since it implies that $\mathbf{c}(u;w) > 0$ in type A when u is a Grassmannian permutation (cf. Example 4.4). In fact, Conjecture 1 in type A is also supported by explicit computations of several thousand cases. At the time of this writing, we have verified that the CSM classes of all Schubert cells in $\mathrm{Fl}(n)$ are positive for $n \leqslant 7$ and for all words of length $\leqslant 15$ in $\mathrm{Fl}(8)$.

In the rest of this section we discuss more evidence for Conjecture 1 in all types. We prove positivity in the following cases:

- $\mathbf{c}(u;w) > 0$ if $u < w$ and $\ell(w) - \ell(u) = 1$ (Corollary 5.2);
- $\mathbf{c}(u;w) > 0$ for all $u \leqslant w$ if w admits a decomposition into distinct simple reflections (Corollary 5.4).

These two results will follow from more general considerations, which seem independently interesting: the first one is an explicit computation of the codimension-1 term in the CSM class of a Schubert cell (Proposition 5.1), and the second one highlights one case in which the operator $\mathcal{T}_k$ *does* preserve positivity (Proposition 5.3).

Proposition 5.1. *Let $\rho = \omega_1 + \cdots + \omega_r$ be the sum of the fundamental weights, and let $w \in W$. Then*

$$c_{\mathrm{SM}}(X(w)^\circ) = [X(w)] + c_1(\mathcal{L}_\rho) \cdot [X(w)] + \textit{lower dimensional terms.}$$

Proof. Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition, and let $Z := Z_{i_1,\ldots,i_k}$ with SNC divisor D_Z, as in § 2.3. By [BK05, Proposition 2.2.2],

$$K_Z = \mathcal{O}_Z(-D_Z) \otimes \theta^*(c_1(\mathcal{L}_{-\rho}))$$

and hence $c_1(T_Z) = [D_Z] + \theta^*(c_1(\mathcal{L}_\rho))$. On the other hand, $c_1(T_Z) = [D_Z] + \sum_{i=1}^k h_i$ by Proposition 2.4(c). Therefore, $h_1 + \cdots + h_k = \theta^*(c_1(\mathcal{L}_\rho))$, and the stated identity follows from (14) and the projection formula. □

Corollary 5.2. *The coefficient $\mathbf{c}(u;w) > 0$ if $u < w$ with $\ell(u) = \ell(w) - 1$.*

Proof. Recall that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ (the Kronecker symbol) and in particular $\langle \rho, \alpha_j^\vee \rangle > 0$ for all simple roots α_j. Consider $u < w$ such that $\ell(u) = \ell(w) - 1$. Then $u = ws_\beta$ for some positive root $\beta \in R^+$ (see e.g. [Hum90, § 5.11]). By the Chevalley formula (1), the coefficient of $[X(ws_\beta)]$ in $c_1(\mathcal{L}_\rho) \cap [X(w)]$ equals $\langle \rho, \beta^\vee \rangle > 0$, concluding the proof. □

Proposition 5.3. *Let $w \in W$ be a Weyl group element, and assume that w admits a decomposition into simple reflections other than s_k.*

(a) *The homology class $T_k([X(v)])$ is a non-negative linear combination of Schubert classes $[X(u)]$ with $u \leqslant vs_k$. In fact,*

$$\mathcal{T}_k([X(v)]) = [X(vs_k)] + [X(v)] + \sum_{u < vs_k, u \neq v} d_k(u;v)[X(u)]$$

with $d_k(u;v) \geqslant 0$ for all $u < vs_k$.

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(b) *Assume in addition that s_k commutes with all simple reflections in a decomposition of v. Then $d_k(u;v) = 0$ for $u < vs_k$, $u \neq v$, that is,*

$$\mathcal{T}_k([X(v)]) = [X(vs_k)] + [X(v)].$$

Proof. Let $S_v := \{s_{i_1}, \dots, s_{i_t}\}$ be the set of reflections appearing in a reduced decomposition of v; this set is independent of the choice of reduced decomposition, since it is preserved by the braid relations in W (see e.g. [Hum90, § 5.1]). Since every decomposition of v into simple reflections can be reduced to a reduced decomposition, the hypothesis of the proposition implies that $s_k \notin S_v$ in part (a), and that further s_k commutes with all $s_{i_j} \in S$ in part (b).

