CONSTRUCTIBLE LATTICES

M. F. JANOWITZ*

(Received 30th June 1970)

Communicated by D. P. Finch

1. Introduction

This paper is a continuation of [2] and will use the terminology and notation introduced therein. Basically, we seek to investigate the structure of indexed lattices in which the set of lengths of the minimal indexing chains is bounded.

In determining the structure of thus type of lattice we shall make frequent use of the notion of (internal) direct product. We say that the lattice L with 0 and I is the *direct product* of the family of lattices $(L_\alpha)_{\alpha \in A}$ and write

$$L = \Pi (L_\alpha; \alpha \in A)$$

if the following conditions hold:

(i) The indexing set A has at least two elements;

(ii) There exists a family of nonzero pairwise disjoint central elements $(z_\alpha)_{\alpha \in A}$ such that

$$\vee_\alpha z_\alpha = I \text{ and } (\forall \alpha \in A) L_\alpha = [0, z_\alpha];$$

(iii) If $x_\alpha \in L_\alpha$ $(\alpha \in A)$ then $\vee_\alpha x_\alpha$ exists in L.

We mention that the mapping $x \to (x \wedge z_\alpha)_{\alpha \in A}$ is then an isomorphism of L onto the usual external direct product of the lattices $(L_\alpha)_{\alpha \in A}$. It will prove convenient to call each L_α a *direct factor* or simply a *factor* of L, to call L *reducible* if it has a central element $z \notin \{0, I\}$, and to call it *irreducible* otherwise.

It turns out that with the assumption of completeness, the class of lattices we seek may be formed from Boolean algebras by means of a finite number of applications of the operations of direct product and disjoint sum. Essentially, we wish to start with the class of all Boolean algebras. The direct product operation does not lead us out of this class, so our next class will be all disjoint sums of Boolean algebras. Now if

* Research supported by NSF Grant GP 11580

$$L = DS(L_\alpha; \alpha \in A)$$

where each summand is a Boolean algebra or a disjoint sum of Boolean algebras, then L itself is a disjoint sum of Boolean algebras. Thus our next class of lattices must be formed by making use of the direct product operation. This discussion should serve to motivate the next definition.

DEFINITION. Let us agree to call a lattice L *0-constructible* if it is a Boolean algebra and *1-constructible* if it is a disjoint sum of Boolean algebras. Suppose that *i-constructibility* has been defined for $i < 2n$. We call a lattice $(2n)$-*constructible* if it is a direct product of i-constructible lattices ($i < 2n$) with at least one factor $(2n-1)$-constructible; it is called $(2n+1)$-*constructible* if it is a disjoint sum of *i-constructible* lattices ($i \leqq 2n$) with at least one $(2n)$-constructible disjoint summand. Finally, L is called *constructible* if it is i-constructible for some non-negative integer i. Every constructible lattice is evidently orthomodular in the sense of [1], p. 53.

We ask the reader to recall that a nonzero element e of an orthomodular lattice L is called *indexed* if there is finite chain $I = e_0 > e_1 > \cdots > e_n = e$ with e_i nearly central in $[0, e_{i+1}]$ for $i = 1, 2, \cdots, n$, and that such a chain is said to have length n. Every indexed element e may be connected with I by a unique such chain of minimal length ([2], Theorem 11), and we agree to say that e has *index n* (denoted $K(e) = n$) if its minimal indexing chain has length n. Finally, we say that the lattice L has *finite index n* and write $K(L) = n$ if L is indexed in the sense of [2], and n is the least upper bound of the set of indices of its indexed elements. If we assign index 1 to the two element Boolean algebra, we see that every Boolean algebra, as well as every disjoint sum of Boolean algebras, has index 1.

2. Constructibility

The first two lemmas provide the key tools that we shall use throughout the paper. It will henceforth be assumed that all lattices in sight are orthomodular.

LEMMA. 1. *Let $L = DS(L_\alpha; \alpha \in A)$. Then L is indexed if and only if each summand L_α is indexed. An element e of L is indexed in L if and only if it is indexed in some summand L_α, and the index of e is then the same in L as it is in the summand.*

PROOF. The lemma follows immediately from the fact that a chain

$$I = e_0 > e_1 > \cdots > e_n = e$$

is an indexing chain in $L \Leftrightarrow$ it is an indexing chain in some disjoint summand of L.

