PDE FOR THE JOINT LAW OF THE PAIR OF A CONTINUOUS DIFFUSION AND ITS RUNNING MAXIMUM

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Abstract

Let X be a d-dimensional diffusion and M the running supremum of its first component. In this paper, we show that for any $t > 0$, the density (with respect to the $(d+1)$-dimensional Lebesgue measure) of the pair $\left(M_t, X_t\right)$ is a weak solution of a Fokker–Planck partial differential equation on the closed set $\left\{(m, x) \in \mathbb{R}^{d+1}, \ m \geq x^1\right\}$, using an integral expansion of this density.

Keywords: Diffusion process; partial differential equation; running supremum process; joint law

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1. Introduction

The goal of this paper is to study the law of the pair (M_t, X_t) where X is a d-dimensional diffusion and M is the running maximum of the first component. In a previous work [9], using Malliavin calculus and specifically Nualart's seminal book [21], we have proved that, for any $t > 0$, the law of $V_t := \left(M_t, X_t\right)$ is absolutely continuous with respect to the Lebesgue measure with density $p_V(.; t)$, and that the support of this density is included in the set $\left\{(m, x) \in \mathbb{R}^{d+1}, \ m \geq x^1\right\}$.

In the present work, we prove that the density p_V is a weak solution of a partial differential equation (PDE). Furthermore, we exhibit a boundary condition on the set $\{(m, x) \in \mathbb{R}^{d+1}, \ m = x^1\}$. This work extends the results given in [8] and in Ngom's thesis [20], obtained in the case where X is a Lévy process, where it is proved that the density is a weak solution to an integro-differential equation.

In the literature, there exist many studies of the law of V_t. When the process X is a Brownian motion, one can refer to [15, 17], where an explicit expression for p_V is given. When X is a one-dimensional linear diffusion, [11] provides an expression for p_V using the scale function, the speed measure, and the density of the law of some hitting times. See also [1, 4] for the particular case of an Ornstein–Uhlenbeck process. For some applications to the local score of a biological sequence, see [19], which presents the case of reflected Brownian motion. The

law of the maximum M_t is studied in [2] for general Gaussian processes. The case of a Lévy process X has been deeply investigated in the literature; see for instance [12, 20]. Moreover, Section 2.4 in Ngom's thesis [20] provides the existence and the regularity of the joint law density of the process (M_t, X_t) for a Lévy process X. In the case where X is a martingale (see e.g. [13, 22] or [10, 16]), the law of the running maximum is provided.

Such studies concerning this running maximum are useful in the areas of finance which involve hitting times, for instance for the pricing of barrier options. It is known that the law of hitting times is closely related to that of the running maximum; see [6, 7, 23]. As an application of our work, consider a firm whose activity is characterized by a set of processes $(X^1, \ldots, X^d)$, one of which, e.g. X^1, is linked to an alarm; namely, when there exists $s \leq t$ such that X_s^1 exceeds a threshold a, which is equivalent to $M_t = \sup_{0 \leq s \leq t} X_s^1 \geq a$, it is important to perform some action. So the firm needs to know the law of such a pair (M_t, X_t); more specifically, the law of the stopping time $\tau_a = \inf\{u, X_u^1 \geq a\}$ is linked to the law of M as follows: $\{\tau_a \leq t\} = \{M_t \geq a\}$. To know the probability of such an alarm, it is useful to know the law of the pair (M_t, X_t).

We provide an infinite expansion of the density of the law of the pair (M_t, X_t), which can lead to a numerical approximation.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a d-dimensional Brownian motion W. Let X be the diffusion process with values in $\mathbb{R}^d$ which solves

$$dX_t = B(X_t)dt + A(X_t)dW_t, \quad t > 0, \tag{1}$$

where X_0 is a random variable independent of the Brownian motion W, with law μ_0, and A (resp. B) is a map from $\mathbb{R}^d$ to the set of $(d \times d)$ matrices (resp. to $\mathbb{R}^d$). Let us denote by $C_b^i(\mathbb{R}^d, \mathbb{R}^n)$ the set of functions on $\mathbb{R}^d$ which are i times differentiable, bounded, with bounded derivatives, taking their values in $\mathbb{R}^n$. Let $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ be the completed right-continuous filtration defined by $\mathcal{F}_t := \sigma\{X_0, W_s, s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the set of negligible sets of $\mathcal{F}$.

Under classical assumptions on A and B (cf. (4) and (5) below), according to [9], for all $t > 0$, the law of $V_t = (\sup_{u \leq t} X_t^1, X_t)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^{d+1}$.

The main results and notation are given in Section 2: in the d-dimensional case, under a quite natural assumption (namely, Hypothesis 2.1 below) on the regularity of p_V around the boundary of Δ, we have that p_V is a weak solution of a Fokker–Planck PDE on the subset of $\mathbb{R}^{d+1}$ defined by $\{(m, x), m \geq x^1\}$. When $A = I_d$, this assumption is satisfied; see Theorem 2.4. The main results are proved in Section 3 under Hypothesis 2.1. Section 4 is devoted to proving that Hypothesis 2.1 is satisfied when $A = I_d$. The main tool is an infinite expansion of p_V given in Proposition 3.2 In Section 5, which treats the one-dimensional case, a Lamperti transformation [18] allows us to get the main result for any $A \in C_b^2(\mathbb{R}, \mathbb{R})$. Finally, the appendix contains some technical tools that are useful for the proofs of main results.

2. Main results and some notation

In this section, we give our main results. (As mentioned in the introduction, the proofs will be given later on.)

2.1. Notation

Let Δ be the open set of $\mathbb{R}^{D+1}$ given by $\Delta := \{(m, x), m \in \mathbb{R}, x \in \mathbb{R}^d, m > x^1, x = (x^1, \ldots, x^d)\}$. From now on, we use Einstein's convention. The infinitesimal generator $\mathcal{L}$

of the diffusion X defined in (1) is the partial differential operator on the space $C^2(\mathbb{R}^d, \mathbb{R})$ given by

$$\mathcal{L} = B^i \partial_{x_i} + \frac{1}{2}(AA^t)^{ij}\partial^2_{x_i, x_j}, \tag{2}$$

where A^t denotes the transposed matrix.

Its adjoint operator is $\mathcal{L}^* f = \frac{1}{2}\Sigma^{ij}\partial^2_{ij}f - [B^i - \partial_j(\Sigma^{ij})]\partial_i f - [\partial_i B^i - \frac{1}{2}\partial^2_{ij}(\Sigma^{ij})]f$, where $\Sigma := AA^t$. In what follows, the operators $\mathcal{L}$ and $\mathcal{L}^*$ are extended to the space $C^2(\mathbb{R}^{d+1}, \mathbb{R})$, for $\Phi \in C^2(\mathbb{R}^{d+1}, \mathbb{R})$, as

$$\mathcal{L}(\Phi)(m, x) = B^i(x)\partial_{x_i}\Phi(m, x) + \frac{1}{2}\Sigma^{ij}(x)\partial^2_{x_i, x_j}\Phi(m, x)$$

and

$$\mathcal{L}^*(\Phi)(m, x) =$$
$$\frac{1}{2}\Sigma^{ij}(x)\partial^2_{ij}\Phi(m, x) - [B^i - \partial_j(\Sigma^{ij})](x)\partial_{x_i}\Phi(m, x) + \left[\frac{1}{2}\partial^2_{x_i, x_j}\Sigma^{ij} - \partial_{x_i}B^i\right](x)\Phi(m, x).$$

We stress that these operators are degenerate, since no derivative with respect to the variable m appears.

Let $A^1(x)$ be the d-dimensional vector $A^1(x) = (A^1_j(x), j = 1, \ldots, d) \in \mathbb{R}^d$ corresponding to the first column of $A(x)$; similarly $A_j(x)$ denotes its jth row.

Recall that M denotes the running maximum of the first component of X, meaning $M_t = \sup_{0 \le s \le t}\{X^1_s\}$, and V is the $\mathbb{R}^{d+1}$-valued process defined by $(V_t = (M_t, X_t), \forall t \ge 0)$. Finally, $\tilde{x} \in \mathbb{R}^{d-1}$ denotes the vector $(x_2, \ldots, x_d)$.

In [9], under Assumptions (4) and (5) below, when the initial value is deterministic, $X_0 = x_0 \in \mathbb{R}^d$, the density of V_t exists and is denoted by $p_V(.; t, x_0)$. If μ_0 is the distribution of X_0, the density of the law of V_t with respect to the Lebesgue measure on $\mathbb{R}^{d+1}$ is

$$p_V(.; t, \mu_0) := \int_{\mathbb{R}^d} p_V(.; t, x_0)d\mu_0(x_0). \tag{3}$$

When there is no ambiguity, the dependency on μ_0 is omitted.

Since $M_t \ge X^1_t$, the support of $p_V(.; t, \mu_0)$ is contained in $\bar{\Delta} := \{(m, x) \in \mathbb{R}^{d+1} | m \ge x^1\}$.

2.2. Main results

The aim of this article is to show that the density p_V is a weak solution of a Fokker–Planck PDE. We assume that the coefficients B and A satisfy

$$B \in C^1_b(\mathbb{R}^d, \mathbb{R}^d) \quad \text{and} \quad A \in C^2_b(\mathbb{R}^d, \mathbb{R}^{d \times d}) \tag{4}$$

and that there exists a constant $c > 0$ such that the Euclidean norm of any vector v satisfies

$$c\|v\|^2 \le v^t A(x)A^t(x)v, \quad \forall v, x \in \mathbb{R}^d. \tag{5}$$

Our first result will be established under the following hypothesis, which is a quite natural assumption on the regularity of p_V in the neighborhood of the boundary of Δ, since the set of times where the process M increases is included in the set $\{t, M_t = X_t\}$.

Hypothesis 2.1. *The density of the law of* $V_t = (M_t, X_t)$, *denoted by* p_V *(see* (3)*), satisfies the following:*

(i) *The map* $(t, m, \tilde{x}) \mapsto \sup_{u>0} p_V(m, m - u, \tilde{x}; t)$ *belongs to* $L^1([0, T] \times \mathbb{R}^d, dtdmd\tilde{x})$.

(ii) *For all* $t > 0$, *almost surely in* $(m, \tilde{x}) \in \mathbb{R}^d$, $\lim_{u \to 0^+} p_V(m, m - u, \tilde{x}; t)$ *exists and is denoted by* $p_V(m, m, \tilde{x}; t)$.

Theorem 2.2. *Assume that A and B fulfil* (4) *and* (5) *and that* (M, X) *fulfils Hypothesis* 2.1. *Then, for any initial law* μ_0 *and* $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$,

$$
\mathbb{E}\left[F(M_t, X_t)\right) = \mathbb{E}\left[F(X_0^1, X_0)\right] + \int_0^t \mathbb{E}\left[\mathcal{L}(F)(M_s, X_s)\right] ds
$$
$$
+ \frac{1}{2} \int_0^t \mathbb{E}\left[\partial_m F(X_s^1, X_s) \|A^1(X_s)\|^2 \frac{p_V(X_s^1, X_s; s)}{p_X(X_s; s)}\right] ds. \tag{6}
$$

Actually, p_X is the solution of the PDE $\partial_t p = \mathcal{L}^* p$, $p(.; 0) = \mu_0$, where

$\mathcal{L}^* f = \frac{1}{2} \Sigma^{ij} \partial_{ij}^2 f - \left[B^i - \partial_j(\Sigma^{ij})\right] \partial_i f - \left[\partial_i B^i - \frac{1}{2} \partial_{ij}^2(\Sigma^{ij})\right] f$. *Let* $a_{ij} := \Sigma^{ij}$,

$a_i := \left[B^i - \partial_j(\Sigma^{ij})\right] \partial_i$, *and* $a_0 := \partial_i B^i - \frac{1}{2} \partial_{ij}^2(\Sigma^{ij})$. *Under Assumptions* (4) *and* (5)*, the operator* $\mathcal{L}^*$ *satisfies all the assumptions of* [14, Theorem 3.5] *(see* (3.2), (3.3), *and* (3.4) *on p. 177). As a consequence of* [14, Theorem 3.5], *line 14, we have* $p_X(x; s) > 0$.

Remark 2.1. (i) When A is the identity matrix of $\mathbb{R}^d$ (denoted by I_d) and $B \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$, Hypothesis 2.1 is fulfilled; see Theorem 2.4 below. When $d = 1$, using a Lamperti transformation [18], one can prove that Hypothesis 2.1 is always fulfilled; see Section 5.

(ii) This result is similar to [8, Theorem 2.1], where the process X is a Lévy process. Proposition 4 in [8] gives a key to the last term in (6) with factor $\frac{1}{2}$. First, roughly speaking, the local behavior of $X_t^1 - X_s^1$ conditionally on $\mathcal{F}_s$ is that of $\|A^1(X_s)\| (W_t^1 - X_s^1)$. So, as in the Brownian case, the running maximum M of X^1 is increasing as soon as it is equal to X^1 and both M and X^1 are increasing. It is well known that the Brownian process W^1 is increasing with probability $\frac{1}{2}$; more specifically, we have $\mathbb{P}\left\{\lim_{t \to s+} \frac{W_t^1 - W_s^1}{t - s} = -\infty\right\} = \mathbb{P}\left\{\lim_{t \to s+} \frac{W_t^1 - W_s^1}{t - s} = +\infty\right\} = \frac{1}{2}$.

The starting point of the proof of Theorem 2.2 is Itô's formula: let F belong to $C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$. The process M is increasing; hence $V = (M, X)$ is a semi-martingale. Applying Itô's formula to $F(V)$ and taking the expectation of both sides, we have

$$
\mathbb{E}[F(V_t)] = \mathbb{E}[F(V_0)] + \int_0^t \mathbb{E}[\mathcal{L}(F)(V_s)] ds + \mathbb{E}\left[\int_0^t \partial_m F(V_s) dM_s\right].
$$

The novelty comes from the third term in the right-hand side of the equation above. The following theorem, proved in Section 3, completes the proof of Theorem 2.2.

Theorem 2.3. *Assume that A and B fulfil* (4) *and* (5) *and that* (M, X) *fulfils Hypothesis* 2.1. *For every* $\Psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$, *let* F_ψ *be the map*

$$
F_\psi : t \mapsto \mathbb{E}\left[\int_0^t \Psi(M_s, X_s) dM_s\right].
$$

Then F_Ψ is absolutely continuous with respect to the Lebesgue measure, and its derivative is

$$\dot{F_\Psi}(t) = \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x}.$$

Observe that, as expressed in Theorem 2.2, *this derivative can be written as*

$$\frac{1}{2} \mathbb{E} \left[\Psi\left(X_t^1, X_t\right) \|A^1(X_t)\|^2 \frac{p_V\left(X_t^1, X_t; t\right)}{p_X(X_t; t)} \right].$$

Remark 2.2. The above proposition provides an explicit formulation of the derivative of the function F_Ψ. Note that the absolute continuity of F_ψ could be established as a direct consequence of the existence of the density of the law of the hitting time $\tau_a = \inf\{s : X_s^1 \geq a\}$ when it exists, using the identity $\{\tau_a \leq t\} = \{M_t \geq a\}$. Conversely, it could be proved that the absolute continuity of F_Ψ yields the existence of the density of the law of the hitting time τ_a, using a sequence of $C_b^2(\mathbb{R}, \mathbb{R})$ functions (F_n) approximating the indicator function $\mathbf{1}_{[a,\infty)}$; namely, this density satisfies $f_{\tau_a}(t) = \frac{1}{2} \int_{\mathbb{R}^{d-1}} p_V(a, a, \tilde{x}; t) d\tilde{x}$.

Theorem 2.4. *Assume that $A = I_d$ and B satisfies Assumption* (4). *Then, for all $t > 0$, the distribution of the pair $\left(M_t, X_t\right)$ fulfils Hypothesis* 2.1. *As a consequence, for all $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$,*

$$\mathbb{E}\left[F\left(M_t, X_t\right)\right] = \mathbb{E}\left[F\left(X_0^1, X_0\right)\right] + \int_0^t \mathbb{E}\left[\mathcal{L}\left(F\right)\left(M_s, X_s\right)\right] ds$$

$$+ \frac{1}{2} \int_0^t \mathbb{E}\left[\partial_m F\left(X_s^1, X_s\right) \frac{p_V\left(X_s^1, X_s; s\right)}{p_X(X_s; s)}\right] ds.$$

Proof. This theorem is a consequence of Theorem 2.2 and Proposition 4.1. $\square$

When $d = 1$, a Lamperti transformation leads to the following corollary.

Corollary 1. *Assume that $d = 1$, and A and B satisfy* (4) *and* (5). *Then the density p_V satisfies Hypothesis* 2.1, *so*

$$\mathbb{E}\left[F\left(M_t, X_t\right)\right] = \mathbb{E}\left[F(X_0, X_0)\right] + \int_0^t \mathbb{E}\left[\mathcal{L}\left(F\right)\left(M_s, X_s\right)\right] ds$$

$$+ \frac{1}{2} \int_0^t \mathbb{E}\left[A^2(X_s) \partial_m F(X_s, X_s) \frac{p_V(X_s, X_s; s)}{p_X(X_s; s)}\right] ds.$$

Remark 2.3. If p_V is regular enough, and if the initial law of X_0 satisfies $\mu_0(dx) = f_0(x)dx$, then Theorem 2.2 means that p_V is a weak solution in the set Δ of $\partial_t p = \mathcal{L}^* p$, where $\mathcal{L}^* f = \frac{1}{2} \Sigma^{ij} \partial_{ij}^2 f - \left[B^i - \partial_j\left(\Sigma^{ij}\right)\right] \partial_i f - \partial_i B^i - \frac{1}{2} \partial_{ij}^2\left(\Sigma^{ij}\right)) f$ with boundary condition

$$B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) = \partial_{x_k}\left(\Sigma^{1,k} p_V\right)(m, m, \tilde{x}; s) + \frac{1}{2} \partial_m\left(\|A^1\|^2 p_V\right)(m, m, \tilde{x}; s). \quad (7)$$

This result is proved in Appendix A.3.