Since $vs_k > v$, by Proposition 4.3 we have

$$\mathcal{T}_k([X(v)]) = [X(vs_k)] + [X(v)] + \sum \langle \alpha_k, \beta^\vee \rangle [X(vs_k s_\beta)], \tag{17}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(v) = \ell(vs_k s_\beta)$. We have to prove that, under the hypothesis of the proposition, $\langle \alpha_k, \beta^\vee \rangle \geqslant 0$ for all β in the range of summation. In fact, all the β in this range satisfy $vs_k s_\beta < vs_k$, and we will verify that $\langle \alpha_k, \beta^\vee \rangle \geqslant 0$ for all such reflections β. By [Hum90, § 5.7], the condition $vs_k > v$ implies that $v(\alpha_k) > 0$, and $vs_k s_\beta < vs_k$ implies that $vs_k(\beta) < 0$, i.e.

$$v(\beta - \langle \beta, \alpha_k^\vee \rangle \alpha_k) = v(\beta) - \langle \beta, \alpha_k^\vee \rangle v(\alpha_k) < 0. \tag{18}$$

If $v(\beta) > 0$, then we are done, because $v(\alpha_k) > 0$. So we assume that $v(\beta) < 0$, which is equivalent to $vs_\beta < v$, and it follows that vs_β admits a reduced expression using only reflections in S_v. We deduce that s_k does not appear in a reduced expression for s_β, and hence that the simple root α_k does not appear in the support of the positive root β. Since $s_k \notin S_v$, it follows that α_k does not appear in the support of $v(\beta)$, and α_k appears with coefficient $+1$ in $v(\alpha_k)$. Then (18) forces $\langle \beta, \alpha_k^\vee \rangle \geqslant 0$, as claimed. This proves part (a).

To prove part (b), we use a similar argument. By (17), it suffices to show that $\langle \alpha_k, \beta^\vee \rangle = 0$ for all reflections s_β such that $vs_k s_\beta < vs_k$. This relation implies that s_β has a reduced decomposition containing only simple reflections in an expression for vs_k. But $\beta \neq \alpha_k$, and no simple reflections s_j with α_j adjacent to α_k in the Dynkin diagram for G can appear in the decomposition of s_β: otherwise such reflections would appear in S_v, contradicting the commutativity hypothesis. This implies that the support of β does not contain any simple root adjacent to α_k; thus, $\langle \alpha_k, \beta^\vee \rangle = 0$, concluding the proof. □

COROLLARY 5.4. *Let $w \in W$ be a Weyl group element, and assume that w admits a decomposition into simple reflections other than s_k. If $\mathbf{c}(v;w) > 0$ for all $v \leqslant w$, then $\mathbf{c}(u;ws_k) > 0$ for all $u \leqslant ws_k$.*

In particular, if $w \in W$ admits a decomposition into distinct simple reflections, then $\mathbf{c}(u;w) > 0$ for all $u \leqslant w$.

Proof. The second statement follows from the first by an immediate induction. To prove the first statement, note that if s_k does not appear in a decomposition for w, then it does not appear in a reduced decomposition for w, and hence it does not appear in a decomposition for v. Thus, the hypothesis of Proposition 5.3 applies to all $v \leqslant w$. By Corollary 3.4, we have

$$\begin{aligned} c_{\mathrm{SM}}(X(ws_k)^\circ) &= \mathcal{T}_k(c_{\mathrm{SM}}(X(w)^\circ)) = \mathcal{T}_k \sum_{v \leqslant w} \mathbf{c}(v;w)[X(v)] \\ &= \sum_{v \leqslant w} \mathbf{c}(v;w) \Bigg([X(vs_k)] + [X(v)] + \sum_{u' < vs_k, u' \neq v} d_k(u';v)[X(u')] \Bigg) \end{aligned}$$