LEMMA 2. *Let* $L = \Pi\ (L_\alpha; \alpha \in A)$. *Then* L *is indexed if and only if each factor* L_α *is indexed. An element* $e \neq 0$ *of* L *is indexed in* L *if and only if it is central in* L *or indexed in some factor* L_α. *If* e *is indexed in* L_α *with* $K(e) = n$ *therein, then* $K(e) = n$ *in* L *if* L_α *is reducible and* $K(e) = n+1$ *otherwise.*

PROOF. Let us write $L_\alpha = [0, z_\alpha]$ with z_α central in L and begin by observing that if

$$z_\alpha > e_1 > \cdots > e_n$$

is an indexing chain in L_α, then

$$I > z_\alpha > e_1 > \cdots > e_n$$

is an indexing chain in L. Since each $x \in L$ may be expressed in the form $x = \vee_\alpha x_\alpha$ with $x_\alpha \in L_\alpha$, it follows that if each factor L_α of L is indexed, so is L.

Suppose now that L is indexed. We must establish that each factor L_α is indexed. It clearly suffices to show that if $e \in L_\alpha$ is indexed in L, it is indexed in L_α. To see this, let

$$I = e_0 > e_1 > \cdots > e_n = e$$

be a minimal indexing chain, and let $e \in L_\alpha$. If $n = 1$, then e is central in L, hence in L_α, so e is indexed therein. If $n > 1$ then by [2], Theorem 11, e_1 is an atom of the center of L, so $e_1 \leqq z_\alpha$. Then

$$z_\alpha \geqq e_1 > \cdots > e_n = e$$

is an indexing chain for e in L_α.

It is an easy matter to dispose of the nature of the indexed elements of L. For if

$$I = e_0 > e_1 > \cdots > e_n = e$$

is a minimal indexing chain, and if $n > 1$, then e_1 is an atom of the center of L, and consequently $e_1 \leq z_\alpha$ for some α. It follows that e is indexed in L_α. On the other hand, if $n = 1$, then e is central in L as desired.

There only remains the consideration of the index of e when e is indexed in L_α. Let

$$z_\alpha > e_1 > \cdots > e_n = e$$

be a minimal indexing chain in L_α. If L_α is irreducible, then by [2], Theorem 9, e_1 is central in a disjoint summand of L_α, z_α is an atom of the center of L, and

$$I > z_\alpha > e_1 > \cdots > e_n = e$$

is a minimal indexing chain of length $n + 1$ in L. On the other hand, if L_α is reducible, then e_1 is central in L_α, hence in L. It follows that

$$I > e_1 > \cdots > e_n = e$$

is a minimal indexing chain in L, so $K(e) = n$ in L.

THEOREM 3. *Every constructible lattice is indexed; furthermore, if L is $(2n)$-constructible or $(2n+1)$-constructible then $K(L) = n+1$.*

PROOF. We will proceed by induction on n. We begin by observing that for $n = 0$, every 0-constructible or 1-constructible lattice has index 1. Suppose that the theorem is true for all integers i such that $0 \leqq i < n$, and let L be $(2n)$-constructible. In view of the definition of constructibility, we may write $L = \Pi\,(L_\alpha; \alpha \in A)$ with each factor L_α 0-constructible or $(2i+1)$-constructible $(0 \leqq i < n)$ and at least one factor $L_{\alpha'}$ which is $(2n-1)$-constructible. By our induction hypothesis each L_α is indexed with $K(L_\alpha) \leqq n$, and in fact $K(L_{\alpha'}) = n$. Since $L_{\alpha'}$ is irreducible, we may apply Lemma 2 to deduce that L is indexed with $K(L) = n+1$. An application of Lemma 1 will now establish the theorem for the case where L is $(2n+1)$-constructible.

3. The Completion by Cuts

We shall follow the notation of [1], pp. 126–7. Given a subset X of L we let X^* be the set of upper bounds of X and X^+ its set of lower bounds. The completion by cuts of L is then

$$\bar{L} = \{X^{*+} : X \subseteq L\}.$$

By [1], Theorem 22, p. 126, $\bar{L}$ is a complete lattice and the mapping $b \to [0, b]$ embeds L in $\bar{L}$ so as to preserve any existing suprema of infima of subsets of L. It is well known that the completion by cuts of a Boolean algebra is itself a Boolean algebra, but MacLaren [4] (among others) has provided an example of an orthomodular lattice whose completion by cuts is *not* orthomodular. Our goal here is to prove that if L has finite index n, then its completion by cuts is constructible, hence an indexed orthomodular lattice.

LEMMA 4. *For any orthomodular lattice L the center of $\bar{L}$ is a complete sublattice of $\bar{L}$.*

PROOF. By [4], Theorem 2.4, p. 600, $\bar{L}$ is orthocomplemented, and it follows immediately from [3], Theorem 8, p. 6, that the center of $\bar{L}$ is closed under arbitrary intersection.

We shall also need the following lemma, whose easy proof is left to the reader.

LEMMA 5. *If $L = DS(L_\alpha; \alpha \in A)$, then $\bar{L} = DS(\bar{L}_\alpha; \alpha \in A)$.*

We are now ready to present the main result of this section.

