Bull. Aust. Math. Soc. **90** (2014), 504–509
doi:10.1017/S0004972714000240

SPECIAL HERMITIAN MANIFOLDS AND THE 1-COSYMPLECTIC HYPERSURFACES AXIOM

MIHAIL B. BANARU

(Received 15 February 2014; accepted 11 March 2014; first published online 15 May 2014)

Abstract

It is proved that if a special Hermitian manifold complies with the 1-cosymplectic hypersurfaces axiom, then it is a Kählerian manifold.

2010 *Mathematics subject classification*: primary 53B35; secondary 53B50.

Keywords and phrases: almost contact metric structure, cosymplectic structure, special Hermitian manifold, Kählerian manifold, type number.

1. Introduction

The classification of the almost Hermitian structures on first-order differential–geometrical invariants can be rightfully attributed to the most significant results obtained by the outstanding American mathematician Alfred Gray and his Spanish colleague Luis M. Hervella. According to this classification, all the almost Hermitian structures are divided into 16 classes. Analytical criteria for each concrete structure to belong to one or another class have been obtained [10].

The class of Hermitian (or $W_3 \oplus W_4$-)manifolds is one of the most important Gray–Hervella classes. This class, consisting of almost Hermitian manifolds with an integrable almost complex structure [10, 13], was studied from the point of view of differential geometry as well as of modern theoretical physics. We remark that the class of Hermitian manifolds contains the classes of special Hermitian manifolds (or W_3-manifolds, using Gray–Hervella notation), locally conformal Kählerian manifolds and Kählerian manifolds.

Note that the class of special Hermitian manifolds has not been studied in as much detail as other so-called 'small' Gray–Hervella classes of almost Hermitian manifolds. Some dozens of significant works are devoted to the nearly Kählerian, almost Kählerian and locally conformal Kählerian manifolds, but much fewer articles are written about special Hermitian manifolds.

We remark also that the present paper is a continuation of researches of the author, who studied Hermitian manifolds, mainly six-dimensional, before (see, for example, [2, 3, 5–7]).

In [6], it has been proved that if an arbitrary special Hermitian manifold complies with the U-cosymplectic hypersurfaces axiom, then it is Kählerian. In the present note, we shall prove a statement that has a similar form. Namely, we shall prove the following.

THEOREM 1.1. *If an arbitrary special Hermitian manifold complies with the 1-cosymplectic hypersurfaces axiom, then it is a Kählerian manifold.*

2. Preliminaries

Let us consider an almost Hermitian manifold, that is, a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold:

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n} [13]. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(2n)$ [13]. Its elements are the frames adapted to the structure (A-frames). These frames look as follows:

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}),$$

where ε_a are the eigenvectors corresponding to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponding to the eigenvalue $-i$. Here the index a ranges from 1 to n, and we state $\hat{a} = a + n$.

Therefore, the matrices of the operator of the almost complex structure and of the Riemannian metric written in an A-frame look as follows, respectively:

$$(J_j^k) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right); \quad (g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$.

We recall that the fundamental (or Kählerian) form of an almost Hermitian manifold is determined by the relation

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

By direct computing, it is easy to obtain that in an A-frame the fundamental form matrix looks as follows:

$$(F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right).$$

An almost Hermitian manifold is called Hermitian if its structure is integrable. The following identity characterises the Hermitian structure [10, 13]:

$$\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = 0,$$

where $X, Y, Z \in \aleph(M^{2n})$.

https://doi.org/10.1017/S0004972714000240 Published online by Cambridge University Press

A special Hermitian structure in addition must comply with the condition $\delta F = 0$, where δ is the codifferentiation operator [13].

The first group of the Cartan structural equations of a special Hermitian manifold written in an A-frame looks as follows:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b,$$
$$d\omega_a = -\omega^b_\alpha \wedge \omega_b + B^c_{ab} \omega_c \wedge \omega^b,$$

where $\{B^{ab}{}_c\}$ and $\{B_{ab}{}^c\}$ are components of the Kirichenko tensors of M^{2n} [1, 7], $a, b, c = 1, \ldots, n$.

Let N be an oriented hypersurface in an Hermitian manifold M^{2n} and let σ be the second fundamental form of the immersion of N into M^{2n}. As is well known [2, 4–6], the almost Hermitian structure on M^{2n} induces an almost contact metric structure on N. We recall also that an almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi, \xi, \eta, g\}$ on this manifold, where ξ is a vector field, η is a covector field, Φ is a tensor of the type $(1, 1)$ and $g = \langle \cdot, \cdot \rangle$ is the Riemannian metric [9, 13]. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1, \quad \Phi(\xi) = 1, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -\mathrm{id} + \xi \otimes \eta,$$
$$\langle \Phi X, \Phi Y \rangle = \langle \Phi X, \Phi Y \rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N),$$

where $\aleph(N)$ is the module of smooth vector fields on N. As an example of an almost contact metric structure, we can consider the cosymplectic structure that is characterised by the following conditions:

$$\nabla \eta = 0, \quad \nabla \Phi = 0,$$

where ∇ is the Levi-Civita connection of the metric [9]. It has been proved that the manifold, admitting the cosymplectic structure, is locally equivalent to the product $M \times R$, where M is a Kählerian manifold [12]. We note that the cosymplectic structures have many remarkable properties and play a fundamental role in contact geometry and mathematical physics [11].