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with $\mathbf{c}(v;w) > 0$ by hypothesis and $d_k(u';v) \geqslant 0$ by Proposition 5.3. The statement is immediate from this expression, since $u \leqslant ws_k$ implies that u has a reduced expression which is a subexpression of one for ws_k; thus, either $u = v \leqslant w$ or $u = vs_k$ with $v \leqslant w$. □

A particular case of Corollary 5.4 is particularly vivid: if $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition and the simple reflections $s_{i_1}, \ldots, s_{i_k}$ *commute* with one another, then an induction argument based on Proposition 5.3(b) implies that

$$c_{\mathrm{SM}}(X(w)^\circ) = \sum_{u \leqslant w} [X(u)]. \tag{19}$$

Notice that if w satisfies this condition, then so does every v preceding it in the Bruhat order. Then (16) and (19) give the CSM class of the Schubert *variety*

$$c_{\mathrm{SM}}(X(w)) = \sum_{u \leqslant w} 2^{\ell(w)-\ell(u)} [X(u)]$$

for every $w \in W$ decomposing into commuting simple reflections.

5.2 Constructible functions and fundamental classes

MacPherson's natural transformation c_* (see § 3.1) is surjective. Thus, for every subvariety V of an algebraic variety Y there exist constructible functions Θ_V such that

$$c_*(\Theta_V) = [V] \in H_*(Y).$$

For example, if V is the closure of a torus orbit V° and V admits a resolution by a toric variety compactifying V°, then one may choose $\Theta_V = \mathbb{1}_{V^\circ}$ (see e.g. the proof of [Alu06a, Théorème 4.2]). For flag manifolds G/B we have the following consequence of the properties of the operators $\mathcal{T}_k$.

Proposition 5.5.

$$c_*\left(\sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} \mathbb{1}_{X(v)^\circ} \right) = [G/B]. \tag{20}$$

Proof. The left-hand side of (20) equals $\sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} c_{\mathrm{SM}}(X(v)^\circ)$. By Theorem 1.1,

$$\begin{aligned} \mathcal{T}_k\left(\sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} c_{\mathrm{SM}}(X(v)^\circ) \right) &= \sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} c_{\mathrm{SM}}(X(vs_k)^\circ) \\ &= -\sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} c_{\mathrm{SM}}(X(v)^\circ). \end{aligned}$$

Therefore, $(\mathcal{T}_k + \mathrm{id}) \sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} c_{\mathrm{SM}}(X(v)^\circ) = 0$; since $\mathcal{T}_k + \mathrm{id} = c(\mathcal{L}_{\alpha_k})\partial_k$ (by (3)) and $c(\mathcal{L}_{\alpha_k})$ is invertible, this implies that

$$c_*\left(\sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} \mathbb{1}_{X(v)^\circ} \right) \in \ker \partial_k$$

for all k. It follows that

$$c_*\left(\sum_{v \in W} (-1)^{\ell(w_0)-\ell(v)} \mathbb{1}_{X(v)^\circ} \right) = C \cdot [X(w_0)] = C \cdot [G/B]$$

for some constant C. Since the coefficient of $[X(w_0)]$ on the left is 1, it follows that $C = 1$, yielding the statement. □

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For more general Schubert varieties $X(w)$, we have evidence relating the values of corresponding constructible functions $\Theta_{X(w)}$ with coefficients $\mathbf{c}(u;v)$ of CSM classes. This evidence suggests that Conjecture 1 should be equivalent to the statement that every Schubert variety $X(w)$ admits a function $\Theta_{X(w)}$ whose sign at $p \in X(w)$ is $(-1)^{e(p)}$, where $e(p)$ equals the codimension of the stratum $X(v)^\circ$ containing p. Explicit computations for low dimensions in type A support this statement, and Proposition 5.5 proves it for $w = w_0$ in all types.

