ON BORNOLOGICAL PRODUCTS

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1. Introduction. It is well known that, provided that the indexing set I is not too large, the product

$$E = \times\{E_\alpha : \alpha \in I\}$$

of a family of bornological locally convex topological vector spaces E_α is bornological. Products of bornological spaces were first studied by Mackey [**3**]. He reduced the problem to the study of $\mathbf{R}^I$, showing that this space is bornological if and only if I satisfies a certain condition, related to a problem in measure theory posed by Ulam [**5**]. We shall therefore call it the Mackey–Ulam condition on I. A similar study of the spaces $\mathbf{R}^I$ is to be found in the paper [**4**] by Simons; see also [**1**, Ch. IV, §6, exercise 3].

Mackey's theorem may be stated as follows. Suppose that I satisfies the Mackey–Ulam condition. If, for each α in I, E_α is locally convex and has the property that every bounded linear mapping from E_α into any locally convex space is continuous, then the product E of the E_α has the same property. In his recent paper [**2**], Iyahen has shown that the methods used by Simons remain valid for semiconvex spaces and that Mackey's theorem continues to hold for these spaces. The purpose of this note is to establish directly a general result of this sort, valid for all topological vector spaces.

Ulam's problem was whether there exists a nonzero measure, defined on all the subsets of a set I and taking the values 0 and 1 only, such that each one-point set has measure zero. If no such measure exists on I, we shall say that I satisfies the *Mackey–Ulam condition.* It is not known if there exists a set I large enough to carry such a measure; if so, its cardinal must be strongly inaccessible. Given any non-zero measure on all subsets of I taking values 0 and 1 only, the subsets of I assigned the value 1 form an ultrafilter $\mathscr{F}$ on I, with the further property that the intersection of any sequence of sets of $\mathscr{F}$ belongs to $\mathscr{F}$ (such an ultrafilter is called a *δ-ultrafilter*, or a Mackey–Ulam ultrafilter on I). Conversely, if $\mathscr{F}$ is a δ-ultrafilter on I, a measure can be defined on all subsets of I by assigning the value 1 to all sets in $\mathscr{F}$ and 0 to all sets not in $\mathscr{F}$. In this natural correspondence between these measures and δ-ultrafilters, the measure of each one-point set is zero if and only if the corresponding ultrafilter $\mathscr{F}$ has no one-point set in it, or equivalently $\bigcap \mathscr{F} = \emptyset$. In this case $\mathscr{F}$ is called *free* (or non-trivial). Thus I satisfies the Mackey–Ulam condition if and only if there is no free δ-ultrafilter on I. This is the form of the condition most convenient in what follows.

2. The main result. Let F be a given topological vector space. Let us call any topological vector space E *F-bornological* if and only if every bounded linear mapping of E into F is continuous.

THEOREM. *Suppose that the indexing set I satisfies the Mackey–Ulam condition and that each space E_α is F-bornological. Then so is the product $E = \times\{E_\alpha : \alpha \in I\}$.*

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Proof. Let f be any bounded linear mapping from E to F and let V be any open neighbourhood of the origin in F. Also let $N = \bigcap\{\lambda V : \lambda > 0\}$. The first step is to show that, if f_α is the restriction of f to E_α, regarded as a subspace of E, then $f_\alpha(E_\alpha) \subseteq N$ for all but finitely many α. If not, there would exist a sequence $(\alpha(n))$ of distinct indices and points $x_{\alpha(n)}$ in $E_{\alpha(n)}$ such that $f_{\alpha(n)}(x_{\alpha(n)}) \notin nV$; then f would map the bounded subset $\{x_{\alpha(n)} : n = 1, 2, \ldots\}$ of E into an unbounded subset of F.

Thus f can be expressed in the form

$$f = g + h = \left\{\sum_{r=1}^{m} f_{\alpha(r)} \circ p_{\alpha(r)}\right\} + h,$$

where p_α denotes the canonical projection of E onto E_α, and h is a linear mapping of E into F taking each E_α into N. Now each $f_{\alpha(r)}$ is bounded and $E_{\alpha(r)}$ is F-bornological, so that $f_{\alpha(r)}$ is continuous. Thus g is continuous and so there is a neighbourhood U of the origin in E mapped by g into V. If we can show that $h(E) \subseteq N$, it will follow that

$$f(U) \subseteq g(U) + h(U) \subseteq V + N = V,$$

and we shall have proved f continuous, as required.

Suppose, then, that $h(E)$ is not contained in N. For each subset J of I, let h_J denote the restriction of h to the subspace

$$E_J = \times\{E_\alpha : \alpha \in J\}$$

of E. Let $\mathscr{A}$ be the set of all J for which $h_J(E_J)$ is not contained in N. By the hypothesis just made, I is in $\mathscr{A}$, so that $\mathscr{A}$ contains at least one filter, namely $\{I\}$. Since the union of any chain of filters on I is a filter on I, it follows from the axiom of choice that $\mathscr{A}$ contains a maximal filter $\mathscr{F}$. In fact $\mathscr{F}$ is maximal also in the set of all subsets of I, so that $\mathscr{F}$ is an ultrafilter on I. To see this, suppose that $J \cup K$ is a partition of I into disjoint subsets. If $J \cap A$ is in $\mathscr{A}$ for every A in $\mathscr{F}$, the maximality of $\mathscr{F}$ in $\mathscr{A}$ shows that $J \in \mathscr{F}$; similarly $K \in \mathscr{F}$ if $K \cap B \in \mathscr{A}$ for every $B \in \mathscr{F}$. The remaining possibility is that there exist sets A, B in $\mathscr{F}$ such that $J \cap A \notin \mathscr{A}$ and $K \cap B \notin \mathscr{A}$. Then $C = A \cap B \in \mathscr{F} \subseteq \mathscr{A}$ and

$$h_C(E_C) = h_{J \cap C}(E_{J \cap C}) + h_{K \cap C}(E_{K \cap C}) \subseteq N + N = N,$$

contrary to the definition of $\mathscr{A}$.