This boundary condition also appears in [4, Proposition 4, Equation (11)] (Ornstein–Uhlenbeck process). Finally, a similar PDE is studied in [14, Chapter 1.2], where the authors establish the existence of a unique strong solution of the PDE, but in the case of a non-degenerate elliptic operator.

3. Proof of Theorem 2.3

We start this section with a roadmap for the proof of Theorem 2.3. First we compute the right derivative of the map $F_\Psi : t \mapsto \mathbb{E}\left[\int_0^t \Psi(M_s, X_s)dM_s\right]$, namely $\lim_{h \to 0^+} T_{h,t}$ with $T_{h,t} = \frac{1}{h}\mathbb{E}_\mathbb{P}\left[\int_t^{t+h} \psi(V_s)dM_s\right]$. A first step is the decomposition

$$T_{h,t} = \frac{1}{h}\mathbb{E}_\mathbb{P}\left[\int_t^{t+h}(\psi(V_s) - \psi(V_t))dM_s\right] + \frac{1}{h}\mathbb{E}_\mathbb{P}[\psi(V_t)(M_{t+h} - M_t)]. \tag{8}$$

Since $\psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$ and the process M is increasing, the first term in (8) is dominated by

$$\mathbb{E}\left[\int_t^{t+h}(\psi(V_s) - \psi(V_t))dM_s\right] \leq \|\nabla\psi\|_\infty \mathbb{E}\left[\sup_{t \leq s \leq t+h}\|V_s - V_t\|(M_{t+h} - M_t)\right].$$

Lemma 3.1 states that $\sup_{t \leq s \leq t+h}\|X_s - X_t\|_p = O(\sqrt{h})$, and Lemma 3.2 yields $\|M_{t+h} - M_t\|_p = o(\sqrt{h})$, so that that this first term is $o(h)$.

Concerning the second term in (8), $M_{t+h} - M_t$ can be written as $\sup_{0 \leq u \leq h}\left(X_{t+h}^1 - X_t^1 - M_t + X_t^1\right)_+$. In order to use the independence of the increments of Brownian motion, we introduce a new process, independent of $\mathcal{F}_t$, which is an approximation of $X_{t+u}^1 - X_t^1$:

$$X_{t,u}^1 := A_k^1(X_t)\hat{W}_u^k \quad \text{where} \quad \hat{W}_u^k := W_{t+u}^k - W_t^k; M_{t,h} := \sup_{0 \leq u \leq h} X_{t,u}^1. \tag{9}$$

Lemma 3.4(ii) will set $\mathbb{E}\left[|M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+|\right] = o(h)$, where $(x)_+ = \max(x, 0)$. Thus,

$$\frac{1}{h}\mathbb{E}[\psi(V_t)(M_{t+h} - M_t)] = \mathbb{E}\left[\psi(V_t)(M_{t,h} - M_t + X_t^1)_+\right] + o(h). \tag{10}$$

Observe that the law of $M_{t,h}$ given $\mathcal{F}_t$ is the law of $\|A^1(X_t)\| \sup_{0 \leq u \leq h} \hat{W}_u^1$; then, using the function $\mathcal{H}$ from (13), an $\mathcal{F}_t$-conditioning yields

$$\frac{1}{h}\mathbb{E}[\psi(V_t)(M_{t+h} - M_t)] = \frac{2}{\sqrt{h}}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right] + o(h). \tag{11}$$

Then

$$T_{h,t} = \frac{2}{\sqrt{h}}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right] + o(h),$$

as given in Proposition 3.1(ii). In Proposition 3.2, under Hypothesis 2.1, we compute $\lim_{h \to 0} T_{h;t}$.

Finally in Section 3.4 we prove that $F_\psi : t \mapsto E\left[\int_0^t \psi(V_s)dM_s\right]$ is an absolutely continuous function with respect to the Lebesgue measure, integral of its right derivative. Actually we prove that F_ψ is a continuous function belonging to the Sobolev space $W^{1,1}(I)$, $I = (0, T)$. This completes the proof of Theorem 2.3.

The main propositions to prove are the following.

Proposition 3.1. *Let B and A fulfil* (4) *and* (5), *and let* $\Psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$. *Recall that* A^1 *is the vector* $(A_j^1, \ j = 1, \ldots, d)$, *and* $\|A^1(x)\|^2 = \sum_{j=1}^d (A_j^1(x))^2$.

(i) *For every* $T > 0$, *there exists a constant* $C > 0$ *(depending on* $\|A\|_\infty$, $\|B\|_\infty$, $\|\nabla A\|_\infty$, $\|\Psi\|_\infty$, *and* $\|\nabla\Psi\|_\infty$, *and* T) *such that for all* $t \in [0, T]$ *and* $h \in [0, 1]$,

$$\left| \mathbb{E}\left[\int_t^{t+h} \Psi(V_s)dM_s - 2\sqrt{h}\left(\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|} \right) \right) \right] \right| \leq Ch\|\nabla\Psi\|_\infty. \tag{12}$$

(ii) *For all* $t > 0$ *and* $h \in [0, 1]$,

$$\lim_{h\to 0+} \frac{1}{h} \left| \mathbb{E}\left[\int_t^{t+h} \Psi(V_s)dM_s \right] - 2\sqrt{h}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|} \right) \right] \right| = 0,$$

where, denoting by Φ_G *the standard Gaussian cumulative distribution function, we write*

$$\mathcal{H}(\theta) := \int_\theta^\infty \frac{1}{\sqrt{2\pi}}(y - \theta)e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} - \theta\Phi_G(-\theta). \tag{13}$$

The following remark will be useful.

Remark 3.1. The definition of $\mathcal{H}$ in (13) implies that $\int_0^\infty \mathcal{H}(u)du = 1/4$. Moreover, $\mathcal{H}'(\theta) = -\Phi_G(-\theta) \leq 0$; in particular, $\mathcal{H}$ is non-increasing.

Proposition 3.2. *Assume that A and B fulfil* (4) *and* (5) *and that* (M, X) *fulfils Hypothesis* 2.1. *Then for all* $\Psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$, *all* $T > 0$, *and all* $t \geq 0$, *the following hold:*

(i) $\quad t \mapsto \sup_{h>0} \dfrac{2\sqrt{h}}{h} \mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\dfrac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|} \right) \right] \in L^1([0, T], \mathbb{R});$

(ii) $\quad \lim_{h\to 0^+} \dfrac{2\sqrt{h}}{h} \mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\dfrac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|} \right) \right]$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x})\|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t)dm d\tilde{x}$$

As a corollary, the function

$$t \to \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x})\|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t)dm d\tilde{x}$$

belongs to $L^1([0, T], \mathbb{R})$.

The proof of Proposition 3.1 will be obtained using the lemmas in the following section.

3.1. Tools for proving Proposition 3.1

Here we provide some estimates of the expectations of the increments of the processes X and M. Assumptions (4) and (5) allow us to introduce a constant K which denotes either

max($\|A\|_\infty$, $\|B\|_\infty$) or max($\|A\|_\infty$, $\|B\|_\infty$, $\|\nabla A\|_\infty$). Let C_p be the constant in the Burkholder–Davis–Gundy inequality (cf. [3, Theorem B.36]).

Lemma 3.1. *Let A and B be bounded. Then, for all $0 < h \leq 1$, for all $p \geq 1$ there exists a constant $C_{p,K}$ (depending only on p and K) such that*

$$\sup_{t>0} \mathbb{E}\left[\sup_{0 \leq s \leq h} \|X_{t+s} - X_t\|^p\right] \leq C_{p,K} h^{p/2}.$$

Proof. Using the fact that $(a+b)^p \leq 2^{p-1}\left[a^p + b^p\right]$, $a, b \geq 0$, one obtains

$$0 \leq \sup_{s \leq h} \|X_{t+s} - X_t\|^p \leq 2^{p-1}\left[\sup_{u \leq h}\left(\left\|\int_t^{t+u} B(X_s)ds\right\|\right)^p + \sup_{u \leq h}\left(\left\|\int_t^{t+u} A_j(X_s)dW_s^j\right\|\right)^p\right].$$

If we take the expectation of both sides, the Burkholder–Davis–Gundy inequality implies

$$\mathbb{E}\left[\sup_{s \leq h} \|X_{t+s} - X_t\|^p\right] \leq 2^{p-1}(1 + C_p)\mathbb{E}\left[\left(\int_t^{t+h} \|B(X_s)\|ds\right)^p + \left(\int_t^{t+h} \|A(X_s)\|^2 ds\right)^{p/2}\right].$$

Assumption (4) on B and A yields

$$\mathbb{E}\left[\sup_{s \leq h} \|X_{t+s} - X_t\|^p\right] \leq 2^{p-1}(1 + C_p)\left(h^p K^p + h^{p/2} K^p\right).$$

$\square$

Lemma 3.2. *Let B and A satisfy Assumptions (4) and (5). Then, for all $0 < h \leq 1$, for all $p \geq 1$ we get*

$$\sup_{t>0} \mathbb{E}\left[|M_{t+h} - M_t|^p\right] \leq C_{p,K} h^{p/2} ; \mathbb{E}\left[|M_{t+h} - M_t|^p\right] = o\left(h^{p/2}\right). \tag{14}$$

Proof. Recall that

$$M_{t+h} - M_t = \left(\sup_{0 \leq u \leq h}\left(X_{t+u}^1 - X_t^1\right) + X_t^1 - M_t\right)_+,$$

recalling $(x)_+ = \max(x, 0)$. For any $a \geq 0$, one has $(x - a)_+ \leq |x| \mathbf{1}_{\{x>a\}}$; thus

$$0 \leq M_{t+h} - M_t \leq \left|\sup_{0 \leq u \leq h}\left(X_{t+u}^1 - X_t^1\right)\right| \mathbf{1}_{\left\{\sup_{0 \leq u \leq h}\left(X_{t+u}^1 - X_t^1\right) > M_t - X_t^1\right\}}.$$

The Cauchy–Schwarz inequality yields

$$0 \leq \mathbb{E}\left[(M_{t+h} - M_t)^p\right]$$

$$\leq \sqrt{\mathbb{E}\left[\left|\sup_{0 \leq u \leq h}\left(X_{t+u}^1 - X_t^1\right)\right|^{2p}\right] \mathbb{P}\left(\left\{\sup_{0 \leq u \leq h}\left(X_{t+u}^1 - X_t^1\right) > M_t - X_t^1\right\}\right)}.$$

Replacing p by $2p$ in Lemma 3.1 leads to the inequality in (14), and the equality

$$\lim_{h \to 0} \sup_{0 \leq u \leq h}\left(X_{t+u}^1 - X_t^1\right) = 0$$

holds almost surely. According to [9, Theorem 1.1] extended to X_0 with law μ_0 on $\mathbb{R}^d$, the pair (M_t, X_t) admits a density; thus $\mathbb{P}\{M_t - X_t^1 = 0\} = 0$ holds almost surely. Therefore $E([M_{t+h} - M_t]^p)$ is bounded by the product of $h^{p/2}$ and a factor going to zero when h goes to 0, and this quantity is $o(h^{p/2})$. $\qquad\square$

For any fixed t we recall the process $(X_{t,u}, \ u \in [0, h])$ and the running maximum of its first component as follows:

$$X_{t,u} := \sum_j A_j(X_t)\hat{W}_u^j, \qquad M_{t,h} := \sup_{0 \le u \le h} X_{t,u}^1. \tag{15}$$

Lemma 3.3. *Under Assumptions* (4) *and* (5), *for every* $p \ge 1$ *there exists a constant* $C_{p,K}$ *such that for all* $t \le T$, *for all* $h \in [0, 1]$,

$$\mathbb{E}\left[\sup_{s \le h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p\right] \le C_{p,K}h^p.$$

Proof. By definition, recalling $\hat{W}_u := W_{t+u} - W_t, u \ge 0$, we obtain

$$X_{s+t}^1 - X_t^1 - X_{t,s}^1 = \int_0^s B^1(X_{u+t})du + \int_0^s \left[A^1(X_{u+t}) - A^1(X_t)\right]d\hat{W}_u.$$

Once again using the inequality $(a + b)^p \le 2^{p-1}(a^p + b^p), \ a, b \ge 0$, we get

$$\sup_{0 \le s \le h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p \le$$

$$2^{p-1}\left[\left(\int_0^h \|B^1(X_{u+t})\|du\right)^p + \sup_{0 \le s \le h}\left\|\int_0^s \left[A^1(X_{u+t}) - A^1(X_t)\right]d\hat{W}_u\right\|^p\right].$$

Taking the expectation of both sides and applying the Burkholder–Davis–Gundy inequality yields, with $D_p = 2^{p-1}(1 + C_p)$,

$$\mathbb{E}\left[\sup_{0 \le s \le h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p\right]$$

$$\le D_p\left(\mathbb{E}\left[\int_0^h \|B^1(X_{u+t})\|du\right]^p + \mathbb{E}\left|\int_0^h \|A^1(X_{u+t}) - A^1(X_t)\|^2 du\right|^{p/2}\right).$$

The first term above is bounded by $K^p h^p$, since B is bounded. The assumption that A belongs to $C_b^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$ and Jensen's inequality imply that the second term is bounded by $K^p h^{p/2-1}\int_0^h E\|X_{u+t} - X_t\|^p du$; thus

$$\mathbb{E}\left[\sup_{0 \le s \le h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p\right] \le D_p K^p h^{p/2-1}\left(h^{p/2+1} + \int_0^h \mathbb{E}\|X_{u+t} - X_t\|^p du\right).$$

From Lemma 3.1 we obtain the uniform upper bound $\mathbb{E}[\|X_{u+t} - X_t\|^p] \le C_{p,K}u^{p/2}$; hence,

$$\mathbb{E}\left[\sup_{s \le h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p\right] \le \frac{D_p K^p C_{p,K}}{\frac{p}{2} + 1}h^p.$$

$\qquad\square$

Lemma 3.4. *Under Assumptions* (4) *and* (5), *the following hold:*

(i) *There exists $C > 0$ such that*

$$\sup_{0 \le t \le T; \ 0 \le h \le 1} h^{-1} \mathbb{E}\left[\left| M_{t+h} - M_t - \left(M_{t,h} - M_t + X_t^1 \right)_+ \right|\right] \le C < \infty.$$

(ii) *We have*

$$\lim_{h \to 0^+} h^{-1} \mathbb{E}\left[\left| M_{t+h} - M_t - \left(M_{t,h} - M_t + X_t^1 \right)_+ \right|\right] = 0.$$

Proof. First, observe that

$$\forall a \in \mathbb{R}, \ \left| (x - a)_+ - (y - a)_+ \right| \le |x - y| \left[\mathbf{1}_{\{x > a\}} + \mathbf{1}_{\{y > a\}} \right], \tag{16}$$

and if f and g are functions on $[0, T]$, then

$$\forall s \in [0, T], \ f(s) - \sup_{0 \le u \le T} g(u) \le f(s) - g(s) \le |f(s) - g(s)| \le \sup_{v \le T} |f(v) - g(v)|;$$

hence $\sup_{s \le T} f(s) - \sup_{u \le T} g(u) \le \sup_{v \le T} |f(v) - g(v)|$. Here the roles of f and g are symmetrical, so $\sup_{s \le T} g(s) - \sup_{u \le T} f(u) \le \sup_{v \le T} |f(v) - g(v)|$, and

$$\left| \sup_{s \le T} g(s) - \sup_{u \le T} f(u) \right| \le \sup_{v \le T} |f(v) - g(v)|. \tag{17}$$

We now consider $M_{t+h} - M_t = \left(\sup_{0 \le u \le h} \left(X_{u+t}^1 - X_t^1 \right) - M_t + X_t^1 \right)_+$; using (16),

$$\left| M_{t+h} - M_t - \left(M_{t,h} - M_t + X_t^1 \right)_+ \right| \le$$
$$\left| \sup_{0 \le u \le h} \left(X_{u+t}^1 - X_t^1 \right) - M_{t,h} \right| \left[\mathbf{1}_{\left\{ \sup_{0 \le u \le h} \left(X_{u+t}^1 - X_t^1 \right) > M_t - X_t^1 \right\}} + \mathbf{1}_{\left\{ M_{t,h} > M_t - X_t^1 \right\}} \right].$$

Then, for any t fixed, we apply the inequality (17) to the maps $g : u \mapsto X_{u+t}^1 - X_t^1$ and $f : u \mapsto X_{t,u}^1$. Then

$$\left| M_{t+h} - M_t - \left(M_{t,h} - M_t + X_t^1 \right)_+ \right| \le$$
$$\sup_{0 \le u \le h} \left| X_{u+t}^1 - X_t^1 - X_{t,u}^1 \right| \left[\mathbf{1}_{\left\{ \sup_{0 \le u \le h} \left(X_{u+t}^1 - X_t^1 \right) > M_t - X_t^1 \right\}} + \mathbf{1}_{\left\{ M_{t,h} > M_t - X_t^1 \right\}} \right].$$

From the Cauchy–Schwarz inequality and the fact that $(a + b)^2 \le 2(a^2 + b^2)$, we get

$$\mathbb{E}\left[\left| M_{t+h} - M_t - \left(M_{t,h} - M_t + X_t^1 \right)_+ \right|\right] \le$$
$$\sqrt{2\mathbb{E}\left[\sup_{u \le h} \left| X_{u+t}^1 - X_t^1 - X_{t,u}^1 \right|^2 \right] \left(\mathbb{P}\left\{ \sup_{0 \le u \le h} \left(X_{u+t}^1 - X_t^1 \right) > M_t - X_t^1 \right\} + \mathbb{P}\left\{ M_{t,h} > M_t - X_t^1 \right\} \right)}.$$

Lemma [3.3](#) with $p = 2$ ensures that the map $h \mapsto h^{-1} \sqrt{2\mathbb{E}\left[\sup_{u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1|^2\right]}$ is uniformly bounded in t. Concerning the second factor, we have the following:

- First, the almost sure continuity with respect to h ensures that the quantities $\lim_{h \to 0} \sup_{0 \leq u \leq h}(X_{u+t}^1 - X_t^1)$ and $\lim_{h \to 0} M_{t,h}$ are equal to 0.