6. Equivariant Chern–Schwartz–MacPherson classes of Schubert cells

In this section we extend our calculation of CSM classes to the T-equivariant situation. We will show that the same difference $\partial_k - s_k$ defines an operator $\mathcal{T}_k^T$ on equivariant homology $H_*^T(G/B)$, sending an (equivariant) CSM class ${c_{\mathrm{SM}}}^T(X(w)^\circ)$ to the class ${c_{\mathrm{SM}}}^T(X(ws_k)^\circ)$. The proof of Lemma 2.2 extends to the equivariant setting, and shows that $(\partial_k - s_k)^2 = \mathrm{id}$. In particular, the operators $\mathcal{T}_k^T$ give a representation of W on equivariant homology. The proof that $\mathcal{T}_k^T$ acts as expected on CSM classes is essentially identical to the proof in the non-equivariant case; we only need to verify that no 'equivariant corrections' are introduced in the recursion formula from Corollary 3.4.

6.1 Equivariant CSM classes

Recall that $T \subset B$ is the maximal torus in the Borel subgroup B. If X is a variety with a T-action, then the equivariant cohomology $H_T^*(X)$ is the ordinary cohomology of the Borel mixing space $X_T := (ET \times X)/T$, where ET is the universal T-bundle and T acts by $t \cdot (e, x) = (et^{-1}, tx)$. It is an algebra over $H_T^*(pt)$, the polynomial ring $\mathbb{Z}[t_1, \ldots, t_s]$, where $t_i, \ldots, t_s$ form generators for the weight lattice of T. We address the reader to [Knu03] or [Ohm06] for basic facts on equivariant cohomology. Since X is smooth, we can and will identify the equivariant homology $H_*^T(X)$ with the equivariant cohomology $H_T^*(X)$. Every closed subvariety $Y \subseteq X$ that is invariant under the T-action determines an equivariant fundamental class $[Y]_T \in H_*^T(X)$.

Ohmoto defined the group of equivariant constructible functions $\mathcal{F}^T(X)$ (for tori and for more general groups) in [Ohm06, § 2]. We recall the main properties that we need.

(i) If $W \subset X$ is a constructible set which is invariant under the T-action, its characteristic function $\mathbb{1}_W$ is an element of $\mathcal{F}^T(X)$. (The group $\mathcal{F}^T(X)$ also contains other elements, but this will be immaterial for us.)

(ii) Every proper T-equivariant morphism $f : Y \to X$ of algebraic varieties induces a homomorphism $f_*^T : \mathcal{F}^T(X) \to \mathcal{F}^T(Y)$. The restriction of f_*^T to characteristic functions of constructible T-invariant sets coincides with the ordinary push-forward f_* of constructible functions. See [Ohm06, § 2.6].

Ohmoto proved [Ohm06, Theorem 1.1] that there is an equivariant version of MacPherson's transformation $c_*^T : \mathcal{F}^T(X) \to H_*^T(X)$ that satisfies $c_*^T(\mathbb{1}_Y) = c^T(T_Y) \cap [Y]_T$ if Y is a projective, non-singular variety, and that is functorial with respect to proper push-forwards. The last statement means that for all proper T-equivariant morphisms $Y \to X$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}^T(Y) & \xrightarrow{c_*^T} & H_*^T(Y) \\ \downarrow{\scriptstyle f_*^T} & & \downarrow{\scriptstyle f_*^T} \\ \mathcal{F}^T(X) & \xrightarrow{c_*^T} & H_*^T(X) \end{array}$$

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We denote by ${c_{\mathrm{SM}}}^T(X(w)^\circ) := c_*^T(\mathbb{1}_{X(w)^\circ})$ the equivariant CSM class of the Schubert cell corresponding to an element $w \in W$.

Finally, we note that both the BGG operator ∂_i and the right Weyl group action s_i are induced by morphisms which commute with the T-action. It follows that they both determine $H_T^*(pt)$-module endomorphisms of $H_T^*(G/B)$, for which we will use the same notation. (The right Weyl group action is also an $H_T^*(pt)$-algebra automorphism.) Further, Proposition 2.1 and formula (3) extend to the equivariant setting after replacing the Chern class $c_1(\mathcal{L}_{\alpha_k})$ by its equivariant version $c_1^T(\mathcal{L}_{\alpha_k})$. We refer to [Knu03, § 3] for details. The analogue of the identity (4) can be found in [Knu03, § 4, Corollary]; the equivariant version includes additional terms.

