

# QUADRATIC ALGEBRAS WITH FEW RELATIONS

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**0. Introduction** Throughout  $V$  will be a finite dimensional vector space over a field  $k$  and  $T(V)$  will denote the tensor algebra over  $V$ . For simplicity the symbol  $\otimes$  will be omitted in the writing of the elements of  $T(V)$ . Let  $\{x_i\}_1^n$  be a basis of  $V$  ordered by  $x_i < x_{i+1}$  for all  $i$ . Then we order the non-commutative monomials  $\{x_{i_1} \dots x_{i_l} \mid l \geq 0 \text{ and } 1 \leq i_s \leq n \text{ for } s = 1, \dots, l\}$  lexicographically from the left. D. Anick [1, p. 652] defines the high term of an element  $b$  in  $T(V)$  to be the highest monomial appearing in  $b$ . As a consequence of [1, 3.2], if the set of the high terms of homogeneous relations is combinatorically free in the sense of no overlap ambiguities, then the connected algebra has global dimension 2. The purpose of this note is to prove this result and more for quadratic algebras under other hypotheses on the relations.

Define the rank of  $r \in V \otimes V$  to be the minimal  $d$  such that  $r = \sum_{i=1}^d s_i t_i$  for  $s_i, t_i \in V$  if  $r \neq 0$ , or to be 0 if  $r = 0$ . If we identify  $V \otimes V$  with  $\text{Hom}(V^*, V)$  in a natural way, then the rank of an element in  $V \otimes V$  is equal to the rank of the corresponding linear map in  $\text{Hom}(V^*, V)$ . For a non-zero subspace  $R \subset V \otimes V$ , define the rank of  $R$  to be the minimal rank of non-zero elements in  $R$ . Let  $T(V)/(R)$  denote the quadratic algebra with relation vector space  $R$ .

**THEOREM 0.1.** *Let  $A$  be the quadratic algebra  $T(V)/(R)$  for some  $R$  with  $0 \neq R \subset V \otimes V$ .*

- (1) *If  $\text{rank } R \geq \dim R + 1$ , then*
  - (a)  *$A$  is Koszul of global dimension 2,*
  - (b) *the Hilbert series of  $A$  is  $H_A(t) = (1 - \dim Vt + \dim R t^2)^{-1}$ .*

*Furthermore, if  $R \cap W \otimes V = 0$ , for some subspace  $W \subset V$ , then*

  - (c)  *$WA \cong W \otimes A$  as a right  $A$ -module, and hence the subalgebra  $k[W]$  is free.*
- (2) *If  $\text{rank } R \geq \dim R + 2$ , then  $A$  is a domain and has no non-trivial normal elements.*

For basic definitions such as Koszul, global dimension and so on, see [3].

**1. Hypotheses.** The main idea here is induction and the hypotheses (H1) and (H2) defined below are technical to make the induction work. Let  $\{x_i\}_1^n$  be a basis of  $V$ . Define projections  $P_i: V \otimes V \rightarrow V$  by  $P_i(r) = f_i$  if  $r = \sum_i x_i f_i$ . *Warning:*  $P_i$  is dependent on the basis  $\{x_i\}_1^n$ . For  $r \in V \otimes V$ ,  $\text{rank } r = \dim \sum_{i=1}^n k P_i(r)$ , which is independent of the choices of  $\{x_i\}_1^n$ . If  $R$  is a subspace of  $V \otimes V$ , then  $P_i(R) := \sum_{r \in R} P_i(r)$  is a subspace of  $V$  for every  $i$ . From now on we fix a nonzero subspace  $R \subset V \otimes V$  and let  $A$  be the quadratic algebra  $T(V)/(R)$ . For any subspace  $W \subset V$ , the multiplication  $W \otimes A \rightarrow WA$  defines a right  $A$ -module homomorphism. Let  $\mathbb{L}$  denote the set of all non-zero subspaces  $W \subset V$  such that  $W \otimes A \rightarrow WA$  is an isomorphism. If  $W \in \mathbb{L}$ , then  $R \cap W \otimes V = 0$ . Note that  $\mathbb{L}$  is