As was mentioned, the almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N, \{\Phi, \xi, \eta, g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on the product $N \times R$ [2, 10]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal.

At the end of this section, note that when we give a Riemannian manifold and its submanifold (in particular, its hypersurface), the rank of the determined second fundamental form is called the type number [14]. We also recall that an almost Hermitian manifold M^{2n} complies with the 1-cosymplectic hypersurfaces axiom (respectively, G-cosymplectic hypersurfaces axiom, U-cosymplectic hypersurfaces axiom) if a cosymplectic hypersurface with type number one (respectively, a totally geodesic cosymplectic hypersurface, a totally umbilical cosymplectic hypersurface) passes through every point of M^{2n}.

3. Proof of Theorem 1.1

Let M^{2n} be an arbitrary Hermitian manifold. As we have noted, an almost contact metric structure is induced on its oriented hypersurface N. The first group of Cartan structural equations of such an almost contact metric structure looks as follows [2, 3, 6]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}_\gamma \omega^\gamma \wedge \omega_\beta + (\sqrt{2}B^{\alpha n}_\beta + i\sigma^\alpha_\beta)\omega^\beta \wedge \omega + \Big(-\frac{1}{\sqrt{2}}B^{\alpha\beta}_n + i\sigma^{\alpha\beta}\Big)\omega_\beta \wedge \omega,$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B^\gamma_{\alpha\beta}\omega_\gamma \wedge \omega^\beta + (\sqrt{2}B^\beta_{\alpha n} - i\sigma^\beta_\alpha)\omega_\beta \wedge \omega + \Big(-\frac{1}{\sqrt{2}}B^n_{\alpha\beta} - i\sigma_{\alpha\beta}\Big)\omega^\beta \wedge \omega,$$

$$d\omega = (\sqrt{2}B^{n\alpha}_\beta - \sqrt{2}B^\alpha_{n\beta} - 2i\sigma^\alpha_\beta)\omega^\beta \wedge \omega_\alpha + (B^n_{n\beta} + i\sigma_{n\beta})\omega \wedge \omega^\beta + (B^{n\beta}_n - i\sigma^\beta_n)\omega \wedge \omega_\beta.$$

Here the indices α, β, γ range from 1 to $n-1$, and the indices a, b, c range from 1 to n.

If the type number of the hypersurface is equal to one, then the matrix of the second fundamental form of the immersion of N into M^{2n} looks as follows [5]:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} 0 & \begin{matrix}0\\ \dots\\ 0\end{matrix} & 0 \\ \hline 0\dots 0 & \sigma_{nn} & 0\dots 0 \\ \hline 0 & \begin{matrix}0\\ \dots\\ 0\end{matrix} & 0 \end{array}\right), \quad p, s = 1, \dots, 2n-1.$$

Evidently $\sigma_{nn} \neq 0$, otherwise the type number is zero.

That is why we can rewrite the Cartan structural equations for this case:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}_\gamma \omega^\gamma \wedge \omega_\beta + \sqrt{2}\,B^{\alpha n}_\beta\,\omega^\beta \wedge \omega + \Big(-\frac{1}{\sqrt{2}}B^{\alpha\beta}_n\Big)\omega_\beta \wedge \omega,$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B^\gamma_{\alpha\beta}\omega_\gamma \wedge \omega^\beta + \sqrt{2}\,B^\beta_{\alpha n}\,\omega_\beta \wedge \omega + \Big(-\frac{1}{\sqrt{2}}B^n_{\alpha\beta}\Big)\omega^\beta \wedge \omega,$$

$$d\omega = (\sqrt{2}B^{n\alpha}_\beta - \sqrt{2}B^\alpha_{n\beta})\,\omega^\beta \wedge \omega_\alpha + B^n_{n\beta}\omega \wedge \omega^\beta + B^{n\beta}_n\,\omega \wedge \omega_\beta.$$

Taking into account that the Cartan structural equations of a cosymplectic structure look as follows [2, 6, 13]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta,$$
$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta,$$
$$d\omega = 0,$$

we get the conditions whose simultaneous fulfilment is a criterion for the structure induced on N to be cosymplectic:

$$(1)\ B^{\alpha\beta}_\gamma = 0, \quad (2)\ \sqrt{2}B^{\alpha n}_\beta = 0, \quad (3)\ -\frac{1}{\sqrt{2}}B^{\alpha\beta}_n = 0,$$
$$(4)\ \sqrt{2}B^{n\alpha}_\beta - \sqrt{2}B^\alpha_{n\beta} = 0, \quad (5)\ B^{n\beta}_n = 0$$

and the formulas obtained by complex conjugation (no need to write them explicitly).