- Second, the law of the pair (M_t, X_t) admits a density with respect to the Lebesgue measure on $\bar{\Delta}$, according to [9, Theorem 1.1], so $\mathbb{P}(\{0 = M_t - X_t^1\}) = 0$, and the limit of the second factor is equal to 0.

This concludes the proof of the lemma. $\qquad\square$

Recall Definition [(15)](#): $X_{t,h} = A_j(X_t)[W_{t+h}^j - W_t^j]$, $M_{t,h} = \sup_{0 \leq \mathbf{u} \leq \mathbf{h}} X_{t,u}^1$, $h \in [0, 1]$.

Lemma 3.5. *Under Assumptions [(4)](#) and [(5)](#), with $\mathcal{H}$ defined in [(13)](#), we have*

$$\mathbb{E}\left[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t\right] = 2\|A^1(X_t)\|\sqrt{h}\mathcal{H}\left(\frac{M_t - X_t^1}{\|A^1(X_t)\|\sqrt{h}}\right).$$

Proof. For any t fixed, conditionally on $\mathcal{F}_t$, the process $(X_{t,u}^1, \ u \in [0, h])$ from [(9)](#) has the same law as $(\sqrt{h}\|A^1(X_t)\|\hat{W}_u, \ u \in [0, 1])$, where $\hat{W}$ is a Brownian motion independent of $\mathcal{F}_t$, and for any h, the random variable $M_{t,h}$ has the same law as $\sqrt{h}\|A^1(X_t)\| \sup_{u \leq 1} \hat{W}_u$.

Following [17, Section 3.1.3], the random variable $\sup_{u \leq 1} \hat{W}_u$ has the same law as $|G|$, where G is a standard Gaussian variable (independent of $\mathcal{F}_t$) with density $\frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathbf{1}_{[0,+\infty[}(z)$. Then, using the function $\mathcal{H}$ introduced in [(13)](#), we have

$$\mathbb{E}\left[(M_{t,h} - (M_t - X_t^1))_+ | \mathcal{F}_t\right] = \int_0^\infty \left(\|A^1(X_t)\|\sqrt{h}z - (M_t - X_t^1)\right)_+ \frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= 2\|A^1(X_t)\|\sqrt{h}\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right).$$

$\qquad\square$

3.2. Proof of Proposition [3.1](#)

Let $t > 0$. The key in this proof is to write the quantity

$$\mathbb{E}\left[\int_t^{t+h} \Psi(V_s)dM_s\right] - 2\sqrt{h}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right]$$

as the sum of three terms,

$$\mathbb{E}\left[\int_t^{t+h} (\Psi(V_s) - \Psi(V_t))dM_s\right] + \mathbb{E}\left[\Psi(V_t)\left((M_{t+h} - M_t) - \mathbb{E}\left[M_{t,h} - M_t + X_t^1\right)_+ | \mathcal{F}_t\right]\right)\right]$$

(18)

$$+ \mathbb{E}\left[\Psi(V_t)\mathbb{E}\left[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t\right] - 2\sqrt{h}\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right].$$

We now prove that each of the terms in the sum (18) are both $o(h)$ and $O(h)$ uniformly in time.

(a) Using Lemma 3.1, the third term is null.

(b) Concerning the second term, using the fact that Ψ is bounded and Lemma 3.1(i), for all $t \in [0, T]$

$$\left| \mathbb{E}\left[\Psi(V_t)[(M_{t+h} - M_t) - \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ \mid \mathcal{F}_t]] \right] \right| \leq$$

$$\|\Psi\|_\infty \left| \mathbb{E}\left[M_{t+h} - M_t - \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ \mid \mathcal{F}_t] \right] \right| \leq Ch \|\Psi\|_\infty,$$

as is required in (12). Moreover, using Lemma 3.4(ii),

$$\lim_{h \to 0} \frac{1}{h} \left| \mathbb{E}\left[\Psi(V_t)[(M_{t+h} - M_t) - \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ \mid \mathcal{F}_t]] \right] \right| = 0.$$

(c) Since $\nabla \Psi$ is bounded and the process M is increasing, the first term is bounded:

$$\mathbb{E}\left[\int_t^{t+h} [\Psi(V_s) - \Psi(V_t)] dM_s \right] \leq \|\nabla \Psi\|_\infty \mathbb{E}\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right].$$

Using the Cauchy–Schwarz inequality,

$$\mathbb{E}\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right] \leq \sqrt{\mathbb{E}\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\|^2 \right] \mathbb{E}\left[(M_{t+h} - M_t)^2 \right]}.$$

Since $\|V_s - V_t\|^2 = (M_s - M_t)^2 + \|X_s - X_t\|^2$, we obtain

$$\sup_{t \leq s \leq t+h} \|V_s - V_t\|^2 \leq (M_{t+h} - M_t)^2 + \sup_{t \leq s \leq t+h} \|X_s - X_t\|^2;$$

hence

$$\mathbb{E}\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right] \leq$$

$$\sqrt{\mathbb{E}[(M_{t+h} - M_t)^2] + \mathbb{E}\left[\sup_{t \leq s \leq t+h} \|X_s - X_t\| \right]^2} \sqrt{\mathbb{E}[(M_{t+h} - M_t)^2]}.$$

Lemmas 3.1 and 3.2 ($p = 2$) yield the fact that the first factor is $o(\sqrt{h})$ and the second is $O(\sqrt{h})$ uniformly with respect to $t \geq 0$. Then $\mathbb{E}\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right]$ is $o(h)$ and $O(h)$ uniformly with respect to $t \geq 0$.   □

3.3. Proof of Proposition 3.2

(i) Recall that A and B fulfil (4), (5) and (M, X) fulfils Hypothesis 2.1. Then, using the density p_V of the law of the pair (M_t, X_t), we have

$$\mathbb{E}\left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \leq$$

$$\|\Psi\|_\infty \|A\|_\infty \int_{\mathbb{R}^{d+1}} \mathcal{H}\left(\frac{m - x^1}{\sqrt{h} \|A^1(x^1, \tilde{x})\|} \right) p_V(m, x^1, \tilde{x}; t) dm \, dx^1 \, d\tilde{x}.$$

The change of variable $x^1 = m - u\sqrt{h}$ yields

$$\frac{\sqrt{h}}{h}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right] \leq \tag{19}$$

$$\|\Psi\|_\infty\|A^1\|_\infty\int_{\mathbb{R}^d\times[0,+\infty[}\mathcal{H}\left(\frac{u}{\|A^1(m - \sqrt{h}u, \tilde{x})\|}\right)p_V(m, m - \sqrt{h}u, \tilde{x}; t)dm\,d\tilde{x}\,du.$$

Since $\mathcal{H}$ is decreasing (Remark 3.1) and $0 \leq h \leq 1$, we have

$$\mathcal{H}\left(\frac{u}{\|A^1(m - \sqrt{h}u, \tilde{x})\|}\right) \leq \mathcal{H}\left(\frac{u}{\|A^1\|_\infty}\right),$$

so

$$\left|\frac{\sqrt{h}}{h}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right]\right| \leq$$

$$\|\Psi\|_\infty\|A^1\|_\infty\int_{\mathbb{R}^d\times[0,+\infty[}\mathcal{H}\left(\frac{u}{\|A^1\|_\infty}\right)\sup_{r>0}p_V(m, m - r, \tilde{x}; t)dm\,d\tilde{x}\,du.$$

Applying Tonelli's theorem, computing the integral with respect to du in the right-hand side with $\int_0^\infty\mathcal{H}(v)dv = 1/4$ (Remark 3.1), we obtain

$$\sup_{h>0}\left|\frac{\sqrt{h}}{h}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right]\right|$$

$$\leq \frac{1}{4}\|\Psi\|_\infty\|A^1\|_\infty^2\int_{\mathbb{R}^d}\sup_{r>0}p_V(m, m - r, \tilde{x}; t)dm\,d\tilde{x}.$$

Using Hypothesis 2.1(i), we obtain that the map

$$t \mapsto \sup_{h>0}\left|\frac{\sqrt{h}}{h}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right]\right|$$

belongs to $L^1([0, T], \mathbb{R})$. Part (i) of Proposition 3.2 is proved.

(ii) For the proof of Part (ii), first note that

$$\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right]$$

$$= \int_{\mathbb{R}^{d+1}}\Psi(m, x)\|A^1(x)\|\mathcal{H}\left(\frac{m - x^1}{\sqrt{h}\|A^1(x)\|}\right)p_V(m, x; t)dm\,dx.$$

After the change of variable $x^1 = m - u\sqrt{h}$, we obtain

$$\frac{\sqrt{h}}{h}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)\right] = \tag{20}$$

$$\int_{\mathbb{R}^d\times\mathbb{R}^+}\Psi(m, m - u\sqrt{h}, \tilde{x})\|A^1(m - u\sqrt{h}, \tilde{x})\|\mathcal{H}\left(\frac{u}{\|A^1(m - \sqrt{h}u, \tilde{x})\|}\right)$$

$$p_V(m, m - \sqrt{h}u, \tilde{x}; t)dm\,d\tilde{x}\,du.$$

Using Lebesgue's dominated convergence theorem, we let h go to 0 in (20) for $t > 0$, and using the fact that Ψ, A, and $\mathcal{H}$ are continuous and Hypothesis 2.1(ii), we obtain

$$\lim_{h \to 0} \frac{\sqrt{h}}{h} \mathbb{E}\left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H}\left(\frac{M_t - X_t}{\sqrt{h}\|A^1(X_t)\|} \right) \right] =$$
$$\int_{\mathbb{R}^d \times [0, +\infty[} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\| \mathcal{H}\left(\frac{u}{\|A^1(m, \tilde{x})\|} \right) p_V(m, m, \tilde{x}; t) dm\, d\tilde{x}\, du.$$

Using the change of variable $z = \frac{u}{\|A^1(m, \tilde{x})\|}$ and Remark 3.1 $\int_0^\infty \mathcal{H}(z)dz = 1/4$ yields

$$\lim_{h \to 0} \frac{\sqrt{h}}{h} \mathbb{E}\left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|} \right) \right]$$
$$= \frac{1}{4} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm\, d\tilde{x}.$$

$\square$

3.4. End of proof of Theorem 2.3

We recall Theorem 8.2 from Brezis [5, p. 204]: let $f \in W^{1,1}(0, T)$; then f is almost surely equal to an absolutely continuous function. As a particular case, any $f \in W^{1,1}(0, T) \cap C(0, T)$ is absolutely continuous. Recall $F_\psi : t \mapsto \mathbb{E}\left[\int_0^t \Psi(V_s) dM_s \right]$.

Lemma 3.6. *Assume that A and B fulfil (4) and (5) and that Ψ is a continuous bounded function. Then F_Ψ is a continuous function on $\mathbb{R}^+$.*

Proof. Let $0 \le s \le t$. Since Ψ is bounded and M is non-decreasing,

$$|F_\Psi(t) - F_\Psi(s)| = \left| \mathbb{E}\left[\int_s^t \Psi(V_u) dM_u \right] \right| \le \|\Psi\|_\infty \mathbb{E}[M_t - M_s].$$

The map $t \mapsto \mathbb{E}[M_t]$ being continuous, F_Ψ is a continuous function on $\mathbb{R}^+$. $\square$

Lemma 3.7. *Assume that A and B fulfil (4) and (5), (M, X) fulfils Hypothesis 2.1, and $\Psi \in C_b^1$. Then for all $T > 0$, the map F_ψ belongs to the Sobolev space $W^{1,1}(]0, T[)$, and its weak derivative is*

$$\dot{F}_\Psi(t) := \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x}.$$

Proof. Let $g : [0, T] \to \mathbb{R}$ be C^1 with compact support $[\alpha, \beta] \subset (0, T)$. This means both functions g and $\dot{g}$ are continuous, hence bounded, and that moreover $g(\alpha) = g(\beta) = 0$. Note that

$$\dot{g}(t) = \lim_{h \to 0} \frac{g(t) - g(t - h)}{h}, \quad \forall t \in (0, T).$$

Moreover,

$$\sup_{t \in [0, T]} \sup_{h \in [0, 1]} |\frac{g(t) - g(t - h)}{h}| \le \|\dot{g}\|_\infty.$$

Observe that, since M is non-decreasing and the coefficients A and B are bounded,

$$\left| F_\psi(t) \right| \le \|\Psi\|_\infty \mathbb{E}[M_T] < \infty.$$

Then, using Lebesgue's dominated convergence theorem, we have

$$\int_0^T \dot{g}(s)F_\psi(s)ds = \int_0^T \lim_{h\to 0} \frac{g(s)-g(s-h)}{h}F_\psi(s)ds = \lim_{h\to 0} \int_0^T \frac{g(s)-g(s-h)}{h}F_\psi(s)ds.$$

Using the change of variable $u = s - h$ in the last integral, we have

$$\int_0^T \frac{g(s)-g(s-h)}{h}F_\Psi(s)ds = h^{-1}\int_0^T g(s)F_\Psi(s)ds - h^{-1}\int_{-h}^{T-h} g(u)F_\Psi(u+h)du$$

$$= \int_0^T g(s)\frac{F_\Psi(s)-F_\Psi(s+h)}{h}ds - h^{-1}\int_{-h}^0 g(s)F_\Psi(s+h)ds + h^{-1}\int_{T-h}^T g(s)F_\Psi(s+h)ds.$$

Recalling that $supp(g) = [\alpha, \beta] \subset (0, T)$, gF_Ψ extended by 0 on $[\alpha, \beta]^c$ is bounded on $[0, T]$, so $\lim_{s\to 0} g(s) = \lim_{s\to T} g(s) = 0$. Then

$$h^{-1}\int_{-h}^0 g(s)F_\Psi(s+h)ds = h^{-1}\int_{T-h}^T g(s)F_\psi(s+h)ds = 0$$

as soon as $0 < h \le T - \beta$; thus

$$\lim_{h\to 0}\left[h^{-1}\int_{-h}^0 g(s)F_\Psi(s+h)ds \right] = \lim_{h\to 0}\left[h^{-1}\int_{T-h}^T g(s)F_\psi(s+h)ds \right] = 0.$$

Applying Lebesgue's dominated convergence theorem yields that F admits a weak derivative:

$$\int_0^T \dot{g}(s)F_\Psi(s)ds = -\int_0^T g(s)\dot{F}_\Psi(s)ds.$$

Using Proposition 3.1(ii), we have

$$\lim_{h\to 0^+}\left(-\frac{F_\Psi(t)-F_\Psi(t+h)}{h} - \frac{2}{\sqrt{h}}\mathbb{E}\left[\Psi(V_t)\|A^1(X_t)\|\mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|} \right) \right] \right) = 0.$$

Using Proposition 3.2(ii),

$$-\dot{F}_\Psi(t+) := \lim_{h\to 0, h>0} \frac{F_\Psi(t)-F_\Psi(t+h)}{h} = -\frac{1}{2}\int_{\mathbb{R}^d} \Psi(m, m, \tilde{x})\|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t)dm d\tilde{x},$$

and by Propositions 3.1(i) and 3.2(i),

$$\sup_{h>0}\left| \frac{F_\Psi(t)-F_\Psi(t+h)}{h} \right| \in L^1([0, T], dt),$$

so $\dot{F}_\Psi \in L^1([0, T], \mathbb{R})$. By [5, Chapter 8, Section 2, p. 202], F_Ψ belongs to $W^{1,1}(]0, T[, \mathbb{R})$. □

We now finish the proof of Theorem 2.3. According to [5, Theorem 8.2, p. 204], F_ψ is equal almost surely to an absolutely continuous function. Since F_Ψ is continuous (Lemma 3.6), the equality holds everywhere. Thus F_Ψ is an absolutely continuous function, and its derivative is its right derivative.

4. The case $A = I_d$

In this rather technical section, we first prove that the density of the pair (M_t, X_t) fulfils Hypothesis 2.1: p_V (from (3)) is continuous on the boundary of $\bar{\Delta}$ and is dominated by an integrable function.

Proposition 4.1. *Assume that B fulfils Assumption* (4) *and* $A = I_d$. *Then* (M, X) *fulfils Hypothesis* 2.1, *meaning that for any probability measure* μ_0 *on* $\mathbb{R}^d$, *the following hold:*

(i) *For every* $T > 0$,

$$\sup_{(h,u) \in [0,1] \times \mathbb{R}_+} p_V(b, b - hu, \tilde{a}; t, \mu_0) \in L^1([0, T] \times \mathbb{R}^d, dtdbd\tilde{a}).$$

(ii) *Almost surely in* $(m, \tilde{x}) \in \mathbb{R}^d$, *for every* $t > 0$,

$$\lim_{u \to 0, \ u > 0} p_V(m, m - u, \tilde{x}; t, \mu_0) = p_V(m, m, \tilde{x}; t, \mu_0).$$

As a by-product, using Theorems 2.2 and 2.3, this proposition completes the proof of Theorem 2.4. The main tool for the proof of this proposition is an integral representation of the density.