6.2 Equivariant CSM classes via the operator $\mathcal{T}_k^T = \partial_k - s_k$

In this section we give the proof of the equivariant version of Theorem 1.1.

Recall the diagram (5) from § 2.3:

$$\begin{array}{ccccc} Z & \xrightarrow{\theta_1} & G/B \times_{G/P_{i_k}} G/B & \xrightarrow{pr_1} & G/B \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p_{i_k}} \\ Z' & \xrightarrow{\theta'} & G/B & \xrightarrow{p_{i_k}} & G/P_{i_k}. \end{array} \tag{21}$$

Recall that Z' is the Bott–Samelson variety for a Weyl group element $w' \in W$ and that Z is the Bott–Samelson variety corresponding to $w's_{i_k}$, where $\ell(w's_{i_k}) > \ell(w')$.

Recall also that G/B is a $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ for some rank-2, equivariant vector bundle over G/P_{i_k}, and that we defined $\mathcal{E} := E \otimes \mathcal{O}_E(1)$, a vector bundle over G/B. This bundle fits into the exact sequence (6) of equivariant vector bundles

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where the action on $\mathcal{O}$ is trivial, and $\mathcal{Q} = T_{p_{i_k}}$. The inclusion $\mathcal{O} \subset \mathcal{E}$ determines an equivariant section $\sigma_{i_k} : G/B \to \mathbb{P}(\mathcal{E}) = G/B \times_{G/P_{i_k}} G/B$, inducing the section σ of π introduced in § 2.3.

Lemma 6.1. *The image of the section σ_{i_k} is the diagonal $\Delta \subset G/B \times_{G/P_{i_k}} G/B$.*

Proof. The image of σ_{i_k} is $\mathbb{P}(\mathcal{O})$, which maps identically to G/B via both pr_1 and pr_2. □

Proposition 6.2. (a) *We have an isomorphism of equivariant bundles*

$$\mathcal{O}_{\mathcal{E}}(1) = pr_2^*\mathcal{O}_E(-1) \otimes pr_1^*\mathcal{O}_E(1).$$

(b) *The diagonal Δ is the zero locus of a homogeneous section of the bundle*

$$\mathcal{O}_{G/B \times_{G/P_{i_k}} G/B}(\Delta) = pr_2^*(\mathcal{Q}) \otimes \mathcal{O}_{\mathcal{E}}(1).$$

(c) *Let $[\Delta]_T \in H_*^T(G/B \times_{G/P_{i_k}} G/B)$ be the equivariant fundamental class determined by Δ. Then*

$$[\Delta]_T = c_1^T(\mathcal{O}_{G/B \times_{G/P_{i_k}} G/B}(\Delta)) \cap [G/B \times_{G/P_{i_k}} G/B]_T.$$

Proof. Part (a) was proved within the proof of Theorem 3.3. The inclusion $\Delta \subset G/B \times_{G/P_{i_k}} G/B$ is given by $\mathbb{P}(\mathcal{O}) \subset \mathbb{P}(\mathcal{E})$. Then by [Ful98, Appendix B.5.6] Δ is the zero locus of a section of the bundle $\mathcal{Q} \otimes \mathcal{O}_{\mathcal{E}}(1)$. This establishes (b).

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Part (c) follows from a general fact: if an equivariant divisor D is the zero scheme of a homogeneous section of an equivariant line bundle $\mathcal{L}$ on G/B, then $[D]_T = c_1^T(\mathcal{L}) \cap [G/B]_T$. This follows from the analogous non-equivariant statement in the corresponding Borel mixing space. □

The next observation is that (9) extends to the equivariant case.