THEOREM 6. *If L is indexed with $K(L) = n+1$, then $\bar{L}$ is either $(2n)$-constructible or $(2n+1)$-constructible.*

PROOF. We begin by observing that if $K(L) = 1$, then every indexed element of L is nearly central. Since every element is the join of a family of indexed elements, and since it is clear from [2], Theorem 9, that the near center of L is closed under the formation of any existing suprema, we see that every element of L is nearly central. Hence by [2], Theorem 9, L is either a Boolean algebra or a disjoint sum of Boolean algebras. It is immediate that $\bar{L}$ is either a Boolean algebra or a disjoint sum of Boolean algebras, so $\bar{L}$ is either 0-constructible or 1-constructible, as claimed. We will proceed by induction on n.

Suppose the theorem is true for all lattices whose index is at most n, and let $K(L) = n+1$. Suppose first that L is reducible. Then every nearly central element of L is in fact central. Let $\{z_\alpha : \alpha \in A\}$ denote the collection of atoms of the center of L. Then every ideal $[0, z_\alpha]$ is central in $\bar{L}$, and by Lemma 4,

$$\bar{I} = \bigvee_{\alpha \in A} [0, z_\alpha]$$

is central in $\bar{L}$. Let J denote the unique complement of $\bar{I}$ in $\bar{L}$. Then $a \in J$ implies $a \perp z_\alpha$ for all $\alpha \in A$. If a were not central, it would follow that a dominates an indexed element b whose index is greater than 1. But if

$$I = b_0 > b_1 > \cdots > b_k = b$$

is a minimal indexing chain, then $k > 1$ forces b_1 to be an atom of the center of L. At this point we have $a \perp b_1$ and

$$a \wedge b_1 \geqq b > 0,$$

a contradiction. We deduce from this that every element of J is central in L. Now $K \leqq J$ implies

$$K = \bigvee \{[0, a];\ a \in K\}$$

so K is central in $\bar{L}$. It is immediate that either $J = (0)$ or the interval $[(0), J]$ is a Boolean factor of $\bar{L}$. Now for each z_α, an obvious modification of the proof of Lemma 2 will show that $[0, z_\alpha]$ is indexed with

$$K([0, z_\alpha]) \leqq n.$$

By induction, the completion by cuts of $[0, z_\alpha]$ is constructible, and in fact, at most $(2n-1)$-constructible. There must exist a minimal indexing chain of the form

$$I = a_0 > a_1 > \cdots > a_n > a_{n+1}$$

and so a_1 is an atom of the center of L and $K[0, a_1]) = n$. Now $[0, a_1]$ admits

https://doi.org/10.1017/S1446788700010776 Published online by Cambridge University Press

a disjoint sum decomposition, so by Lemma 5, its completion by cuts is irreducible. Hence by our induction hypothesis, the completion by cuts of $[0, a_1]$ is $(2n-1)$-constructible. We now use the fact that the interval from (0) to $[0, z_\alpha]$ in $\bar{L}$ coincides with the completion by cuts of $[0, z_\alpha]$ to deduce that $\bar{L}$ is $(2n)$-constructible.

Suppose now that L is irreducible. Then by [2], Theorem 9,

$$L = DS(L_\alpha; \alpha \in A)$$

with each L_α reducible. By Lemma 1, each summand is indexed with index at most $n+1$, and at least one summand has index equal to $n+1$. Applying the above argument, we see that the completion by cuts of each summand is at most $(2n)$-constructible, with at least one being $(2n)$-constructible. By Lemma 5,

$$\bar{L} = DS(\bar{L}_\alpha; \alpha \in A),$$

so $\bar{L}$ is $(2n+1)$-constructible.

COROLLARY 7. *The completion by cuts of a constructible lattice is constructible.*

COROLLARY 8. *If L has finite index n, then $\bar{L}$ is orthomodular with finite index n.*

COROLLARY 9. *A complete orthomodular lattice is constructible if and only if it has a finite index.*

References

[1] G. Birkhoff, *Lattice theory* (Amer. Math. Colloq. Publ. XXV, 3rd Edition, Providence, R.I., 1967).

[2] M. F. Janowitz, 'The near center of an orthomodular lattice', *J. Austral. Math. Soc.* 14 (1972), 20–29.

[3] M. F. Janowitz, 'A note on normal ideals', *J. Sci. Hiroshima Univ. Ser. A-I* 30 (1966), 1–9.

[4] M. D. MacLaren, 'Atomic orthocomplemented lattices', *Pacific J. Math.* 14 (1964), 597–612.

University of Massachusetts
Amherst, Massachusetts
U.S.A.