Proposition 4.2 *For any probability measure* μ_0 *on* $\mathbb{R}^d$, *for all* $t > 0$,

$$p_V = p^0 - \sum_{k=m,1,\ldots,d} (p^{k,\alpha} + p^{k,\beta}), \tag{21}$$

where the various p terms are defined as follows (here ∂_k *is the derivative with respect to* $k = m, x^1, \ldots, x^d$ *and* $B^m = B^1$ *):*

$$p^0(m, x; t) := \int_{\mathbb{R}^d} p_{W^{*1}, W}(m - x_0^1, x - x_0; t) \mu_0(dx_0),$$

$$p^{k,\alpha}(m, x; t) := \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b<m} B^k(a) \partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s) p_V(b, a; s) db da ds,$$

$$p^{k,\beta}(m, x; t) := \int_0^t \int_{\mathbb{R}^{(d+1)}} \mathbf{1}_{b<m} B^k(a) \partial_k p_{W^{*1}, W}(b - a^1, x - a; t - s) p_V(m, a; s) db da ds,$$

where $p_{W^{*1}, W}(., .; t)$ *is the density of the distribution of* $(\sup_{s \leq t} W_s^1, W_t)$ *for* $t \geq 0$; *see Appendix* A.2.

4.1. Integral representation of the density: proof of Proposition 4.2

Let $t > 0$ be fixed. First, we assume that $\mu_0 = \delta_{x_0}$, x_0 being fixed in $\mathbb{R}^d$. By Lemma 4.2 below, and using the fact that B is bounded, for all $t \in [0, T]$, the functions $p^{k,\gamma} \in^\infty L([0, T], L^1(\mathbb{R}^{d+1}))$ for $\gamma = \alpha, \beta$.

Let $F \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$ with compact support. We will prove that

$$\mathbb{E}_{\mathbb{P}}[F(M_t, X_t)] = \int_{\mathbb{R}^{d+1}} F(m, x) \left(p^0 - \sum_{k=m,x^1,\ldots,x^d} (p^{k,\alpha} + p^{k,\beta})(m, x, t) \right) dmdx. \tag{22}$$

Using Malliavin calculus we obtain the following decomposition.

Lemma 4.1. *We have*

$$
\mathbb{E}_{\mathbb{P}}\left[F(M_t, X_t)\right] = \int_{\mathbb{R}^{d+1}} F(x_0^1 + b, x_0 + a) p_{W^{*1}, W}(b, a; t) db da
$$
$$
+ \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_m F\left(X_s^1 + b, X_s + a\right) \mathbf{1}_{\left\{M_s < X_s^1 + b\right\}} B^1(X_s) p_{W^{*1}, W}(b, a; t-s) db da\right] ds
$$
$$
+ \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_k F\left(\max\left(M_s, X_s^1 + b\right), X_s + a\right) B^k(X_s) p_{W^{*1}, W}(b, a; t-s) db da\right] ds.
$$

Proof. Let Z be the exponential martingale solution of

$$
Z_t = 1 + \int_0^t Z_s B^k(x_0 + W_s) dW_s^k. \tag{23}
$$

As before, Einstein's convention is used. Let $\mathbb{Q} = Z\mathbb{P}$; according to Girsanov's theorem, using (23), $W_B := \left(W_t^k - \int_0^t B^k(x_0 + W_s) ds; k = 1, \ldots, d\right)_{t \geq 0}$ is a $\mathbb{Q}$-continuous martingale such that $\langle W_B \rangle_t = t$ for all $t \geq 0$. That means that under $\mathbb{Q}$, $\overline{W}_B$ is a d-dimensional Brownian motion. Then the distribution of X (resp. (M, X)) under $\mathbb{P}$ is the distribution of $W + x_0$ (resp. $(W^{1*} + x_0, W + x_0)$) under $\mathbb{Q}$, and

$$
\mathbb{E}_{\mathbb{P}}\left[F(M_t, X_t)\right] = \mathbb{E}_{\mathbb{Q}}\left[F(x_0^1 + W_t^{1*}, x_0 + W_t)\right] = \mathbb{E}_{\mathbb{P}}\left[F(x_0^1 + W_t^{1*}, x_0 + W_t) Z_t\right]. \tag{24}
$$

Let $G := F(x_0^1 + W_t^{1*}, x_0 + W_t)$ and $u := ZB(x_0 + W)$; using (23),

$$
\mathbb{E}_{\mathbb{P}}[F(M_t, X_t)] = \mathbb{E}_{\mathbb{P}}\left[F(x_0^1 + W_t^{1*}, x_0 + W_t)\right] + \mathbb{E}_{\mathbb{P}}\left[G\delta(u)\right]. \tag{25}
$$

As a first step we will apply (50) (see the appendix) to the second term in (25). Thus we have to check that the pair $(G, u) \in \mathbb{D}^{1,2} \times \mathbb{L}^{1,2}$. Since F is bounded and smooth, we have $G \in \mathbb{D}^{1,2}$; and according to Lemma A.1, the process u belongs to $\mathbb{L}^{1,2}$.

Using (53) $\left(\tau := \inf\left\{s, W_s^{1*} = W_t^{1*}\right\}\right)$, the pair $\left(W_t^{1*}, W_t\right)$ belongs to $\mathbb{D}^{1,2}$ with Malliavin gradient

$$
D_s W_t^{1*} = \left(\mathbf{1}_{[0,\tau]}(s), 0, \ldots, 0\right), \quad D_s W_t^k = \left(\delta_{j=k}, \; j = 1, \ldots, d\right) \mathbf{1}_{[0,t]}(s), \quad k = 1, \ldots, d.
$$

Using the chain rule,

$$
\langle DG, u \rangle_{\mathbb{H}} = \int_0^t \partial_m F(x_0^1 + W_t^{*1}, x_0 + W_t) \mathbf{1}_{\left\{W_s^{1*} < W_t^{1*}\right\}} B^1(x_0 + W_s) Z_s ds
$$
$$
+ \int_0^t \partial_k F(x_0^1 + W_t^{*1}, x_0 + W_t) B^k(x_0 + W_s) Z_s ds.
$$

We are now in a position to apply (50), $E_{\mathbb{P}}[G\delta(u)] = E_{\mathbb{P}}[\langle DG, u \rangle_{\mathbb{H}}]$:

$$
\mathbb{E}_{\mathbb{P}}[G\delta(u)] = \mathbb{E}_{\mathbb{P}}\left[\int_0^t \partial_m F(x_0^1 + W_t^{1*}, x_0 + W_t) \mathbf{1}_{\{W_s^{1*} < W_t^{1*}\}} B^1(x_0 + W_s) Z_s ds\right]
$$
$$
+ \mathbb{E}_{\mathbb{P}}\left[\int_0^t \partial_k F(x_0^1 + W_t^{1*}, x_0 + W_t) B^k(x_0 + W_s) Z_s ds\right]. \tag{26}
$$

Plugging the identity (26) into the right-hand side of (25) and using Fubini's theorem to commute the integrals in ds and $d\mathbb{P}$, we obtain

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[F(M_t, X_t)\right] = {} & \mathbb{E}_{\mathbb{P}}\left[F\left(x_0^1 + W_t^{1*}, x_0 + W_t\right)\right] \\
& + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\partial_m F\left(x_0^1 + W_t^{1*}, x_0 + W_t\right)\mathbf{1}_{\left\{W_s^{1*} < W_t^{1*}\right\}} Z_s B^1(x_0 + W_s)\right] ds \\
& + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\partial_k F\left(x_0^1 + W_t^{1*}, x_0 + W_t\right) Z_s B^k(x_0 + W_s)\right] ds.
\end{aligned} \tag{27}
$$

As a second step, we use the independence of the increments of the Brownian motion in order to obtain the density of $\left(W_{t-s}^{1*}, W_{t-s}\right)$. Recall (9): $\hat{W}_{t-s} := W_t - W_s$ and $\left(\hat{W}^1\right)_{t-s}^* = \max_{s \le u \le t}\left(W_u^1 - W_s^1\right)$. Then

$$
W_t^{1*} = \max\left(W_s^{1*}, W_s^1 + \max_{s \le u \le t}\left(W_u^1 - W_s^1\right)\right) = \max\left(W_s^{1*}, W_s^1 + \left(\hat{W}^1\right)_{t-s}^*\right),
$$

so the expression (27) becomes

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[F(M_t, X_t)\right] = {} & \mathbb{E}_{\mathbb{P}}\left[F\left(x_0^1 + W_t^{1*} x_0 +, W_t\right)\right] + \\
& \int_0^t \mathbb{E}_{\mathbb{P}}\left[\partial_m F\left(x_0^1 + W_s^1 + \left(\hat{W}^1\right)_{t-s}^*, x_0 + W_s + \hat{W}_{t-s}\right)\mathbf{1}_{\left\{W_s^{1*} < W_s^1 + \left(\hat{W}^1\right)_{t-s}^*\right\}} Z_s B^1(x_0 + W_s)\right] ds \\
& + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\partial_k F\left(\max\left(x_0^1 + W_s^{1*}, x_0^1 + W_s^1 + \left(\hat{W}^1\right)_{t-s}^*\right), x_0 + W_s + \hat{W}_{t-s}\right) Z_s B^k(x_0 + W_s)\right] ds.
\end{aligned}
$$

The random vector $\left(\left(\hat{W}^1\right)_{t-s}^*, \hat{W}_{t-s}\right)$ is independent of the σ-field $\mathcal{F}_s$ and has the same distribution as the pair $\left(W_{t-s}^{1*}, W_{t-s}\right)$. Let $p_{W^{*1}, W}(., .; t-s)$ be the density of its law, and express the expectation with this density:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[F(M_t, X_t)\right] = {} & \int_{\mathbb{R}^{d+1}} F\left(x_0^1 + b, x_0 + a\right) p_{W^{*1}, W}(b, a; t) db da \\
& + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_m F\left(x_0^1 + W_s^1 + b, x_0\right.\right. \\
& \qquad \left.\left. + W_s + a\right)\mathbf{1}_{\left\{W_s^{1*} < W_s^1 + b\right\}} Z_s B^1(x_0 + W_s) p_{W^{*1}, W}(b, a; t-s) db da\right] ds \\
& + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_k F\left(x_0^1 + \max\left(W_s^{1*}, W_s^1 + b\right), x_0\right.\right. \\
& \qquad \left.\left. + W_s + a\right) Z_s B^k(x_0 + W_s) p_{W^{*1}, W}(b, a; t-s) db da\right] ds.
\end{aligned}
$$

Using Girsanov's Theorem for $Z.\mathbb{P} = \mathbb{Q}$, since the law of (M, X) under $\mathbb{P}$ is the law of $\left(x_0^1 + W^{1*}, x_0 + W\right)$, under $\mathbb{Q}$, using the equality (24), we have:

$$
\mathbb{E}_{\mathbb{P}}\left[F\left(M_t, X_t\right)\right] = \int_{\mathbb{R}^{d+1}} F\left(x_0^1 + b, x_0 + a\right) p_{W^{*1}, W}(b, a; t) db da
$$

$$
+ \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_m F\left(X_s^1 + b, X_s + a\right) \mathbf{1}_{\left\{M_s < X_s^1 + b\right\}} B^1(X_s) p_{W^{*1}, W}(b, a; t - s) db da\right] ds
$$

$$
+ \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_k F\left(\max\left(M_s, X_s^1 + b\right), X_s + a\right) B^k(X_s) p_{W^{*1}, W}(b, a; t - s) db da\right] ds.
$$

$\square$

We are now in a position to complete the proof of Proposition 4.2. Using some suitable translations of the variables (a, b), we have $\mathbb{E}_{\mathbb{P}}\left[F\left(M_t, X_t\right)\right] = \sum_{k=0}^{d} I_k + I_m$, where

$$
I_0 = \int_{\mathbb{R}^{d+1}} F(b, a) p_{W^{*1}, W}\left(b - x_0^1, a - x_0; t\right) db da, \tag{28}
$$

$$
I_m = \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_m F(b, a) \mathbf{1}_{\{M_s < b\}} B^1(X_s) p_{W^{*1}, W}\left(b - X_s^1, a - X_s; t - s\right) db da\right] ds,
$$

and for $k = 1, \ldots, d$,

$$
I_k = \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \partial_k F(\max(M_s, b), a) B^k(X_s) p_{W^{*1}, W}\left(b - X_s^1, a - X_s; t - s\right) db da\right] ds.
$$

Since B, F, and its derivatives are bounded, all these integrals are finite. Using (54) in the appendix, the function $p_{W^{*1}, W}(., .; t)$ is C^∞ on $\bar{\Delta} = \left\{(b, a), \ b \geq a_+^1, \ (a, b) \in \mathbb{R}^{d+1}\right\}$. The aim is now to identify the terms p^0, $p^{k, \alpha}$, $p^{k, \beta}$, $\mathbf{k} = m, 1, \ldots, d$, defined in Proposition 4.2.

Step 1. First we identify $p^0(b, a; t)$ as the factor of $F(b, a)$ in the integrand of I_0:

$$
p^0(b, a; t) = p_{W^{*1}, W}\left(b - x_0^1, a - x_0; t\right).
$$

Step 2. We now deal with I_k, $k = 2, \ldots, d$. Integrating by parts with respect to a^k between $-\infty$ and ∞ in I_k for $k = 2, \ldots, d$ yields

$$
I_k = -\int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} F(\max(M_s, b), a) B^k(X_s) \partial_k p_{W^{*1}, W}\left(b - X_s^1, a - X_s; t - s\right) db da\right] ds
$$

$$
= -\int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{\{b > M_s\}} F(b, a) B^k(X_s) \partial_k p_{W^{*1}, W}\left(b - X_s^1, a - X_s; t - s\right) db da\right] ds
$$

$$
- \int_0^t \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{\{b < M_s\}} F(M_s, a) B^k(X_s) \partial_k p_{W^{*1}, W}\left(b - X_s^1, a - X_s; t - s\right) db da\right] ds.
$$

We identify $-p^{k, \alpha}(b, a, t)$ inside the integral on the set $\{b > M_s\}$. For the integral on the set $\{b < M_s\}$, we introduce the density of (M_s, X_s) and identify $-p^{k, \beta}(m, a; t)$ as the factor of $F(m, a)$.

Step 3. Finally, we identify the terms $p^{m,\gamma}$ and $p^{1,\gamma}$, $\gamma = \alpha, \beta$, which come from the sum of I_m and I_1. Note that $p_{W^{*1},W}(b - X_s^1, a - X_s; t - s) = 0$ on the set $\{b < a^1\}$. Integrating by parts with respect to b between $\max(a^1, M_s)$ and ∞ in I_m yields

$$I_m = -\int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^d} F\left(\max(a^1, M_s), a\right) B^1(X_s) p_{W^{*1},W}\left(\max(a^1, M_s) - X_s^1, a - X_s; t - s\right) da\right] ds$$

$$-\int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{M_s < b} F(b, a) B^1(X_s) \partial_m p_{W^{*1},W}\left(b - X_s^1, a - X_s; t - s\right) db da\right] ds. \qquad (29)$$

Integrating by parts with respect to a^1 between $-\infty$ and b in I_1 yields

$$I_1 = \int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^d} F(\max(M_s, b), b, \tilde{a}) B^1(X_s) p_{W^{*1},W}(b - X_s^1, b - X_s^1, \tilde{a} - \tilde{X}_s; t - s) db d\tilde{a}\right] ds$$

$$-\int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^{d+1}} F(\max(M_s, b), a) B^1(X_s) \partial_1 p_{W^{*1},W}(b - X_s^1, a - X_s; t - s) db da\right] ds. \qquad (30)$$

We then have the following:

   (i) The term $p^{m,\beta}(b, a, t)$ comes from the second term in I_m (Equation (29)) as the factor of $F(b, a)$:

$$-\int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{M_s < b} F(b, a) B^1(X_s) \partial_m p_{W^{*1},W}(b - X_s^1, a - X_s; t - s) db da\right] ds$$

   (ii) The terms $-p^{1,\alpha}(b, a, t)$ and $-p^{1,\beta}(b, a; t)$ come from the second term in I_1 (Equation (30)):

$$-\int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^{d+1}} F(\max(b, M_s), a) B^1(X_s) \partial_1 p_{W^{*1},W}(b - X_s^1, a - X_s; t - s) db da\right] ds.$$

   Inside the integral on the set $\{M_s < b\}$ we identify $-p^{1,\alpha}(b, a, t)$ as the factor of $F(b, a)$, and inside the integral on the set $\{M_s > b\}$ we identify $-p^{1,\beta}(b, a; t)$ as the factor of $F(M_s, a)$.

   (iii) The term $-p^{m,\alpha}(b, a, t)$ comes from the sum of first terms in I_1 (Equation (30)) and I_m (Equation (29)).

Now we replace the variable b by a_1 and replace $db d\tilde{a}$ by da in the first terms of I_m and I_1:

$$I_m^1 = -\int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^d} F\left(\max(a^1, M_s), a\right) B^1(X_s) p_{W^{*1},W}\left(\max(a^1, M_s) - X_s^1, a - X_s; t - s\right) da\right] ds,$$

$$I_1^1 = \int_0^t \mathbb{E}_\mathbb{P}\left[\int_{\mathbb{R}^d} F\left(\max(M_s, a^1), a\right) B^1(X_s) p_{W^{*1},W}\left(a^1 - X_s^1, a - X_s; t - s\right) da\right] ds.$$

Note that

$$-p_{W^{*1},W}\left(\max(a^1, M_s) - X_s^1, a - X_s; t - s\right) + p_{W^{*1},W}\left(a^1 - X_s^1, a - X_s; t - s\right)$$

$$= \left[-p_{W^{*1},W}\left(M_s - X_s^1, a - X_s; t - s\right) + p_{W^{*1},W}\left(a^1 - X_s^1, a - X_s; t - s\right)\right] \mathbf{1}_{M_s > a^1}$$

$$= -\int_{a^1}^{M_s} \partial_m p_{W^{*1},W}\left(b - X_s^1, a - X_s, t - s\right) db \, \mathbf{1}_{M_s > a^1}.$$

Then the sum of I_m^1 and I_1^1 is

$$-\int_0^t \mathbb{E}_{\mathbb{P}}\left[\iint_{\mathbb{R}^{d+1}} F(M_s, a)B^1(X_s)\partial_m p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s)\mathbf{1}_{M_s>b>a^1}\, da\, db\right] ds.$$

We introduce the density of the law of the pair (M_s, X_s) and we identify $-p^{m,\alpha}(m, a; t)$ as the factor of $F(m, a)$.