The next step is to show that $\mathscr{F}$ is a δ-ultrafilter, i.e. that, if each J_n belongs to $\mathscr{F}$, so does their intersection J. For this, it is sufficient to show that J meets every set of the ultrafilter $\mathscr{F}$. Suppose, on the contrary, that there is a set A of $\mathscr{F}$ disjoint from J, and put

$$K_n = A \cap \bigcap_{r=1}^{n} J_r.$$

Then, for each n, K_n is in $\mathscr{F}$ and so there exists a point x_n in E_{K_n} such that $h(x_n) \notin nV$. Now the sets K_n are decreasing and their intersection is empty, by hypothesis. Thus, for each α in I, $p_\alpha(x_n) = 0$ for all but finitely many integers n, and so $\{x_n\}$ is a bounded set in E. But

$h = f - g$ is a bounded linear mapping of E into F, carrying this bounded set into the unbounded set $\{h(x_n)\}$. This is a contradiction and shows that $J \cap A$ cannot be empty, as required.

We can now invoke the Mackey–Ulam condition on I; there is an element β of I such that $\{\beta\} \in \mathscr{F}$. But then $\{\beta\} \in \mathscr{A}$, contrary to the fact that h maps E_β into N. Thus $h(E) \subseteq N$ and the proof is complete.

3. Further remarks. If, in the theorem, the E_α are all assumed to be locally convex and F is allowed to range over all locally convex spaces, we obtain the usual result about a product of bornological spaces. If, instead, F ranges over all semiconvex spaces, we obtain Iyahen's result **[2]**. If F is unrestricted, we obtain the following theorem. Suppose that the indexing set satisfies the Mackey–Ulam condition and that each E_α has the property that every bounded linear map from E_α to any topological vector space is continuous; then the product of the E_α has the same property.

If there exists a set I so large that it does not satisfy the Mackey–Ulam condition, the space $\mathbf{R}^I$ is not bornological and this fact can be used to show easily that any product of (nonzero) locally convex spaces indexed by I also fails to be bornological. The situation for general topological vector spaces is complicated by the fact that there may not exist any nonzero continuous linear mappings from the E_α into F. (This will be the case, for example, when E_α has zero dual and F is locally convex.) However, it is possible to demonstrate the following result, which gives a partial converse for the theorem.

PROPOSITION. *Suppose that F is a topological vector space and that, for each α in I, E_α is a topological vector space for which there exists a nonzero bounded linear mapping t_α of E_α into F. Suppose also that F, regarded as a set, satisfies the Mackey–Ulam condition, but that I does not. Then the product E of the E_α is not F-bornological.*

Proof. We shall construct a nonzero bounded linear map f from E to F, vanishing on each E_α. If such a map were continuous, it would be a finite linear combination of $f_\alpha \circ p_\alpha$, in the notation of the proof of the theorem. But this would make f zero: hence f is not continuous and E is not F-bornological.

To construct f, take a free δ-ultrafilter $\mathscr{F}$ on I. For each $\alpha \in I$ and each $x = (x_\alpha) \in E$, put

$$\phi(\alpha, x) = t_\alpha(x_\alpha),$$

and let $\mathscr{G}(x)$ be the image by the mapping $\alpha \to \phi(\alpha, x)$ of the filter $\mathscr{F}$. Then $\mathscr{G}(x)$ is easily verified to be the base of a δ-ultrafilter on F. This δ-ultrafilter cannot be free and so it contains a single-point set, $\{f(x)\}$, say. The linearity of the mapping $x \to \phi(\alpha, x)$ now ensures that f is a linear map of E into F.

Now take any $\alpha \in I$; if every set of $\mathscr{F}$ contained α, the filter generated by $\{\alpha\}$ would be a refinement of $\mathscr{F}$ and so identical with it, and $\mathscr{F}$ would not be free. Hence there is a $J \in \mathscr{F}$ with $\alpha \notin J$. Now if $x \in E_\alpha$, $x_\beta = 0$ for $\beta \neq \alpha$ and so

$$f(x) \in \phi(J, x) = \{0\}.$$

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Thus f, restricted to E_α, is zero for each α. On the other hand, there are points $z_\alpha \in E_\alpha$ with $t_\alpha(z_\alpha) \neq 0$. Thus if $z = (z_\alpha) \in E$, $\phi(J, z)$ does not contain 0 for any $J \in \mathscr{F}$ and so $f(z) \neq 0$. Thus f is not identically zero on E.

It remains to show that f is bounded. Any bounded set in E is contained in one of the form

$$B = \times\{B_\alpha : \alpha \in I\},$$

where each B_α is bounded in E_α. Let V be any neighbourhood of the origin in F and put

$$J_n = \{\alpha : t_\alpha(B_\alpha) \subseteq nV\}.$$

Since each t_α is bounded, every α is in some J_n. It follows that $J_m \in \mathscr{F}$ for some m, since otherwise the δ-ultrafilter $\mathscr{F}$ would contain $I \backslash J_n$ for each n and also the intersection of these sets, which is empty. This means that, for all $x \in B$,

$$f(x) \in \phi(J_m, x) \subseteq mV,$$

and the proof is complete.

The proposition shows, for example, that even if E_0 is F-bornological for every topological vector space F (e.g. if E_0 is metrisable), nevertheless E_0^I is not E_0-bornological when I fails to satisfy the Mackey–Ulam condition.

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