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Sub-prime radical classes determined by zerorings

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It is shown that the correspondence which associates with each radical class T of abelian groups the (radical) class of prime radical rings with additive groups in T gives a complete classification of those radical classes of rings which are determined (as lower radicals) by zerorings.

In this note we investigate radical subclasses of the Baer lower (= prime) radical class B. B is the lower radical class defined by the class of all zerorings, so we are interested in the extent to which rings in the lower radical class over a class of zerorings are describable in terms of their membership of B and the (additive) structure of the zero-rings concerned.

In what follows, G^0 is the zeroring on an abelian group G, R^+ the additive group of a ring R and $L(C)$ the lower radical class defined by a class C of rings. (For details concerning radicals, see [3].)

Let T be a radical class of abelian groups, $T^0 = \{G^0 \mid G \in T\}$, $T^* = \{R \mid R^+ \in T\}$. Then T^* is a radical class and thus there are two ways of associating a radical subclass of B with T: one can consider $L(T^0)$ or $B \cap T^*$. (For further details, see [4].) It is clear that $L(T^0) \subseteq B \cap T^*$ in all cases, so the obvious problem is to determine when we have equality here. Armendariz [1] showed that equality holds when T

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is subgroup-closed and this was extended by the author [4] to the case where T is merely closed under pure subgroups. In this paper we demonstrate equality for every T.

Since for any radical class R the class of zerorings in R has the form T^0, and since, moreover, a nilpotent ring A belongs to a radical class R if and only if $A^{+0} \in R$ ([5], Theorem 2.5) our result provides a characterization of the lower radical classes defined by classes of nilpotent rings.

We make use of a notion introduced by Sands [7] in his investigation of the interaction between radicals and Morita contexts. A class C of rings is *principally left hereditary* if $Ra \in C$ for all $a \in R \in C$.

All rings considered are associative; the symbol $\triangleleft$ indicates an ideal.

PROPOSITION 1. *Let* T *be a radical class of abelian groups. Then* $B \cap T^*$ *is principally left hereditary.*

Proof. If $a \in R \in B \cap T^*$, then Ra^+ is a homomorphic image of R^+ *via* the correspondence $r \mapsto ra$, so $Ra \in T^*$. Also B is subring-closed and hence $Ra \in B$. //

PROPOSITION 2. *Let* R *be a principally left hereditary radical subclass of* B, M *the class of nilpotent rings in* R. *Then* $R = L(M)$.

Proof. By Theorem 3 of [6], $L(M)$ consists of all rings A such that every non-zero homomorphic image A'' has a non-zero accessible subring S in M; that is, there exists a finite chain

$$0 \neq S = I_1 \triangleleft I_2 \triangleleft \dots \triangleleft I_n \triangleleft A'' .$$

Since R is homomorphically closed, it is enough to show that non-zero rings in R have non-zero accessible subrings in M and clearly only non-nilpotent rings need be considered. Such a ring R does not coincide with its right annihilator $(R : 0)$, so $R/(R : 0)$ is a non-zero ring in B and accordingly has a non-zero nilpotent ideal I. Let $\bar{a}$ be a non-zero element of I represented in R by a. Then

$$Ra/[Ra \cap (R : 0)] \cong [Ra+(R : 0)]/(R : 0) \subseteq I ,$$

so $Ra/[Ra \cap (R : 0)]$ is nilpotent, whence it follows that Ra is

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nilpotent. Also $Ra \neq 0$, and, since R is principally left hereditary, $Ra \in M$. Finally, let $J/(R : 0) = I$. Then J is nilpotent and $Ra \subseteq J$. By Proposition 8 of [2], Ra is an accessible subring of R . //

THEOREM. *Let* T *be a radical class of abelian groups. Then* $L(T^0) = B \cap T^*$.

Proof. A nilpotent ring A belongs to a radical class R if and only if A^{+0} does ([5], Theorem 2.5) and $\{G \mid G^0 \in R\}$ is a radical class of abelian groups ([4], Proposition 1.1). Combining these observations with Propositions 1 and 2, we see that $B \cap T^* = L(U^0)$ for some radical class U of abelian groups. But $U^0 \subseteq B \cap T^*$ implies $U \subseteq T$, while $T^0 \subseteq B \cap T^* = L(U^0) \subseteq B \cap U^*$ implies $T \subseteq U$. This proves the theorem. //

Note that we have also shown that a radical subclass of B is principally left hereditary if and only if it has the form $B \cap T^*$.

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