These three steps complete the proof of Proposition 4.2 when $\mu_0 = \delta_{x_0}$.

Finally, when μ_0 is the law of X_0, we have $p_V(m, w; t, \mu_0) = \int_{\mathbb{R}^d} p_V(m, x; t, \delta_{x_0})\mu_0(dx_0)$. Then we can complete the proof of Proposition 4.2 for any initial law μ_0 by integrating with respect to μ_0 the expression obtained in (21) for $p_V(m, x; t, \delta_{x_0})$. $\qquad\square$

4.2. Proof of Proposition 4.1

Using some ideas used in [14, Section V.3.2], let us introduce the following linear maps on $L^\infty\big([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx)\big)$, for $k = m, 1, \ldots, d$:

$$\mathcal{I}^{k,\alpha}[p](m, x; t) := \int_0^t \iint_{\mathbb{R}^{d+1}} \mathbf{1}_{b<m}B^k(a)\partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s)p(b, a; s)\, db\, da\, ds, \tag{31}$$

$$\mathcal{I}^{k,\beta}[p](m, x; t) := \int_0^t \iint_{\mathbb{R}^{d+1}} \mathbf{1}_{b<m}B^k(a)\partial_k p_{W^{*1}, W}(b - a^1, x - a; t - s)p(m, a; s)\, db\, da\, ds.$$

Let us introduce the following inductively defined functions:

$$p_0(m, x; t, \mu_0) = \int_{\mathbb{R}^d} p_{W^{1*}, W}(m - x_0^1, x - x_0; t)\mu_0(dx_0), \quad p_n = -\sum_{k=m,1,\ldots,d}\left(p_n^{k,\alpha} + p_n^{k,\beta}\right), \tag{32}$$

and for $k = m, 1, \ldots, d, j = \alpha, \beta$, and $n \geq 1$,

$$p_{n+1}^{k,j}(m, x; t) := \mathcal{I}^{k,j}[p_n](m, x; t).$$

Let us define the operator

$$\mathcal{I} := -\sum_{j=\alpha,\beta;\, k=m,1,\ldots,d} \mathcal{I}^{k,j}. \tag{33}$$

Moreover, one can observe that this means $p_{n+1} = \mathcal{I}(p_n)$, and Proposition 4.2 leads to $p_V = p_0 + \mathcal{I}(p_V)$. Let

$$P_n := \sum_{k=0}^n p_k, \quad n \geq 0. \tag{34}$$

Proposition 4.3. *Assume the vector B is bounded; then for all T the sequence $(P_n)_n$ converges in $L^\infty([0, T], L^1(\mathbb{R}^{d+1}, dxdm))$ to p_V. Moreover, $p_V = \sum_{n=0}^\infty p_n$.*

The proof is a consequence of the following two lemmas.

Lemma 4.2. *Let $j = \alpha$, β, $k = m, 1, \ldots, d$, and $T > 0$. The linear maps $\mathcal{I}^{k,j}$ are continuous on $L^\infty\big([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx)\big)$: there exists a constant C such that for all $p \in L^\infty\big([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx)\big)$,*

$$\sup_{s \in [0,t]} \|\mathcal{I}^{k,j}[p](., .; s)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq C \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \in [0,s]} \|p(., .; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} ds. \quad (35)$$

As a consequence,

$$\sup_{s \in [0,t]} \|\mathcal{I}[p](., .; s)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq 2(d+1)C \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \in [0,s]} \|p(., .; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} ds. \quad (36)$$

Proof. Let $T > 0$, $p \in L^\infty\big([0, T] \times L^1(\mathbb{R}^{d+1}, dmdx)\big)$, and $t \in [0, T]$, and let ϕ_{d+1} be the Gaussian law density restricted to the subset $\{b > a_+^1\}$ (up to a constant):

$$\phi_{d+1}(b, b - a^1, \tilde{a}; 2t) := \frac{1}{\sqrt{2\pi t}^{d+1}} \mathbf{1}_{b > a_+^1} e^{-\frac{b^2 + (b - a^1)^2 + \|\tilde{a}\|^2}{4t}}. \quad (37)$$

(i) Let $j = \alpha$ and $k = m, 1, \ldots, d$; according to the definition of $\mathcal{I}^{k,\alpha}$ and the boundedness of B,

$$\left| \mathcal{I}^{k,\alpha}[p](m, x; t) \right| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} |\partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s) p(b, a; s)| db da ds.$$

Using Lemma A.2, there exists a constant D such that for $k = m, 1, \ldots, d$,

$$|\partial_k p_{W^{*1}, W}(b, a; t)| \leq \frac{D}{\sqrt{t}} \phi_{d+1}(b, b - a^1, \tilde{a}; 2t). \quad (38)$$

So

$$\left| \mathcal{I}^{k,\alpha}[p](m, x; t) \right|$$
$$\leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \frac{D}{\sqrt{t-s}} \phi_{d+1}(m - a^1, m - x^1, \tilde{x} - \tilde{a}; t - s) |p(b, a; s)| db da ds.$$

We perform an integration with respect to (m, x) using Tonelli's theorem and omitting the indicator functions. Since ϕ_{d+1} is the density of a Gaussian law, we get the bound

$$\left\| \mathcal{I}^{k,\alpha}[p](., .; t) \right\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq D\|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \frac{1}{\sqrt{t-s}} |p(b, a; s)| db da ds$$
$$\leq 2^{(d+1)/2} D\|B\|_\infty \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \leq s} \|p(., .; u)\|_{L^1(\mathbb{R}^{d+1}, dbda)} ds,$$

which implies the inequality (35) when $j = \alpha$.

(ii) Let $j = \beta$ and $k = m, 1, \ldots, d$. According to the definition of $\mathcal{I}^{k,\beta}$ and the boundedness of B,

$$\left| \mathcal{I}^{k,\beta}[p](m, x; t) \right| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} |\partial_k p_{W^{*1}, W}(b - a^1, x - a; t - s) p(m, a; s)| db da ds.$$

Using (38) yields

$$\left| \mathcal{I}^{k,\beta}[p](m, x; t) \right| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \frac{D}{\sqrt{t-s}} \phi_{d+1}\big(b - a^1, b - x^1, \tilde{x} - \tilde{a}; 2(t - s)\big) |p(m, a; s)| db da ds.$$

We integrate with respect to x, then with respect to b, using Tonelli's theorem and omitting the indicator functions, and using the fact that ϕ is the density of a Gaussian law. So the bound with respect to a multiplicative constant is

$$\left\| \mathcal{I}^{k,\beta}[p](., .; t) \right\|_{L^1\left(\mathbb{R}^{d+1}, dmdx\right)} \leq D\|B\|_\infty 2^{(d+1)/2} \int_0^t \int_{\mathbb{R}^{d+1}} \frac{1}{\sqrt{t-s}} |p(m, a; s)| dmda ds$$

$$\leq D\|B\|_\infty 2^{(d+1)/2} \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \leq s} \|p(., .; u)\|_{L^1\left(\mathbb{R}^{d+1}, dmda\right)} ds,$$

which implies the inequality (35) for $j = \beta$.

Finally, the estimate (36) is obtained by adding the estimates (35) for $j = \alpha, \beta$ and $k = m, 1, \ldots, d$. □

The following lemma is a consequence of (36) in Lemma 4.2.

Lemma 4.3. *For all n,*

$$\sup_{u \leq t} \|p_n(., .; u)\|_{L^1\left(\mathbb{R}^{d+1}, dmdx\right)} \leq (2(d+1)C)^n t^{n/2} \frac{\Gamma(1/2)^n}{\Gamma(1 + n/2)}, \tag{39}$$

$$\sup_{u \leq t} \|(p_V - P_n)(., .; u)\|_{L^1\left(\mathbb{R}^{d+1}, dmdx\right)} \leq (2(d+1)C)^{n+1} t^{(n+1)/2} \frac{\Gamma(1/2)^{n+1}}{\Gamma((n+3)/2)}. \tag{40}$$

Proof. (i) For all $t > 0$, $p_0(.; t)$ is a probability density function, so (39) is satisfied for $n = 0$. We now assume that (39) is satisfied for n. Using $p_{n+1} = \mathcal{I}[p_n]$, (36), and the inductive assumption,

$$\sup_{u \leq t} \|p_{n+1}(., .; u)\|_{L^1\left(\mathbb{R}^{d+1}, dmdx\right)} \leq (2(d+1)C)^{n+1} \frac{\Gamma(1/2)^n}{\Gamma(1 + n/2)} \int_0^t \frac{\sqrt{s^n}}{\sqrt{t-s}} ds.$$

We perform the change of variable $s = tu$ and use

$$\int_0^1 u^{a-1}(1 - u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$$

to obtain

$$\sup_{u \leq t} \|p_{n+1}(., .; u)\|_{L^1\left(\mathbb{R}^{d+1}, dmdx\right)} \leq (2(d+1)C)^{n+1} t^{(n+1)/2} \frac{\Gamma(1/2)^n}{\Gamma(1 + n/2)} \frac{\Gamma(1/2)\Gamma(1 + n/2)}{\Gamma((n+3)/2)},$$

which proves (39) for all n.

(ii) Noting that $P_0 = p_0$ and $p_V - p_0 = \mathcal{I}[p_V]$ and applying (36) to p_V, we obtain

$$\sup_{u \leq t} \|(p_V - P_0)(., .; u)\|_{L^1\left(\mathbb{R}^{d+1}, dmdx\right)} \leq 2(d+1)Ct^{1/2}.$$

But $\Gamma(1/2)/\Gamma(3/2) = 2$, so (40) is satisfied for $n = 0$.

We now suppose that (40) is satisfied for n. Using

$$p_V - P_{n+1} = p_0 + \mathcal{I}(p_V) - (p_0 + \mathcal{I}(P_n)) = \mathcal{I}(p_V - P_n),$$

the bound (36), and the induction assumption, we have

$$\sup_{u \leq t} \|[p_V - P_{n+1}](.,.;u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq 2(d+1)C \int_0^t (2(d+1)C)^{n+1} \frac{\Gamma(1/2)^{n+1}}{\Gamma((3+n)/2)} \frac{\sqrt{s^{n+1}}}{\sqrt{t-s}} ds.$$

We now perform the change of variable $s = tu$ and

$$\int_0^1 u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

with $a = (n+3)/2$, $b = \frac{1}{2}$, to obtain

$$\sup_{u \leq t} \|[p_V - P_{n+1}](.,.;u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq (2(d+1)C)^{n+2} t^{(n+2)/2} \frac{\Gamma(1/2)^{n+2}}{\Gamma((4+n)/2)},$$

which proves (40) for $n+1$ and thus for all n. $\qquad\square$

The series $\sum_n \frac{x^n}{\Gamma(n/2+1)}$ is convergent for any x, so Proposition 4.3 is a consequence of Lemmas 4.2 and 4.3.

4.2.1. *Upper bound of p_V (i.e., Hypothesis 2.1(i)).* For all $T > 0$, $x_0 \in \mathbb{R}^d$, and $p \in L^\infty([0, T], L^1(\mathbb{R}^{d+1}, dmdx))$ whose support is contained in $\{(m, x), \ m > x_0^1, m > x^1\}$, let us define

$$N(p; t, x_0) := \sup_{(m,x) \in \mathbb{R}^{d+1}, \ m > x^1, m > x_0^1} \frac{|p(m, x; t)|}{\phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t)}. \tag{41}$$

Proposition 4.4. *For every $T > 0$ there exists a constant C_T, and for all n there exist constants $C_n = \frac{[\|B\|_\infty D(2(d+1))2^{d/2}\Gamma(1/2)]^n}{\Gamma(1+n/2)}$, such that for all $x_0 \in \mathbb{R}^d$, $0 < t \leq T$, the following hold:*

(i) $\left| p_n(m, x; t, x_0) \right| \leq C_n t^{n/2} \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0, 2t) \mathbf{1}_{m > \max(x^1, x_0^1)}$;

(ii) $\left| p_V(m, x; t, x_0) \right| \leq C_T \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0, 2t) \mathbf{1}_{m > \max(x^1, x_0^1)}$;

(iii) *for every initial probability measure μ_0 on $\mathbb{R}^d$,*

$$\sup_{u > 0} p_V(m, m - u, \tilde{x}, t; \mu_0) \in L^1([0, T] \times \mathbb{R}^d, dtdmd\tilde{x}).$$

Note that Part (iii) is actually Hypothesis 2.1(i).

Proof. Part (ii) is a consequence of Part (i), since $p_V = \sum_{n=0}^\infty p_n$, and the series $\sum_n \frac{1}{\Gamma(1+n/2)} x^n$ has an infinite radius of convergence (Proposition 4.3).

We prove Part (i) by induction on n, using Part (ii) of Lemma A.2:

$$\begin{aligned} p_0(m, x; t, x_0) &\leq \frac{e^{-\frac{(m-x^1)^2}{4t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t} - \frac{(m-x_0^1)^2}{4t}}}{\sqrt{(2\pi)^{d+1} t^{d+1}}} \mathbf{1}_{m > \max(x^1, x_0^1)} \\ &= \phi_{d+1}(m - x^1, m - x_0^1, \tilde{x} - \tilde{x}_0; 2t) \mathbf{1}_{m > \max(x^1, x_0^1)}, \end{aligned}$$

so $N(p_0; t, x_0) \leq 1$, which is Part (i) for $n = 0$, $C_0 = 1$.

We assume Part (i) is true for p_n, meaning $N(p_n; t, x_0) \leq C_n t^{n/2}$. By definition, $p_{n+1} = \mathcal{I}[p_n]$; Lemma 4.4, proved below, yields

$$N(p_{n+1}; t, x_0) = N(\mathcal{I}[p_n]; t, x_0) \leq 2(d+1)2^{d/2}\|B\|_\infty D C_n \int_0^t \frac{s^{n/2}}{\sqrt{2\pi(t-s)}} ds.$$

We perform the change of variable $s = tu$ to obtain

$$N(p_{n+1}; t, x_0) \leq \frac{2(d+1)2^{d/2}\|B\|_\infty D}{\sqrt{2\pi}} C_n(\sqrt{t})^{n+1} \int_0^1 \frac{u^{n/2}}{\sqrt{1-u}} ds.$$

Using

$$\int_0^1 \frac{u^{n/2}}{\sqrt{1-u}} du = \frac{\Gamma((n+2)/2)\Gamma(1/2)}{\Gamma((n+3)/2)}$$

and the definition of C_n, we have

$$N(p_{n+1}; t, x_0) \leq C_{n+1}(\sqrt{t})^{n+1};$$

this completes the proof of Proposition 4.4(i).

Then, for all $x_0 \in \mathbb{R}^d$ and using $x^1 = m - u$,

$$\sup_{u > 0} p_V(m, m - u, \tilde{x}; t) \leq C_T \phi_{d+1}(m - x_0^1, 0, \tilde{x} - \tilde{x}_0; 2t) \in L^1([0, T] \times \mathbb{R}^d, dt dm d\tilde{x}).$$

Since $p_V(m, x; t, \mu_0) = \int_{\mathbb{R}^d} p_V(m, x; t, x_0)\mu_0(dx_0)$, Part (iii) is true. □

Lemma 4.4. *Let $T > 0$, $x_0 \in \mathbb{R}^d$, $p \in L^\infty([0, T], dt, L^1(\mathbb{R}^{d+1}, dm dx))$ be such that the support of $p(., .; t)$ is included in $\{(m, x),\ m > x_0^1, m > x^1\}$ and for all $s \in]0, T]$ $N(p; s, x_0) < \infty$. Then for $j = \alpha$, $k = m, 1, \ldots, d$, the support of the function $\mathcal{I}^{j,k}[p](.; t)$ is included in $\{(m, x),\ m > x_0^1, m > x^1\}$. Moreover, for all $t \in [0, T]$, we have*

$$N(\mathcal{I}[p]; t, x_0) \leq 2(d+1)2^{d/2}\|B\|_\infty D \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} N(p; s, x_0) ds, \quad \forall t \in [0, T].$$

Proof. Let $T > 0$, $x_0 \in \mathbb{R}^d$, $p \in L^\infty([0, T], dt, L^1(\mathbb{R}^{d+1}, dm dx))$ be such that for all $t > 0$, the support of $p(.; t)$ is included in $\{(m, x),\ m > x_0^1, m > x^1\}$.

(i) For $j = \alpha$, $k = m, 1, \ldots, d$, using the definition of $\mathcal{I}^{\alpha,k}$ yields

$$\mathcal{I}^{k,\alpha}[p](m, x; t) :=$$
$$\int_0^t \int_{\mathbb{R}^{d+1}} B^k(a)\partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s)\mathbf{1}_{x_0^1 < b < m, m > x^1} p(b, a; s) db da ds.$$

So the support of $\mathcal{I}^{\alpha,k}[p](.; t)$ is included in $\{(m, x) \in \mathbb{R}^{d+1},\ m > \max(x_0^1, x^1)\}$. From now on, we consider only (m, x) such that $m > \max(x^1, x_0^1)$.