Let Z be a variety with a T-action, and let $D \subseteq Z$ be a divisor with simple normal crossings and equivariant components D_i. Then

$$c_{\mathrm{SM}}{}^T(Z \smallsetminus D) = \frac{c^T(T_Z)}{\prod_j (1 + D_i^T)} \cap [Z]_T, \tag{22}$$

where $D_i^T = c_1^T(\mathcal{O}(D_i))$ (so that $D_i^T \cap [Z]_T = [D_i]_T$).

This may be proven by the same method used in the proof of [Alu99, Theorem 1].

Now let $Z = Z_{i_1,\dots,i_k}$ and let $D = D_Z$ be the SNC divisor defined in § 2.3. Recall that $D = \pi^{-1}(D_{Z'}) \cup D_k$. The following is the equivariant analogue of Lemma 3.2.

LEMMA 6.3. *The following identity holds in $H_*^T(G/B)$:*

$$c_{\mathrm{SM}}{}^T(X(w)^\circ) = \theta_*((1 + h_k^T) \cdot \pi^*(c_{\mathrm{SM}}{}^T(Z'^\circ))),$$

where $h_k^T = (\theta')^* c_1^T(\mathcal{O}_{\mathcal{E}}(1))$.

Proof. By Proposition 6.2(b), $\Delta^T = c_1^T(pr_2^*(\mathcal{Q}) \otimes \mathcal{O}_{\mathcal{E}}(1))$. Pulling back by θ', we obtain $D_k^T = c_1^T(\pi^*(\theta'^* \mathcal{Q}) \otimes \mathcal{O}_{\mathcal{E}}(1))$, and this implies that

$$c^T(T_Z) = \pi^*(c^T(T_{Z'}))(1 + h_k^T)(1 + D_k^T),$$

arguing exactly as in the proof of Proposition 2.4(c). The stated identity follows then from (22) by the same argument proving Lemma 3.2 from (9). □

THEOREM 6.4. *Let $\mathcal{T}_k^T : H_T^*(G/B) \to H_T^*(G/B)$ be the operator*

$$\mathcal{T}_k^T = (1 + c_1^T(\mathcal{L}_{\alpha_k}))\partial_k - \mathrm{id} = \partial_k - s_k.$$

Then $\mathcal{T}_k^T(c_{\mathrm{SM}}{}^T(X(w)^\circ)) = c_{\mathrm{SM}}{}^T(X(ws_k)^\circ)$.

Proof. The same proof applies as in the non-equivariant case, taking into account that all maps used are T-equivariant and that the statement of Proposition 2.1 extends without changes to the equivariant setting. □

6.3 Properties of equivariant CSM classes

In this section we will list a few basic properties of the equivariant CSM classes. Some of these follow from general facts due to Ohmoto [Ohm06] and Weber [Web12]; we will give short independent proofs based on the basic properties of the operator $\mathcal{T}_k^T$. We will also formulate an equivariant generalization of the positivity conjecture (Conjecture 1).

The equivariant versions of the identities (2) and (4), as seen in e.g. [Knu03, § 3], yield the following explicit formula for the class $\mathcal{T}_k^T[X(w)]_T$:

$$\begin{cases} -[X(w)]_T & \text{if } \ell(ws_k) < \ell(w), \\ (1 + w(\alpha_k))[X(ws_k)]_T + [X(w)]_T + \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)]_T & \text{if } \ell(ws_k) > \ell(w), \end{cases} \tag{23}$$

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where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_k s_\beta)$, and $w(\alpha_k)$ denotes the natural W action on roots. Recall that both operators ∂_i and s_i are $H_T^*(pt)$-linear; therefore, so is $\mathcal{T}_k^T$.

Theorem 6.4 combined with the identity (23) and the fact that $c_{\mathrm{SM}}{}^T(X(\mathrm{id})^\circ) = [X(\mathrm{id})]_T = [e_{\mathrm{id}}]_T$ (where e_{id} is the B-fixed point) give an effective way to compute equivariant CSM classes of Schubert cells in G/B. Since the equivariant classes $[X(u)]_T$ form a basis of $H_*^T(G/B)$ over $H_T^*(pt) = \mathbb{Z}[t_1, \ldots, t_s]$, we have an equivariant analogue of (15):

$$c_{\mathrm{SM}}{}^T(X(w)^\circ) = \sum_{u \leqslant w} \mathbf{c}^T(u; w)[X(u)]_T, \tag{24}$$

where $\mathbf{c}^T(u; w)$ are well-defined polynomials in the t_i. The equivariant cohomology $H_T^*(pt)$ acts on $H_*^T(G/B)$ by cap product; therefore, each weight t_i decreases the homological degree by 1.