closed under the operation of taking non-zero subspaces; i.e., if  $W \in \mathbb{L}$  and  $0 \neq W' \subset W$ , then  $W' \in \mathbb{L}$ . If  $A$  is a domain, then every 1-dimensional subspace  $W \subset V$  is in  $\mathbb{L}$ . Let  $\Phi$  be a non-empty collection of non-zero subspaces of  $V$ . We say that  $\Phi$  satisfies (H1) if, for every  $W \in \Phi$ , there is a basis  $\{x_i\}_1^n$  of  $V$  such that  $\{x_i\}_1^s$  is a basis of  $W$  for some  $s < n$ , and for every non-zero subspace  $R' \subset R$ ,

$$\text{there is } i_0 > s \text{ such that } P_{i_0}(R') \text{ is a non-zero subspace of } W' \text{ for some } W' \in \Phi. \quad (C1)$$

If, in addition to (C1),

$$\text{there is } i_1 > s \text{ such that } i_1 \neq i_0 \text{ and for all } (0, 0) \neq (a, b) \in k^2, (aP_{i_0} - bP_{i_1})(R') \in \mathbb{L}, \quad (C2)$$

then we say  $\Phi$  satisfies (H2).

If  $\Phi$  satisfies (H2), then for every  $0 \neq r \in R$ , the dimension of  $\sum_{i>s} kP_i(r)$  is at least 2. In particular,  $\text{rank } r \geq 2$ .

Let  $\bar{\Phi} = \{W' \mid 0 \neq W' \subset W, \text{ for some } W \in \Phi\}$ . Then  $\bar{\Phi}$  is closed under the operation of taking non-zero subspaces. It is routine to check that  $\Phi$  satisfies (H1) (or (H2)) if and only if  $\bar{\Phi}$  satisfies (H1) (or (H2)). Therefore, we may assume that  $\Phi$  is closed under the operation of taking non-zero subspaces if necessary. If this is the case, then (C1) can be replaced by the following condition:

$$\text{there is } i_0 > s \text{ such that } P_{i_0}(R') \in \Phi. \quad (C1')$$

Suppose that  $\Phi_i$  satisfies (H1) (or (H2)), for all  $i \in I$ , where  $I$  is an index set. Then the union  $\bigcup_{i \in I} \Phi_i$  satisfies (H1) (or (H2)). Therefore, for a fixed  $R$ , there is a maximal  $\Phi_{H1}$  satisfying (H1) (a maximal  $\Phi_{H2}$  satisfying (H2) respectively) that contains every set  $\Phi$  satisfying (H1) ((H2) respectively). It is obvious that  $\Phi_{H2} \subset \Phi_{H1}$ . In general it is difficult to determine  $\Phi_{H1}$  and  $\Phi_{H2}$ , but it is relatively easy to find one  $\Phi$ , (if it exists), satisfying (H1) or (H2).

EXAMPLE 1.1. Suppose that  $R$  has dimension  $p$  less than  $n$  and suppose that there is a basis  $\{r_i\}_1^p$  of  $R$  and a basis  $\{x_i\}_1^n$  of  $V$  such that

$$r_i = \sum_{j < n} x_j f_{ji} + x_n x_i, \quad \forall i \leq p.$$

Let  $W = \sum_{i=1}^p kx_i$ . Then  $\Phi := \{W\}$  satisfies (H1) for  $i_0 = n$ . □

We will use  $=_*$  for equality in the (free) tensor algebra  $T(V)$ , use ordinary  $=$  for equality in  $A = T(V)/(R)$ , and use the same letter  $f$  for an element in  $T(V)$  and for the image in  $A$ . Since we are working on graded algebras, we will choose elements to be homogeneous. For a graded module  $M = \bigoplus_i M_i$ ,  $M_{>n}$  (or  $M_{<n}$ ) is defined to be  $\bigoplus_{i>n} M_i$  (or  $\bigoplus_{i<n} M_i$ ). The following obvious fact will be used several times.