Let p be a function such that for all $s \in]0, T]$, $N(p; x_0, s) < \infty$. The definition of $\mathcal{I}^{k,\alpha}$, the boundedness of B, the fact that $\partial_k p_{W^*, W}$ satisfies (38), and the definition (41) of $N(p; t, x_0)$ imply that

$$\left|\mathcal{I}^{k,\alpha}[p](m, x; t)\right| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} N(p; s, x_0) \frac{D}{\sqrt{(t-s)}} \mathbf{1}_{m > x_1} \mathbf{1}_{m > b > \max(a^1, x_0^1)}$$
$$\phi_{d+1}(m - a^1, m - x^1, \tilde{x} - \tilde{a}; t - s)\phi_{d+1}(b - x_0^1, b - a^1, \tilde{a} - \tilde{x}_0; s) db da ds.$$

We integrate in $\tilde{a}$ using Lemma A.3(ii) with $u = \tilde{x}$, $v = \tilde{a}$, $w = \tilde{x}_0$ and the fact that ϕ_{d+1} is a Gaussian probability density function:

$$\left| \mathcal{I}^{\alpha,k}[p](m, x; t) \right| \le 2^{(d-1)/2} \|B\|_\infty D \tag{42}$$

$$\int_0^t \int_{\mathbb{R}^2} N(p; s, x_0) \mathbf{1}_{m > b > \max\left(a^1, x_0^1\right)} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d-1}}} \frac{e^{-\frac{\left(m - a^1\right)^2}{4(t-s)} - \frac{\left(m - x^1\right)^2}{4(t-s)}}}{\sqrt{(2\pi)^2(t-s)^3}} \frac{e^{-\frac{\left(b - x_0^1\right)^2}{4s} - \frac{\left(b - a^1\right)^2}{4s}}}{\sqrt{(2\pi)^2 s^2}} \, db \, da^1 \, ds.$$

Using Part (i′) of Lemma A.2 with $u = m$, $v = a^1$, $w = b$, $k = 1$, we integrate in da^1 up to b:

$$\int_{-\infty}^b \frac{e^{-\frac{\left(m - a^1\right)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \frac{e^{-\frac{\left(b - a^1\right)^2}{4s}}}{\sqrt{(2\pi s)}} \, da^1 = \frac{e^{-\frac{(m-b)^2}{4t}}}{\sqrt{2\pi t}} \Phi_G\left(\sqrt{\frac{s}{2t(t-s)}}(b - m) \right),$$

where

$$\Phi_G(u) = \int_{-\infty}^u e^{-z^2/2} dz \le \frac{1}{2} e^{-u^2/2}$$

for $u = b - m < 0$ by Lemma A.3(iii). This yields the bound

$$\frac{e^{-\frac{(m-b)^2}{4t}}}{\sqrt{2\pi t}} e^{-\frac{s(b-m)^2}{4t(t-s)}}$$

and

$$2 \int_{-\infty}^b \frac{e^{-\frac{\left(m - a^1\right)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \frac{e^{-\frac{\left(b - a^1\right)^2}{4s}}}{\sqrt{(2\pi s)}} \, da^1 \le \frac{e^{-\frac{(m-b)^2}{4t}}}{\sqrt{2\pi t}} e^{-\frac{s(m-b)^2}{4t(t-s)}} = \frac{e^{-\frac{(m-b)^2}{4t}}}{\sqrt{2\pi t}}.$$

Plugging this inequality into (42) yields the following, with $C_{d,B} = 2^{(d+1)/2} \|B\|_\infty D$:

$$\frac{\left| \mathcal{I}^{\alpha,k}[p](m, x; t) \right|}{C_{d,B}} \le \int_0^t \int_{\mathbb{R}} N(p; s, x_0) \mathbf{1}_{m > b > x_0^1} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^d}} \frac{e^{-\frac{(m-b)^2}{4(t-s)} - \frac{\left(m - x^1\right)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)^2 s}} e^{-\frac{\left(b - x_0^1\right)^2}{4s}} \, db \, ds.$$

Omitting the indicator functions, Lemma A.3(ii) with $u = m$, $v = b$, $w = x_0^1$, $k = 1$ implies

$$\int_{b < m} \frac{e^{-\frac{(m-b)^2}{4(t-s)}} e^{-\frac{\left(b - x_0^1\right)^2}{4s}}}{\sqrt{2\pi(t-s) 2\pi s}} \, db \le \sqrt{\frac{2}{2\pi t}} e^{-\frac{\left(m - x_0^1\right)^2}{4t}}.$$

Inserting this result, we obtain

$$\left| \mathcal{I}^{\alpha,k}[p](m, x; t) \right| \le \sqrt{2} C_{d,B} \int_0^t N(p; s, x_0) \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}} \frac{e^{-\frac{\left(m - x_0^1\right)^2}{4t} - \frac{\left(m - x^1\right)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \, ds.$$

For $0 < s < t$, we have

$$e^{-\frac{\left(m - x^1\right)^2}{4(t-s)}} \le e^{-\frac{\left(m - x^1\right)^2}{4t}},$$

so

$$\left|\mathcal{I}^{\alpha,k}[p](m,x;t)\right| \leq \sqrt{2}C_{d,B}\int_0^t N(p;s,x_0)\frac{e^{-\frac{\|\tilde{x}-\tilde{x_0}\|^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}}e^{-\frac{\left(m-x_0^1\right)^2}{4t}-\frac{\left(m-x^1\right)^2}{4t}}\,ds.$$

Using the definition of ϕ_{d+1} we find

$$\left|\mathcal{I}^{\alpha,k}[p](m,x;t)\right| \leq \sqrt{2}C_{d,B}\int_0^t N(p;s,x_0)\frac{\phi_{d+1}\left(m-x_0^1,m-x^1,\tilde{x}-\tilde{x_0};2t\right)}{\sqrt{2\pi(t-s)}}\,ds,$$

and with the definition of N, with respect to a multiplicative constant we have

$$N(\mathcal{I}^{\alpha,k}[p],x_0,t) \leq \sqrt{2}C_{d,B}\int_0^t N(p;s,x_0)\frac{1}{\sqrt{2\pi(t-s)}}\,ds.$$

(ii) For $j=\beta$, $k=m,1,\ldots,d$, using the definition of $\mathcal{I}^{\beta,k}$ and the fact that the support of p is included in $\left\{(m,x),\ m>\max(x_0^1,x^1)\right\}$ yields

$$\mathcal{I}^{\beta,k}[p](m,x;t)=$$
$$\int_0^t\int_{\mathbb{R}^{d+1}}\mathbf{1}_{m>b>x^1,m>x_0^1,b>a^1}B^k(a)\partial_k p_{W^{1*},W}\left(b-a^1,x-a,t-s\right)p(m,a,s)\,da\,db\,ds.$$

Thus the support of $\mathcal{I}^{\beta,k}[p](.;t)$ is included in $\left\{(m,x),\ m>\max(x^1,x^0)\right\}$. From now on we consider only (m,x) satisfying $m>\max(x^1,x_0^1)$. From the definition of $\mathcal{I}^{\beta,k}$ and the boundedness of B, as well as the inequality (38) satisfied by $\partial_k p_{W^*,W}$, namely

$$|\partial_k p_{W^{1*},W}\left(b-a^1,x-a,t-s\right)| \leq \frac{D}{\sqrt{t-s}}\phi_{d+1}\left(b-a^1,b-x^1,\tilde{x}-\tilde{a},2(t-s)\right),$$

and the definition of $N(p;t,x_0)$, we obtain

$$\left|\mathcal{I}^{\beta,k}[p](m,x;t)\right| \leq \|B\|_\infty D\int_0^t\int_{\mathbb{R}^{d+1}}\mathbf{1}_{m>b>x^1,b>a^1}N(p;s,x_0)$$

$$\frac{e^{-\frac{\left(b-a^1\right)^2}{4(t-s)}-\frac{\left(b-x^1\right)^2}{4(t-s)}-\frac{\|\tilde{x}-\tilde{a}\|^2}{4(t-s)}}}{\sqrt{(2\pi)^{d+1}(t-s)^{d+2}}}\frac{e^{-\frac{\left(m-x_0^1\right)^2}{4s}-\frac{\left(m-a^1\right)^2}{4s}-\frac{\|\tilde{x}_0-\tilde{a}\|^2}{4s}}}{\sqrt{(2\pi)^{d+1}s^{d+1}}}\,da\,db\,ds.$$

We integrate in $\tilde{a}$ using Lemma A.3(ii) with $u=\tilde{x}$, $v=\tilde{a}$, and $w=\tilde{x}_0$:

$$\left|\mathcal{I}^{\beta,k}[p](m,x;t)\right| \leq C_{d,B}. \tag{43}$$

$$\int_0^t\int_{\mathbb{R}^2}\mathbf{1}_{m>b>x^1,b>a^1}\frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d-1}}}N(p;s,x_0).\frac{e^{-\frac{\left(b-a^1\right)^2}{4(t-s)}-\frac{\left(b-x^1\right)^2}{4(t-s)}}}{\sqrt{(2\pi)^2(t-s)^3}}\frac{e^{-\frac{\left(m-x_0^1\right)^2}{4s}-\frac{\left(m-a^1\right)^2}{4s}}}{\sqrt{(2\pi)^2 s^2}}\,da^1\,db\,ds.$$

Using Lemma A.3(i') for $u=b$, $v=a^1$, $w=m$, $k=1$, we have

$$\int_{-\infty}^b\frac{e^{-\frac{\left(b-a^1\right)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}}\frac{e^{-\frac{\left(m-a^1\right)^2}{4s}}}{\sqrt{2\pi s}}\,da^1 = \sqrt{2}\frac{e^{-\frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}}\Phi_G\left(\sqrt{\frac{t}{4s(t-s)}}\left[b-\left(\frac{s}{t}b+\frac{t-s}{t}m\right)\right]\right)$$

$$=\frac{e^{-\frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}}\Phi_G\left(\sqrt{\frac{t-s}{4st}}[b-m]\right) \leq \frac{e^{-\frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}}e^{-\frac{t-s}{4st}[b-m]^2} = \frac{e^{-\frac{(b-m)^2}{4s}}}{\sqrt{2\pi t}},$$

the last bound coming from Lemma A.3(iii) since $b - m < 0$.

We plug this estimate into (43):

$$\left| \mathcal{I}^{\beta,k}[p](m, x; t) \right| \leq$$

$$C_{d,B} \int_0^t \int_{\mathbb{R}} \mathbf{1}_{m > b > x^1} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^d}} N(p; s, x_0) \frac{e^{-\frac{(b - x^1)^2}{4(t - s)}}}{\sqrt{2\pi(t - s)^2}} \frac{e^{-\frac{\left(m - x_0^1\right)^2}{4s} - \frac{(b - m)^2}{4s}}}{\sqrt{2\pi s}} db ds.$$

We integrate with respect to b on $\mathbb{R}$, and we use Lemma A.3(ii) with $u = x^1$, $v = b$, $w = m$, $k = 1$:

$$\left| \mathcal{I}^{\beta,k}[p](m, x; t) \right| \leq \sqrt{2} C_{d,B} \int_0^t \frac{e^{-\frac{\left(m - x^1\right)^2}{4t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}} N(p; s, x_0) \frac{e^{-\frac{\left(m - x_0^1\right)^2}{4s}}}{\sqrt{2\pi(t - s)}} ds.$$

When $0 < s < t$, we have

$$e^{-\frac{\left(m - x_0^1\right)^2}{4s}} \leq e^{-\frac{\left(m - x_0^1\right)^2}{4t}},$$

so

$$\left| \mathcal{I}^{\beta,k}[p](m, x; t) \right| \leq \sqrt{2} C_{d,B} \int_0^t \frac{e^{-\frac{\left(m - x^1\right)^2}{4t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t} - \frac{\left(m - x_0^1\right)^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}} N(p; s, x_0) \frac{1}{\sqrt{2\pi(t - s)}} ds.$$

Under the integral we identify the factor $\phi_{d+1}\left(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t\right)$, so

$$\left| \mathcal{I}^{\beta,k}[p](m, x; t) \right| \leq \sqrt{2} C_{d,B} \phi_{d+1}\left(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t\right) \int_0^t N(p; s, x_0) \frac{1}{\sqrt{2\pi(t - s)}} db ds.$$

Finally, using the definition of N (Equation (41)), we have proved

$$N\left(\mathcal{I}^{\beta,k}[p], x_0, t \right) \leq \sqrt{2} C_{d,B} \int_0^t N(p; s, x_0) \frac{1}{\sqrt{2\pi(t - s)}} ds,$$

which completes the proof of Lemma 4.4.   □

4.2.2. *Proof of Hypothesis 2.1(ii), case $A = I_d$.*

Proposition 4.5. *For any probability measure μ_0 on $\mathbb{R}^d$, for all $(m, \tilde{x}, t) \in \mathbb{R}^d \times]0, T]$, $u \mapsto p_V(m, m - u, \tilde{x}, t)$ admits a limit when u goes to 0, $u > 0$.*

Proof. The proof is a consequence of the following three lemmas.

Lemma 4.5. *Recall that $p^0(m, x; t) = \int_{\mathbb{R}^d} p_{W^{*1}, W}\left(m - x_0^1, x - x_0; t\right) \mu_0(dx_0)$. We have*

$$\lim_{u \to 0, u > 0} p^0(b, b - u, \tilde{a}; t) = p^0(b, b, \tilde{a}; t), \quad \forall u > 0, \quad (b, \tilde{a}) \in \mathbb{R}^d, \quad \forall t > 0.$$

Proof. We have

$$p^0(b, b - u, \tilde{x}; t) = \int_{\mathbb{R}^d} 2 \frac{b + u - x_0^1}{\sqrt{(2\pi)^d \mathbf{t}^{d+1}}} e^{-\frac{\left(b + u - x_0^1\right)^2}{2t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{2t}} \mathbf{1}_{b \geq x_0^1, \, u \geq 0} \mu_0(dx_0).$$

Then, since the integrand is dominated by

$$\frac{D}{\sqrt{(2\pi)^d t^{d+1}}}$$

and μ_0 is a probability measure, using Lebesgue's dominated convergence theorem yields

$$\lim_{u \to 0, u > 0} p^0(b, b - u, \tilde{x}; t) = p^0(b, b, \tilde{x}; t), \quad \forall \ (b, \tilde{x}) \in \mathbb{R}^d, \forall t > 0.$$

$\square$

Lemma 4.6. *For $k = m, 1, \ldots, d$, recall that*

$$p^{k,\alpha}(m, x; t) = \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b<m} B^k(a) \partial_k p_{W^{*1}, W}(m - a^1, x - a, t - s) p_V(b, a; s) db \, da \, ds.$$

The map $u \mapsto p^{k,\alpha}(m, m - u, \tilde{x}; t)$ converges to $p^{k,\alpha}(m, m, \tilde{x}; t)$ when u goes to 0^+.

Proof. The proof will be a consequence of Lebesgue's dominated convergence theorem. First, the map $u \mapsto \mathbf{1}_{b<m} \partial_k p_{W^{*1}, W}(m - a^1, m - u - a^1, \tilde{x} - \tilde{a}; t - s) p_V(b, a; s)$ converges to $\mathbf{1}_{b<m} \partial_k p_{W^{*1}, W}(m - a^1, ma^1, \tilde{x} - \tilde{a}; t - s) p_V(b, a; s)$ when u goes to 0^+.