Proposition 6.5. *The following properties hold.*

(a) *Each coefficient $\mathbf{c}^T(u; w)$ is a polynomial of degree at most $\dim X(u)$ in the subring $\mathbb{Z}[\alpha_1, \ldots, \alpha_r] \subseteq H_T^*(pt)$ generated by the positive simple roots $\alpha_1, \ldots, \alpha_r$.*

(b) *The constant term of the polynomial $\mathbf{c}^T(u; w)$ (i.e. the term independent of the α_i) equals the non-equivariant coefficient $\mathbf{c}(u; w)$. In particular, $\mathbf{c}^T(\mathrm{id}; w) = 1$ for all $w \in W$.*

(c) *Let $w = s_{i_1} \ldots s_{i_k}$ be a reduced decomposition. Then the coefficient of $[X(w)]_T$ in $c_{\mathrm{SM}}{}^T(X(w)^\circ)$ equals*

$$\mathbf{c}^T(w; w) = \prod_{t=1}^{k} (1 + s_{i_0} \cdots s_{i_{t-1}}(\alpha_{i_t}))$$

with the convention that $s_{i_0} = \mathrm{id}$. In particular,

$$\mathbf{c}^T(w_0; w_0) = \prod (1 + \alpha),$$

where the product ranges over all positive roots α.

(d) *Let $w = s_{i_1} \ldots s_{i_k}$ be a reduced decomposition. Then the term of degree 0 in $c_{\mathrm{SM}}(X(w)^\circ)$ equals*

$$c_{\mathrm{SM}}(X(w)^\circ)_{\deg 0} = (-1)^{\ell(w)}(s_{i_k} \ldots s_{i_1}) \cdot [e_{\mathrm{id}}]_T = [e_w]_T,$$

where $[e_w]_T$ denotes the equivariant fundamental class of the T-fixed point corresponding to w. In particular, one obtains an equivariant version of the classical Poincaré–Hopf theorem:

$$c^T_{\dim G/B}(T_{G/B}) \cap [G/B]_T = \sum_{w \in W} [e_w]_T.$$

Proof. Parts (a), (b), and the first statement in part (c) follow from the fact that $c_{\mathrm{SM}}{}^T(X(w)^\circ) = \mathcal{T}^T_{w^{-1}}[e_{\mathrm{id}}]_T$, and by successive applications of formula (23). For the second statement in part (c), notice that for any $w \in W$, $\ell(ws_k) > \ell(w)$ implies that $w(\alpha_k) > 0$ (see e.g. [Hum90, § 5.4]), and thus the sequence of positive roots obtained from a reduced decomposition of the longest element w_0 coincides with the full set of positive roots. We now focus on part (d). The operator ∂_k increases the homological degree by 1 (i.e. it sends $H_m^T(G/B)$ to $H_{m+1}^T(G/B)$), while the right Weyl group action by s_k is homogeneous. It follows that the degree-0 term of $c_{\mathrm{SM}}{}^T(X(w)^\circ)$ depends only on the contributions of the operators $-s_{i_j}$; this proves the first identity. The second identity follows because $s_k[e_w]_T = -[e_{ws_k}]_T$ for any $w \in W$. This can be easily

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checked by localization, using the definition of the right action (cf. [Knu03, § 3]) and the fact that

$$\iota_u^*[e_w]_T = \begin{cases} \prod_{\alpha \in R^+} w(-\alpha) & \text{if } u = w; \\ 0 & \text{otherwise,} \end{cases}$$

where ι_u^* denotes the localization at the T-fixed point e_u. (The minus sign is due to the fact that the tangent space $T_{e_{\mathrm{id}}}(G/B)$ is $\mathrm{Lie}(G)/\mathrm{Lie}(B)$ and its weights are the *negative* roots.) The last statement follows because $c^T(T_{G/B}) \cap [G/B]_T = {c_{\mathrm{SM}}}^T(G/B) = \sum_{w \in W} {c_{\mathrm{SM}}}^T(X(w)^\circ)$ by additivity of c_*^T. □

Remark 6.6. • Parts (a) and (b) of Proposition 6.5 also follow from Ohmoto's construction of equivariant CSM classes in [Ohm06].