https://doi.org/10.1017/S0004972714000240 Published online by Cambridge University Press

We see that all kinds of the components of the Kirichenko tensors vanish:

$$(1)\ B^{\alpha\beta}{}_{\gamma} = 0, \quad (2)\ B^{\alpha\beta}{}_{n} = 0, \quad (3)\ B^{n\beta}{}_{n} = 0, \quad (4)\ B^{\alpha n}{}_{\beta} = 0.$$

So,

$$B^{ab}{}_{c} = 0, \quad B_{ab}{}^{c} = 0. \tag{3.1}$$

We obtain that the Kirichenko tensors satisfy (3.1) at an arbitrary point of a cosymplectic hypersurface with type number one in an Hermitian manifold. Therefore, if M^{2n} complies with the 1-cosymplectic hypersurfaces axiom, then (3.1) holds at its every point. But the fulfilment of (3.1) is a criterion for an arbitrary special Hermitian manifold to belong to the class of Kählerian manifolds [8]. Hence, if a special Hermitian manifold complies with the 1-cosymplectic hypersurfaces axiom, then it is Kählerian. □

In [4], it has been proved that if an arbitrary special Hermitian manifold complies with the G-cosymplectic hypersurfaces axiom, then it is also Kählerian. In view of the fact that a totally geodesic hypersurface is a hypersurface with zero type number, we can reformulate this statement in the following way.

THEOREM 3.1 [4]. *If an arbitrary special Hermitian manifold complies with the 0-cosymplectic hypersurfaces axiom, then it is a Kählerian manifold.*

So, we can generalise Theorem 1.1.

THEOREM 3.2. *If an arbitrary special Hermitian manifold complies with the t-cosymplectic hypersurfaces axiom, and t is at most one, then it is a Kählerian manifold.*

Acknowledgement

The author sincerely thanks Professor Galina Anatol'evna Banaru (Smolensk State University) for profound and useful discussions on the subject of this paper.

References

[1] A. Abu-Saleem and M. Banaru, 'Some applications of Kirichenko tensors', *An. Univ. Oradea Fasc. Mat.* **17** (2010), 201–208.
[2] M. Banaru, 'Two theorems on cosymplectic hypersurfaces of six-dimensional Hermitian submanifolds of Cayley algebra', *J. Harbin Inst. Tech.* **8** (2000), 38–40.
[3] M. Banaru, 'A note on six-dimensional G_1-submanifolds of octave algebra', *Taiwanese J. Math.* **6** (2002), 383–388.
[4] M. Banaru, 'On W_3-manifolds satisfying G-cosymplectic hypersurfaces axiom', *Proc. XXIV Young Scientists Conf.*, Moscow State University, 2002, 13–15 (in Russian).
[5] M. Banaru, 'On minimality of a Sasakian hypersurface in a W_3-manifold', *Saitama Math. J.* **20** (2002), 1–7.
[6] M. Banaru, 'Hermitian manifolds and U-cosymplectic hypersurfaces axiom', *J. Sichuan Univ. Nat. Sci.* **26** (2003), 261–263.
[7] M. Banaru, 'On the type number of six-dimensional planar Hermitian submanifolds of Cayley algebra', *Kyungpook Math. J.* **43** (2003), 27–35.

[8] M. Banaru, 'On the Gray–Hervella classes of AH-structures on six-dimensional submanifolds of Cayley algebra', *Annuaire Univ. Sofia Fac. Math. Inform.* **95** (2004), 125–131.
[9] D. E. Blair, 'Riemannian geometry of contact and symplectic manifolds', *Progress in Mathematics* (Birkhauser, Boston–Basel–Berlin, 2002).
[10] A. Gray and L. M. Hervella, 'The sixteen classes of almost Hermitian manifolds and their linear invariants', *Ann. Mat. Pura Appl. (4)* **123** (1980), 35–58.
[11] A. L. Kholodenko, *Applications of Contact Geometry and Topology in Physics* (World Scientific, New Jersey–London–Singapore, 2013).
[12] V. F. Kirichenko, 'Sur la gèomètrie des varietes approximativement cosymplectiques', *C. R. Acad. Sci. Paris Ser. 1* **295** (1982), 673–676.
[13] V. F. Kirichenko, *Differential–Geometrical Structures on Manifolds* (MSPU, Moscow, 2003) (in Russian).
[14] H. Kurihara, 'The type number on real hypersurfaces in a quaternionic space form', *Tsukuba J. Math.* **24** (2000), 127–132.

MIHAIL B. BANARU, Chair of Mathematics and Informatics,
Smolensk State University, Przhevalski str., 4, Smolensk 214 000,
Russian Federation
e-mail: mihail.banaru@yahoo.com