Second, it is dominated by

$$q^{k,\alpha}(m, \tilde{x}, a, b; s, x_0) :=$$
$$|B^k(a)| \mathbf{1}_{b<m} \sup_{u > 0} \left| \partial_k p_{W^{*1}, W}(m - a^1, m - u - a^1, \tilde{x} - \tilde{a}; t - s) p_V(b, a; s, x_0) \right|.$$

We seek to prove that

$$\int_0^T \int_{\mathbb{R}^{2d+1}} q^{k,\alpha}(m, \tilde{x}, a, b; s, x_0) ds \, db \, da \, \mu_0(dx_0) < +\infty. \tag{44}$$

According to the estimate (38) of $\partial_k p_{W^{*1}, W}$ and the estimate (ii) of Proposition 4.4, we obtain

$$q^{k,\alpha}(m, \tilde{x}, a, b; s, x_0) \leq \|B\|_\infty \mathbf{1}_{m>b>a^1} \frac{D}{\sqrt{t - s} \sqrt{2\pi(t-s)}^{d+1}} \exp\left[-\frac{(m - a^1)^2}{4(t-s)} - \frac{\|\tilde{x} - \tilde{a}\|^2}{4(t-s)} \right]$$
$$\frac{C_T}{\sqrt{2\pi s}^{d+1}} \exp\left[-\frac{(b - x_0^1)^2}{4s} - \frac{(b - a^1)}{4s} - \frac{\|\tilde{x}_0 - \tilde{a}\|^2}{4s} \right].$$

We integrate with respect to $\tilde{a}$ using Lemma A.3(ii) for $k = d + 1$, $u = \tilde{x}$, $v = \tilde{a}$, and $w = \tilde{x}_0$:

$$\int_{\mathbb{R}^{d-1}} q^{k,\alpha}(m, \tilde{x}, a^1, b; s, x_0^1) d\tilde{a} \leq \mathbf{1}_{m>b>a^1} \frac{\|B\|_\infty C_T D 2^{(d-1)/2}}{\sqrt{t-s} \sqrt{2\pi(t-s)}^2 \sqrt{2\pi s}^2} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^{d-1}}$$
$$\exp\left[-\frac{(m - a^1)^2}{4(t-s)} - \frac{(b - x_0^1)^2}{4s} - \frac{(b - a^1)^2}{4s} \right].$$

We integrate with respect to a_1 between $-\infty$ and b using Lemma A.3(i$'$) for $u = m$, $v = a^1$, and $w = b$:

$$\int_{\mathbb{R}^d} \mathbf{1}_{a^1 < b} q^{k,\alpha}(m, \tilde{x}, b, a; s; x_0) da \leq \mathbf{1}_{b < m} \frac{\|B\|_\infty C_T D2^{d/2}}{\sqrt{t-s}\sqrt{2\pi(t-s)}\sqrt{2\pi s}} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t} - \frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}^d}$$
$$\exp\left[-\frac{(b - x_0^1)^2}{4s}\right] \Phi_G\left(\sqrt{\frac{t}{2s(t-s)}}\left(b - \left[\frac{s}{t}m + \frac{(t-s)}{t}b\right]\right)\right).$$

Note that

$$\sqrt{\frac{t}{2s(t-s)}}\left(b - \left[\frac{s}{t}m + \frac{(t-s)}{t}b\right]\right) = \sqrt{\frac{s}{2t(t-s)}}(b - m),$$

and using Lemma A.3(iii),

$$\int_{\mathbb{R}^d} q^{k,\alpha}(m, \tilde{x}, b, a; s, x_0) da \leq \mathbf{1}_{b < m} \frac{\|B\|_\infty C_T D2^{d/2}}{\sqrt{t-s}\sqrt{2\pi(t-s)}\sqrt{2\pi s}} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t} - \frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}^d}$$
$$\exp\left[-\frac{(b - x_0^1)^2}{4s}\right] \exp\left[-\frac{s}{t(t-s)}\frac{(b-m)^2}{4}\right].$$

We observe that $\frac{1}{t} + \frac{s}{t(t-s)} = \frac{1}{t-s}$, so that

$$\exp\left[-\frac{(b-m)^2}{4t}\right] \exp\left[-\frac{s}{t(t-s)}\frac{(b-m)^2}{4}\right] = \exp\left[-\frac{(b-m)^2}{4(t-s)}\right].$$

We integrate with respect to b (neglecting the indicator function) using Lemma A.3(ii) for $u = m$, $v = b$, and $w = x_0^1$, and $\exp\left[-\frac{(m - x_0^1)^2}{4t}\right] \leq 1$:

$$\int_{\mathbb{R}^{d+1}} q^{k,\alpha}(m, \tilde{x}, b, a; s, x_0) dadb \leq \mathbf{1}_{m > x_0^1} \frac{\|B\|_\infty C_T D2^{(d+1)/2}}{\sqrt{t-s}} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{t}^{d+1}}.$$

Since μ_0 is a probability measure, we have

$$\int_0^t \int_{\mathbb{R}^{2d+1}} q^{k,\alpha}(m, \tilde{x}, b, a; s, x_0) dadb \mu_0(dx_0) ds < +\infty.$$

This is (44) and completes the proof of Lemma 4.6. $\qquad\square$

Lemma 4.7. *For $k = m, 1, \ldots, d$, recall that*

$$p^{k,\beta}(m, x; t) = \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^{*1}, W}(b - a^1, x - a, t - s) p_V(m, a; s) db da ds.$$

The map $u \mapsto p^{k,\beta}(m, m - u, \tilde{x}; t)$ converges to 0 when u goes to 0^+.

Proof. Using the estimate (38) of $\partial_k p_{W^{*1}, W}$ and the estimate (ii) of Proposition 4.4 for p_V, we dominate the integrand which defines $p^{k,\beta}(m, m - u, \tilde{x}; t)$ by

$$q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) :=$$

$$\mathbf{1}_{m - u < b < m, a^1 < b} \frac{e^{-\frac{\left(b - a^1\right)^2}{4(t-s)} - \frac{(b - m + u)^2}{4(t-s)} - \frac{\|\tilde{x} - \tilde{a}\|^2}{4(t-s)} - \frac{\left(m - x_0^1\right)^2}{4s} - \frac{\left(m - a^1\right)^2}{4s} - \frac{\|\tilde{x}_0 - \tilde{a}\|^2}{4s}}}{\sqrt{t-s}\sqrt{2\pi(t-s)}^{d+1}\sqrt{2\pi s}^{d+1}}$$

up to a multiplicative constant, meaning that

$$\left| p^{k,\beta}(m, m-u, \tilde{x}; t) \right| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{2d+1}} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s)dbdads\mu_0(dx_0). \tag{45}$$

We integrate with respect to $\tilde{a}$ using Lemma A.3(ii) with $u = \tilde{x}$, $v = \tilde{a}$, and $w = \tilde{x}_0$:

$$\int_{R^{d-1}} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s)d\tilde{a} \leq \sqrt{2}^{d-1} \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^{d-1}} \frac{e^{-\frac{\left(b-a^1\right)^2}{4(t-s)} - \frac{(b-m+u)^2}{4(t-s)} - \frac{\left(m-x_0^1\right)^2}{4s} - \frac{\left(m-a^1\right)}{4s}}}{\sqrt{t-s}\sqrt{2\pi(t-s)}^2\sqrt{2\pi s}^2}.$$

We integrate with respect to a^1 between $-\infty$ and b using Lemma A.3(i$'$) for $u = b$, $v = a^1$, and $w = m$:

$$\int_{R^d} \mathbf{1}_{b>a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s)da \leq$$

$$\frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^d} \frac{e^{-\frac{(b-m)^2}{4t} - \frac{(b-m+u)^2}{4(t-s)} - \frac{\left(m-x_0^1\right)^2}{4s}}}{\sqrt{t-s}\sqrt{2\pi(t-s)}\sqrt{2\pi s}} \Phi_G\left(\sqrt{\frac{t}{s(t-s)2}}\left(b - \frac{s}{t}b - \frac{t-s}{t}m\right)\right)$$

$$= \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^d} \frac{e^{-\frac{(b-m)^2}{4t} - \frac{(b-m+u)^2}{4(t-s)} - \frac{\left(m-x_0^1\right)^2}{4s}}}{\sqrt{t-s}\sqrt{2\pi(t-s)}\sqrt{2\pi s}} \Phi_G\left(\sqrt{\frac{t-s}{2st}}(b-m)\right).$$

Since $b - m < 0$, using Lemma A.3(iii) we have

$$\int_{R^d} \mathbf{1}_{b>a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s)da \leq \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^d} \frac{e^{-\frac{(b-m)^2}{4t} - \frac{(b-m+u)^2}{4(t-s)} - \frac{\left(m-x_0^1\right)^2}{4s}}}{\sqrt{t-s}\sqrt{2\pi(t-s)}\sqrt{2\pi s}} e^{-\frac{t-s}{4st}(b-m)^2}.$$

Note that

$$e^{-\frac{(b-m)^2}{4t}} e^{-\frac{t-s}{4st}(b-m)^2)} = e^{-\frac{(b-m)^2)}{4s}}.$$

We integrate this last bound with respect to b between $m - u$ and m using Lemma A.3(i$'$) for the triplet $(m - u, b, m)$ and the fact that

$$\frac{s}{t}(m-u) + \frac{t-s}{t}m = \frac{s(m-u) + m(t-s)}{t},$$

to obtain

$$\int_{R^{d+1}} \mathbf{1}_{m-u<b<m, b<a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s)dadb \leq \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^{d+1}} \frac{e^{-\frac{\left(m-x_0^1\right)^2}{4s}}}{\sqrt{t-s}}.$$
$$\left[\Phi_G\left(\sqrt{\frac{t}{2s(t-s)}}\left(m - \frac{s(m-u)+m(t-s)}{t}\right)\right) - \Phi_G\left(\sqrt{\frac{t}{2s(t-s)}}\left(m - u - \frac{s(m-u)+m(t-s)}{t}\right)\right)\right].$$

Then,

$$\int_{R^{d+1}} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s)dadb \leq$$

$$\frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^{d+1}} \frac{e^{-\frac{\left(m-x_0^1\right)^2}{4s}}}{\sqrt{t-s}} \left[\Phi_G\left(\sqrt{\frac{s}{2t(t-s)}}u\right) - \Phi_G\left(-\sqrt{\frac{t-s}{2t(t-s)}}u\right)\right].$$

Note that

$$\lim_{u \to 0} \Phi_G \left(\sqrt{\frac{s}{2t(t-s)}} u \right) - \Phi_G \left(-\sqrt{\frac{t-s}{2t(t-s)}} u \right) = 0$$

and

$$\left| \Phi_G \left(\sqrt{\frac{s}{2t(t-s)}} u \right) - \Phi_G \left(-\sqrt{\frac{t-s}{2t(t-s)}} u \right) \right| \le 1.$$

Since μ is a probability measure, using Lebesgue's dominated convergence theorem,

$$\lim_{u \to 0^+} \int_0^t \int_{R^{2d+1}} \mathbf{1}_{m-u<b<m,b<a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) da\, db\, \mu_0(dx_0) ds = 0.$$

Finally, the estimate (45) yields $\lim_{u \to 0^+} p^{k,\beta}(m, m-u, \tilde{x}; t) = 0$. $\qquad\square$

5. The case $d = 1$

Proposition 5.1. *Consider the real diffusion X given by $dX_t = B(X_t)dt + A(X_t)dW_t$ where A, B fulfil* (4) *and* (5)*. Then the probability density function p_V satisfies Hypothesis* 2.1*, so for any initial law μ_0 and $F \in C_b^2(\mathbb{R}^2, \mathbb{R})$,*

$$\mathbb{E}\left[F(M_t, X_t)\right] = \mathbb{E}\left[F(X_0, X_0)\right] + \int_0^t \mathbb{E}\left[\mathcal{L}(F)(M_s, X_s)\right] ds$$

$$+ \frac{1}{2} \int_0^t \mathbb{E}\left[\partial_m F(X_s, X_s) \|A(X_s)\|^2 \frac{p_V(X_s, X_s; s)}{p_X(X_s; s)}\right] ds. \tag{46}$$

Proof. We perform a Lamperti transformation [18]. Without loss of generality, A can be chosen positive. In the case $d = 1$, Assumption (5), which states that there exists $c > 0$ such that for any $x \in \mathbb{R}$, $A^2(x) \ge c$, can be expressed as follows:

$$\exists c > 0 \text{ such that for any } x \in \mathbb{R}, \ A(x) \ge c. \tag{47}$$

Let φ be such that $\varphi' = \frac{1}{A}$ and $\varphi(0) = 0$, so that φ' is uniformly bounded and $\varphi \in C^2(\mathbb{R})$, as is the function A. Moreover, φ' being strictly positive, φ is strictly increasing, hence invertible, and we denote by φ^{-1} its inverse function. Under the initial condition $\varphi(0) = 0$, using Itô's formula, $Y = \varphi(X)$ satisfies

$$dY_t = \left[\frac{B}{A} \circ \varphi^{-1}(Y_t) - \frac{1}{2} A' \circ \varphi^{-1}(Y_t)\right] dt + dW_t, \qquad Y_0 = \varphi(X_0). \tag{48}$$

Let $A_\varphi = 1$ and $B_\varphi := \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2} A' \circ \varphi^{-1}$, which belongs to $C_b^1(\mathbb{R})$ as a consequence of the fact that $B \in C_b^1$ and $A \in C_b^2$. Obviously, $\varphi' > 0$ implies that φ is increasing, and $Y_t^* = \varphi(X_t^*) = \varphi(M_t)$.

Theorem 1.1 in [9] can easily be extended to the case where X admits an initial law μ_0; thus the law of the pair (Y_t^*, Y_t) admits a density with respect to the Lebesgue measure. Moreover, [9, Lemma 2.2] sets out

$$p_V(b, a; t) = \frac{p_{Y^*,Y}(\varphi(b), \varphi(a); t)}{A(b)A(a)}.$$

Now, applying Theorem 2.4 to the pair $(B_\varphi, 1)$, we find that the density $p_{Y^*,Y}$ satisfies Hypothesis 2.1.

Since the functions A and φ are continuous,

$$\lim_{u \to 0^+} p_V(b, b - u; t) = \frac{p_{Y^*, Y}(\varphi(b), \varphi(b); t)}{A^2(b)},$$

which means p_V satisfies Hypothesis 2.1(ii).

Now, using (47),

$$\sup_{u>0} p_V(b, b - u; t) \leq \frac{1}{c^2} \sup_{u>0} p_{Y^*, Y}(\varphi(b), \varphi(b - u); t).$$

Since φ is increasing, if $u > 0$, then $\varphi(b - u) < \varphi(b)$; writing $v = \varphi(b) - \varphi(b - u)$, we have $v > 0$, and

$$\sup_{u>0} p_V(b, b - u; t) \leq \frac{1}{c^2} \sup_{v>0} p_{Y^*, Y}(\varphi(b), \varphi(b) - v; t).$$

After the change of variable $m = \varphi(b)$, so that $db = A(b)dm$, we have

$$\int_0^T \int_{\mathbb{R}} \sup_{u>0} p_{Y^*, Y}(\varphi(b), \varphi(b) - u; t) db\, dt = \int_0^T \int_{\mathbb{R}} A(\varphi^{-1}(m)) \sup_{u>0} p_{Y^*, Y}(m, m - u; t) dm\, dt.$$

Since A is bounded and $p_{Y^*, Y}$ satisfies Hypothesis 2.11(i),

$$\int_0^T \int_{\mathbb{R}} A(\varphi^{-1}(m)) \sup_{u>0} p_{Y^*, Y}(m, m - u; t) dm\, dt < \infty,$$

and p_V satisfies Hypothesis 2.1(i)–(ii). $\qquad\square$

6. Conclusion

This paper establishes a PDE of which the density of the running-maximum–diffusion-process pair $[M_t, X_t]$ is a weak solution, under a quite natural assumption on the regularity of p_V around the boundary of Δ. This assumption is fulfilled when the matrix coefficient of the diffusion A is the identity matrix or when the dimension $d = 1$. This PDE is degenerate; thus, the classical results on uniqueness cannot be applied here. The case of a non-constant matrix A is an open problem. Such a generalization could be useful in practical applications, such as the management of barrier options, in models including stochastic volatility.

Appendix A. Tools

A.1. Malliavin calculus tools

The material in this subsection is taken from [21, Section 1.2].

Let $\mathbb{H} = L^2([0, T], \mathbb{R}^d)$ be endowed with the usual scalar product $\langle ., . \rangle_{\mathbb{H}}$ and the associated norm $\|.\|_{\mathbb{H}}$. For all $h \in \mathbb{H}$, $W(h) := \int_0^T h(t) dW_t$ is a centered Gaussian variable with variance equal to $\|h\|_{\mathbb{H}}^2$. If $(h_1, h_2) \in \mathbb{H}^2$ and $\langle h_1, h_2 \rangle_{\mathbb{H}} = 0$, then the random variables $W(h_i)$, $i = 1, 2$, are independent.

Let $\mathcal{S}$ denote the class of smooth random variables F defined by

$$F = f(W(h_1), \ldots, W(h_n)), \ n \in \mathbb{N}, \ h_1, \ldots, h_n \in \mathbb{H}, \ f \in C_b(\mathbb{R}^n). \tag{49}$$

Definition A.1. *The derivative of the smooth variable F defined in* (49) *is the* $\mathbb{H}$*-valued random variable given by* $DF := \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n)) h_i$.

We denote the domain of the operator D in $L^2(\Omega)$ *by* $\mathbb{D}^{1,2}$, *meaning that* $\mathbb{D}^{1,2}$ *is the closure of the class of smooth random variables* $\mathcal{S}$ *with respect to the norm*

$$\|F\|_{1,2} = \left\{ \mathbb{E}\big[|F|^2\big] + \mathbb{E}\big[\|DF\|_{\mathbb{H}}^2\big] \right\}^{1/2}.$$

Definition A.2. $\mathbb{L}^{1,2}$ *is the set of processes* $(u_s, s \in [0, T])$ *such that*

$$u \in L^2\big(\Omega \times [0, T], \mathbb{R}^d\big),$$

and for all $s \in [0, T]$, u_s *belongs to* $\mathbb{D}^{1,2}$ *and*

$$\|u\|_{\mathbb{L}^{1,2}}^2 = \|u\|_{L^2([0,T]\times\Omega)}^2 + \|Du\|_{L^2([0,T]^2\times\Omega)}^2 < \infty.$$

Definition A.3. *Let* $u \in \mathbb{L}^{1,2}$; *then the divergence* $\delta(u)$ *is the unique random variable of* $L^2(\Omega)$ *such that* $\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\mathbb{H}}]$ *for every smooth random variable* $F \in \mathcal{S}$. *We apply* [21, Definition 1.3.1] *with* $u \in \mathbb{L}^{1,2}$ *and* $G \in \mathbb{D}^{1,2}$:

$$\mathbb{E}[G\delta(u)] = \mathbb{E}[\langle DG, u\rangle_{\mathbb{H}}]. \tag{50}$$

Let $x_0 \in \mathbb{R}^d$. *We introduce the exponential martingale*

$$Z_t^{x_0} := \exp\left[\sum_{k=1}^{d}\left(\int_0^t B^k(x_0+W_s)dW_s^k - \frac{1}{2}\int_0^t \big(B^k(x_0+W_s)\big)^2 ds\right)\right]. \tag{51}$$

When there is no ambiguity, we will omit the exponent x_0.

Lemma A.1. *Let* $B \in C_b^1(\mathbb{R}^d, \mathbb{R})$; *then for all* $x_0 \in \mathbb{R}^d$, *the process*

$$(B(x_0 + W_s)Z_s^{x_0}, \ s \in [0, T])$$

belongs to $\mathbb{L}^{1,2}$.