• It is well known that the sequence of roots $\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \ldots s_{i_{t-1}}(\alpha_{i_t})$ coincides with the sequence of inversions of $w^{-1} = s_{i_k} \ldots s_{i_1}$; see e.g. [Hum90, p. 14]. In particular, this confirms that the sequence is independent of the choice of the reduced decomposition. Also, these are the weights of the tangent space $T_{e_w}(X(w))$; one can use this formula to recover a formula from [RV15] to localize ${c_{\mathrm{SM}}}^T(X(w)^\circ)$ at the T-fixed point e_w. In the particular case when $w = w_0$, the coefficient $\mathbf{c}^T(w_0; w_0)$ can also be obtained from the general fact that

$$c^T(T_{G/B}) = \prod_{\alpha \in R^+} (1 + c_1^T(\mathcal{L}_\alpha)),$$

where $\mathcal{L}_\alpha$ is the line bundle defined in § 2.1. (This follows e.g. by using localization.)

• Finally, part (d) can also be deduced from a localization result of Weber, which holds in a more general situation; see [Web12, Theorem 20].

The following statement generalizes Conjecture 1.

CONJECTURE 2. For all $u \leqslant w$, the coefficient $\mathbf{c}^T(u; w)$ is a polynomial with positive coefficients in the simple roots α_i and positive constant term.

This statement is supported by explicit computations for low dimensions in type A. It would imply equivariant positivity of CSM classes for Schubert cells in all flag manifolds G/P. (Computations in the Grassmann manifold $\mathrm{Gr}(4, \mathbb{C}^8)$ led Weber to suggest equivariant positivity for Schubert cells in Grassmannians [Web12, § 12].) Proposition 6.5 also provides partial evidence for Conjecture 2, across all Lie types.

Example 6.7. Let Γ_T be the matrix whose (u, w)-entry is $\mathbf{c}^T(u; w)$. For Fl(3), listing the elements of S_3 in the order $(123), (132), (213), (231), (312), (321)$, we have

$$\Gamma_T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1+\alpha_2 & 0 & 2+\alpha_1+\alpha_2 & 1+\alpha_2 & 2+\alpha_1+\alpha_2 \\ 0 & 0 & 1+\alpha_1 & 1+\alpha_1 & 2+\alpha_1+\alpha_2 & 2+\alpha_1+\alpha_2 \\ 0 & 0 & 0 & (1+\alpha_1)(1+\alpha_1+\alpha_2) & 0 & (1+\alpha_1)(1+\alpha_1+\alpha_2) \\ 0 & 0 & 0 & 0 & (1+\alpha_2)(1+\alpha_1+\alpha_2) & (1+\alpha_2)(1+\alpha_1+\alpha_2) \\ 0 & 0 & 0 & 0 & 0 & (1+\alpha_1)(1+\alpha_2)(1+\alpha_1+\alpha_2) \end{pmatrix},$$

verifying Conjecture 2 in this case. Explicit computations also confirm that Conjecture 2 holds for all Schubert cells in Fl(4), Fl(5), and Fl(6).

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Acknowledgements

We are grateful to Mark Shimozono for pointing out some algebraic identities between the operators ∂_k and s_k; to an anonymous referee for a careful reading of the paper; and to Dave Anderson, Seung-Jin Lee, Jörg Schürmann, and Chanjiang Su for useful discussions. P. A. is grateful to the Perimeter Institute and to Caltech for hospitality while this research was carried out.

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https://doi.org/10.1112/S0010437X16007685 Published online by Cambridge University Press