Proof. Let x_0 be fixed. In this proof we omit the exponent x_0. Note that

$$Z_t^2 = \exp\left(2\sum_{k=1}^{d}\int_0^t B^k(x_0+W_s)dW_s^k - \frac{4}{2}\int_0^t \|B(x_0+W_s)\|^2 ds + \frac{4-2}{2}\int_0^t \|B(x_0+W_s)\|^2 ds\right)$$

$$\leq e^{T\|B\|_\infty^2}\exp\left(2\sum_{k=1}^{d}\int_0^t B^k(x_0+W_s)dW_s^k - \frac{4}{2}\int_0^t \|B(x_0+W_s)\|^2 ds\right).$$

Then, Z_t belongs to $L^2(\Omega)$ for all $t \in [0, T]$, since

$$\sup_{t\in[0,T]} \mathbb{E}(Z_t^2) \leq e^{T\|B\|_\infty^2}. \tag{52}$$

Note that $Z_t = 1 + \sum_{k=1}^{d} \int_0^t B^k(x_0 + W_s)Z_s dW_s^k$, $t \in [0, T]$. Using Lemma 2.2.1 and Theorem 2.2.1 of [21], as well as the definition of $\mathbb{L}^{1,2}$ applied to the $\mathbb{R}^{d+1}$-valued process $Y = (W, Z)$ with a null drift coefficient, the $(d + 1) \times d$ matrix Σ defined by

$$[\sigma^{j,k}(y), 1 \leq j, k \leq d] = I_d, \qquad \sigma^{d+1,k}(y) = B^k(x_0^1 + y^1, \ldots, x_0^d + y^d)z, \qquad k = 1, \ldots, d,$$

we obtain that Z belongs to $\mathbb{L}^{1,2}$. Since B is continuously differentiable with bounded derivatives, the process $(B(W_s + x_0)Z_s, \ s \in [0, T])$ belongs to $\mathbb{L}^{1,2}$. $\qquad \square$

The following remark will frequently be used: by [9, p. 135, line 15] or [21, Exercise 1.2.11, p. 36], we have

$$D_s^1 W_t^{1*} = \mathbf{1}_{[0,\tau]}(s), \quad \text{where } \tau := \inf\{s, W_s^{1*} = W_t^{1*}\}. \tag{53}$$

A.2. Estimates for the Brownian motion case

Let us recall the density of the distribution of the pair $(W_t^{*,1}, W_t^1)$, where W^1 is a one-dimensional Brownian motion and $W^{*,1}$ its running maximum (see, e.g., [17, Section 3.2] or [15]):

$$p_{W^{1*},W^1}(b, a; t) = 2\frac{2b - a}{\sqrt{2\pi t^3}} \exp{-\frac{(2b-a)^2}{2t}} \mathbf{1}_{b > \sup(a,0)}.$$

Thus, using the independence of the components of the process W, the law of (W_t^{1*}, W_t) has a density with respect to the Lebesgue measure on $\mathbb{R}^{d+1}$, denoted by $p_{W^{1*},W}(.;t)$:

$$p_{W^{1*},W}(b, a; t) = 2\frac{(2b - a^1)}{\sqrt{(2\pi)^d t^{d+2}}} e^{-\frac{(2b-a^1)^2}{2t} - \frac{\sum_{k=2}^{d} |a^k|^2}{2t}} \mathbf{1}_{b \geq 0, b \geq a^1},$$
$$b \in \mathbb{R}, \ a = (a^1, \ldots, a^d) \in \mathbb{R}^d. \tag{54}$$

Lemma A.2. *(i) For all $t > 0$, $p_{W^{*1},W}(.;t)$ is the restriction to $\bar{\Delta}$ of a $C^\infty(\mathbb{R}^{d+1})$ function, and there exists a universal constant D such that for $x = b, a^1, a^2, \ldots, a^d$,*

$$\left|\partial_x p_{W^{*1},W}(b, a; t)\right| \leq \frac{D}{\sqrt{(4\pi)^d t^{d+2}}} e^{-\frac{b^2 + (b-a^1)^2}{4t} - \sum_{k=2}^{d} \frac{(a^k)^2}{4t}} \mathbf{1}_{b > \max(a^1,0)}. \tag{55}$$

As a consequence,

$$\sum_{x=b,a^1,\ldots,a^d} \left|\partial_x p_{W^{*1},W}(b, a; t - s)\right| \in L^1([0, t] \times \mathbb{R}^{d+1}, dbdads).$$

(ii) We have

$$p_0(m, x; \ t, x_0) \leq \frac{e^{-\frac{(m-x^1)^2}{4t} - \frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t} - \frac{(m-x_0^1)^2}{4t}}}{\sqrt{(2\pi)^{d+1} t^{d+1}}} \mathbf{1}_{m > \max(x^1, x_0^1)}$$
$$= 2^{(d+1)/2} \phi_{d+1}(m - x^1, m - x_0^1, \tilde{x} - \tilde{x}_0; \ 2t) \mathbf{1}_{m > \max(x^1, x_0^1)},$$

Proof. (i) Let p_W be the density of a d-dimensional Brownian motion, and the density of the law of W_t for all $t > 0$, with $p_W(.; t) \in C^\infty(\mathbb{R}^d)$ given by

$$p_W(x; t) = \frac{1}{\sqrt{2^d \pi^d t^d}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \ldots, x^d) \in \mathbb{R}^d.$$

Its derivative with respect to x^1 is

$$\partial_{x^1} p_W(x; t) = -\frac{x^1}{\sqrt{2^d \pi^d t^{d+2}}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \ldots, x^d) \in \mathbb{R}^d.$$

Its second derivatives are

$$\partial^2_{x^1 x^k} p_W(x; t) = \frac{x^1 x^k}{\sqrt{2^d \pi^d t^{d+4}}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \ldots, x^d) \in \mathbb{R}^d, \quad k = 2, \ldots, d,$$

$$\partial^2_{x^1 x^1} p_W(x; t) = \frac{(x^1)^2 - t}{\sqrt{2^d \pi^d t^{d+4}}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \ldots, x^d) \in \mathbb{R}^d.$$

Using [14, (2.1), p. 106] we obtain the analogue of [14, (2.2), p. 107]: there exists a constant C such that

$$\left| \partial^2_{x^1 x^1} p_W(x; t) \right| + \left| \partial^2_{x^1 x_k} p_W(x; t) \right| \leq \frac{C}{t} p_W(x; 2t), \quad k = 1, \ldots, d, \quad t > 0, \quad x \in \mathbb{R}^d. \quad (56)$$

Recall (54):

$$p_{W^{*1}, W}(b, a; t) = 2 \frac{2b - a^1}{\sqrt{(2\pi)^d t^{d+2}}} e^{-\frac{(2b - a^1)^2}{2t} - \sum_{k=2}^d \frac{(a^k)^2}{2t}} \mathbf{1}_{b \geq a^1_+}, \quad \forall (b, a) \in \mathbb{R}^{d+1}, \quad t > 0.$$

We observe that

$$p_{W^{*1}, W}(b, a; t) = -2 \partial_{x^1} p_W(2b - a^1, a^2, \ldots, a^d; t) \mathbf{1}_{b \geq a^1_+}. \quad (57)$$

Then $p_{W^{*1}, W}(., .; t)$ is the restriction to $\overline{\Delta}$ of a C^∞ function. Moreover, using the chain rule, with x being $(b, a^1, \ldots, a^d)$, we have

$$\left| \partial_x p_{W^{*1}, W}(b, a; t) \right| \leq \frac{4C}{t} p_W(2b - a^1, a^2, \ldots, a^d; 2t) \mathbf{1}_{b \geq a^1_+}. \quad (58)$$

On the set $\{(b, a), \ b > \max(0, a^1)\}$ we have

$$(2b - a^1)^2 = (b + b - a^1)^2 \geq (b)^2 + (b - a^1)^2. \quad (59)$$

Plugging the estimate (59) into (58) yields (55) with $D = 2^3 C$.

(ii) Recalling the definition

$$p_0(m, x; t, x_0) = p_{W^{1*}, W}(m - x_0^1, x - x_0; t)$$

$$= 2 \frac{m - x^1 + m - x_0^1}{\sqrt{(2\pi)^d t^{d+2}}} e^{-\frac{(m - x^1 + m - x_0^1)^2}{2t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{2t}} \mathbf{1}_{m \geq x^1 \vee x_0^1},$$

we deduce the standard bound which uses $xe^{-x^2} \leq e^{-x^2/2}$ and $(m - x^1 + m - x_0^1)^2 \geq (m - x^1)^2 + (m - x_0^1)^2$:

$$p_0(m, x; t, x_0) \leq \frac{e^{-\frac{(m-x^1)^2}{4t} - \frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t} - \frac{(m-x_0^1)^2}{4t}}}{\sqrt{(2\pi)^{d+1}t^{d+1}}} \mathbf{1}_{m>x^1 \vee x_0^1}$$

$$= 2^{(d+1)/2} \phi_{d+1}(m - x^1, m - x_0^1, \tilde{x} - \tilde{x}_0; 2t) \mathbf{1}_{m>x^1 \vee x_0^1}.$$

Lemma A.3. *For all $0 < s < t$, $k \geq 1$ and all $u, v, w \in \mathbb{R}^k$, the following hold:*

(i) $\dfrac{\|u - v\|^2}{t - s} + \dfrac{\|v - w\|^2}{s} = \dfrac{t}{s(t - s)} \left\| v - \left(\dfrac{s}{t} u + \dfrac{t - s}{t} w \right) \right\|^2 + \dfrac{\|u - w\|^2}{t};$

(i′) $k = 1$, $\displaystyle\int_{-\infty}^{b} \dfrac{e^{-\frac{(u-v)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \dfrac{e^{-\frac{(w-v)^2}{4s}}}{\sqrt{2\pi s}} dv = \sqrt{2} \dfrac{e^{-\frac{(u-w)^2}{4t}}}{\sqrt{2\pi t}} \Phi_G\left(\sqrt{\dfrac{t}{s(t-s)}} \left[b - \left(\dfrac{s}{t} u + \dfrac{t-s}{t} w \right) \right] \right);$

(ii) $\displaystyle\int_{\mathbb{R}^k} \dfrac{e^{-\frac{\|u-v\|^2}{4(t-s)}}}{\sqrt{(2\pi(t-s))^k}} \dfrac{e^{-\frac{\|w-v\|^2}{4s}}}{\sqrt{(2\pi s)^k}} dv = 2^{k/2} \dfrac{e^{-\frac{\|u-w\|^2}{4t}}}{\sqrt{(2\pi t)^k}};$

(iii) for $u > 0$, $1 - \Phi_G(u) := \displaystyle\int_u^{+\infty} \dfrac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = \Phi_G(-u) \leq \dfrac{e^{-\frac{u^2}{2}}}{2}.$

Proof. Part (i) is proved by a development of both sides, then an identification of the coefficients of the squared norms and scalar products $\|u\|^2$, $\|v\|^2$, $\|w\|^2$, $u.v$, $u.w$, $v.w$. So we deduce (i′) as the integral of

$$\frac{e^{-\frac{t}{4s(t-s)}(v - (\frac{s}{t}u + \frac{t-s}{t}w))^2 - \frac{(u-w)^2}{4t}}}{\sqrt{2\pi(t-s)}\sqrt{2\pi s}}$$

with respect to v up to b.

Part (ii) is a consequence of Part (i), then an integration on $\mathbb{R}^k$ of the Gaussian density with respect to dv.

(iii) The function

$$u \mapsto \Phi_G(u) - \frac{e^{-\frac{u^2}{2}}}{2}$$

is null at 0, it has a null limit when u goes to $-\infty$, and its derivative is

$$u \mapsto -\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} + u\frac{e^{-\frac{u^2}{2}}}{2}.$$

Its derivative vanishes at $\sqrt{2/\pi}$ and is negative for $u \leq \sqrt{2/\pi}$ and positive after. Thus, $u \mapsto \Phi_G(u) - \frac{e^{-\frac{u^2}{2}}}{2}$ is negative for $u \leq 0$. $\square$

A.3. Proof of Remark 2.3, boundary conditions of the PDE

Here we assume that p_V is regular enough. Let $\mu_0(dx) = f_0(x)dx$. Using Theorem 2.2, (6) means that for all $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R}^d)$,

$$\int_{\bar{\Delta}} F(m, x)p_V(m, x; t)dmdx = \int_{\mathbb{R}^d} F(m, m, \tilde{x})f_0(m, \tilde{x})dmd\tilde{x}$$
$$+ \int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x)p_V(m, x; s)dmdxds$$
$$+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \|A^1(m, \tilde{x})\|^2 \partial_m F(m, m, \tilde{x})p_V(m, m, \tilde{x}; s)dmd\tilde{x}ds, \tag{60}$$

recalling $\mathcal{L} = B^i \partial_{x_i} + \frac{1}{2} \Sigma^{ij} \partial^2_{x_i, x_j}$, where $\Sigma = AA^t$.

(i) Integrating by parts with respect to a convenient dx_k in

$$\int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x)p_V(m, x; s)dmdxds$$

and noting that the support of $p_V(., .; s)$ is $\bar{\Delta}$, we see that the boundary terms concern only the component x^1:

$$\int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x)p_V(m, x; s)dmdxds = - \int_0^t \int_{\bar{\Delta}} F(m, x)\partial_{x^k}(B^k p_V)(m, x; s)dmdxds$$
$$- \frac{1}{2} \int_0^t \int_{\bar{\Delta}} \partial_{x^l}F(m, x)\partial_{x^k}[\Sigma^{k,l}(m, x)p_V(m, x; s)]dmdxds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \left(F(m, m, \tilde{x})B^1(m, \tilde{x}) + \frac{1}{2}\partial_{x^k}F(m, m, \tilde{x})\Sigma^{1,k}(m, \tilde{x}) \right) p_V(m, m, \tilde{x}; s)dmd\tilde{x}ds.$$

We again perform integration by parts on the second term on the right-hand side above:

$$- \frac{1}{2} \int_0^t \int_{\bar{\Delta}} \partial_{x^l}F(m, x)\partial_{x^k}\Sigma^{k,l}(m, x)p_V(m, x; s)]dmdxds$$
$$= \frac{1}{2} \int_0^t \int_{\bar{\Delta}} F(m, x)\partial^2_{x^k, x^l}[\Sigma^{k,l}p_V](m, x; s)dmdxds$$
$$- \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} F(m, m, \tilde{x})\partial_{x^k}[\Sigma^{1,k}p_V](m, m, \tilde{x}; s)dmd\tilde{x}ds.$$

Gathering these equalities yields

$$\int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x)p_V(m, x; s)dmdxds = \int_0^t \int_{\bar{\Delta}} F(m, x)\mathcal{L}^* p_V(m, x; s)dmdxds$$
$$- \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} F(m, m, \tilde{x})\partial_{x^k}[\Sigma^{1,k}p_V](m, m, \tilde{x}; s)dmd\tilde{x}ds \tag{61}$$
$$+ \int_0^t \int_{\mathbb{R}^d} \left(F(m, m, \tilde{x})B^1(m, \tilde{x})p_V(m, m, \tilde{x}; s) + \frac{1}{2}[\partial_{x^k}F\Sigma^{1,k}p_V](m, m, \tilde{x}; s) \right) dmd\tilde{x}ds.$$

(ii) Using $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$ with compact support in Δ (so $F(m, m, \tilde{x}) = 0$), we deduce the equality in Δ:

$$\partial_t p_V(m, x; s) = \mathcal{L}^* p_V(m, x; s), \quad \forall s > 0, \quad (m, x) \in \Delta. \tag{62}$$

We use (60), (61), and (62) applied to $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$ with compact support in $\bar{\Delta}$:

$$
\begin{aligned}
0 = {} & \int_{\mathbb{R}^d} F(m, m, \tilde{x}) f_0(m, \tilde{x}) dm d\tilde{x} - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} F(m, m, \tilde{x}) \partial_{x^k} [\Sigma^{1,k} p_V](m, m, \tilde{x}; s) dm d\tilde{x} ds \\
& + \int_0^t \int_{\mathbb{R}^d} \left(F(m, m, \tilde{x}) B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) + \frac{1}{2} [\partial_{x^k} F \Sigma^{1,k} p_V](m, m, \tilde{x}; s) \right) dm d\tilde{x} ds \\
& + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \|A^1(m, \tilde{x})\|^2 \partial_m F(m, m, \tilde{x}) p(m, m, \tilde{x}; s) dm d\tilde{x} ds.
\end{aligned}
\tag{63}
$$

We now perform integration by parts on the last two terms:

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} & \left([\partial_{x^k} F . \Sigma^{1,k} . p_V](m, m, \tilde{x}; s) + \|A^1(m, \tilde{x})\|^2 \partial_m F(m, m, \tilde{x}) p_V(m, m, \tilde{x}; s) \right) dm d\tilde{x} ds = \\
& -\int_0^t \int_{\mathbb{R}^d} \left([F . \partial_{x^k} (\Sigma^{1,k} p_V)](m, m, \tilde{x}; s) + \partial_m (\|A^1\|^2 p_V)(m, m, \tilde{x}; s) \right) dm d\tilde{x} ds
\end{aligned}
\tag{64}
$$

Plugging (64) into (63) yields the boundary condition, namely a PDE of which p_V is a solution in the weak sense:

$$
\begin{aligned}
B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) = {} & \tfrac{1}{2} \sum_{k \geq 1} \partial_{x^k} (\Sigma^{1,k} p_V)(m, m, \tilde{x}; s) \\
& + \tfrac{1}{2} \sum_{k \geq 1} \partial_{x^k} (\Sigma^{1,k} p_V)(m, m, \tilde{x}; s) + \tfrac{1}{2} \partial_m (\|A^1\|^2 p_V)(m, m, \tilde{x}; s),
\end{aligned}
$$

simplified as

$$
B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) = \sum_{k \geq 1} \partial_{x^k} (\Sigma^{1,k} p_V)(m, m, \tilde{x}; s) + \frac{1}{2} \partial_m (\|A^1\|^2 p_V)(m, m, \tilde{x}; s)
$$

with the initial condition $p_V(m, m, \tilde{x}; 0) = f_0(m, \tilde{x})$. $\qquad\square$

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