LEMMA 1.2. Let  $A$  be the quadratic algebra  $T(V)/(R)$ , for some  $0 \neq R \subset V \otimes V$ . Suppose that the multiplication induces an isomorphism  $W \otimes I \cong WI$ , for some subspaces

$W, I \subset A$ . If  $\{g_j\}$  is a linearly independent set of  $I$  and  $\{f_j\}$  is a subset of  $W$  such that  $\sum_j f_j g_j = 0$ , then  $f_j = 0$  for all  $j$ . □

**PROPOSITION 1.3.** *Let  $A$  be the quadratic algebra  $T(V)/(R)$  for some  $R$  with  $0 \neq R \subset V \otimes V$  and suppose that there is a  $\Phi$  satisfying (H1). Then  $\Phi \subset \mathbb{L}$ .*

*Proof.* Given a  $W \in \Phi$ , we pick a basis  $\{x_i\}_1^n$  as in the definition of (H1). In particular,  $\{x_i\}_1^s$  is a basis of  $W$ . It suffices to show that if  $\sum_{i=1}^s x_i t_i = 0$  then  $t_i = 0$ , for all  $i \leq s$ . Since  $A$  is graded, we may assume that  $t_i$  is either 0 or homogeneous of degree  $d$ , for all  $i$ .

If  $d = 0$ , then  $t_i \in k$  and hence  $t_i = 0$ , because  $\{x_i\}_1^s$  are linearly independent. If  $d = 1$ , then  $\sum_{i=1}^s x_i t_i = 0$  implies that  $\sum_{i=1}^s x_i t_i = * r' \in R$ . For any  $i_0 > s$ ,  $P_{i_0}(r') = * P_{i_0}(\sum_{i=1}^s x_i t_i) = * 0$ . By (C1),  $r' = 0$ . Hence  $t_i = * P_i(\sum_{i=1}^s x_i t_i) = * P_i(r') = * 0$  for all  $i \leq s$ .

Now suppose that  $d > 1$ . If  $\sum_{i=1}^s x_i t_i = 0$ , then

$$\sum_{i=1}^s x_i t_i = * \sum_{w=1}^q f_w^u r^u g_w^u + \sum_{j=1}^q r_j g_j, \tag{E1}$$

for some  $r^u, r_j \in R, f_w^u, g_j \in T(V)_{>0}$  and  $g_w^u \in T(V)$ . We assume that  $q$  in (E1) is as small as possible. Write  $f_w^u = * \sum_i x_i f_{iw}^u$  and  $r_j = * \sum_i x_i r_{ij}$ . Hence

$$t_i = * \sum_{w=1}^q f_{iw}^u r^u g_w^u + \sum_{j=1}^q r_{ij} g_j, \quad \forall i \leq s, \tag{E2}$$

$$0 = * \sum_{w=1}^q f_{iw}^u r^u g_w^u + \sum_{j=1}^q r_{ij} g_j, \quad \forall i > s. \tag{E3}$$

Passing to  $A$  we have  $t_i = \sum_{j=1}^q r_{ij} g_j$  for  $i \leq s$  and  $0 = \sum_{j=1}^q r_{ij} g_j$  for  $i > s$ . If  $q > 0$ , then  $\{g_j\}_1^q$  are linearly independent in  $A$ . By (C1), there is  $i_0 > s$  such that  $P_{i_0}(\sum_{j=1}^q k r_j)$  is a non-zero subspace of  $W'$ , for some  $W' \in \Phi$ . By induction,  $W' A_{<d} \cong W' \otimes A_{<d}$ . By Lemma 1.2,  $0 = \sum_{j=1}^q r_{i_0 j} g_j$  implies that  $r_{i_0 j} = 0$  for all  $j$ . This contradicts the fact that  $P_{i_0}(\sum_{j=1}^q k r_j)$  is non-zero. Therefore  $q = 0$ . It follows from (E2) that  $t_i = 0$  for all  $i \leq s$ , as desired. □

The following is an immediate consequence of Lemma 1.2 and Proposition 1.3.

**COROLLARY 1.4.** *Let  $A$  be the quadratic algebra  $T(V)/(R)$ , for some  $R$  with  $0 \neq R \subset V \otimes V$ . Suppose that there exists a  $\Phi$  satisfying (H1). Let  $W = \sum_{i=1}^s k x_i$  be in  $\Phi$ , where  $\{x_i\}_{i=1}^n$  is a basis of  $V$ . Let  $\{g_j\}_1^q \subset A$  and  $\{r_j\}_1^q \subset R$  be linearly independent sets of*

elements, and write  $r_j = \sum_{i=1}^n x_i r_{ij}$ . Then, if  $i_0 > s$  is as in (C1), one has  $\sum_{j=1}^q r_{i_0 j} g_j \neq 0$ . In particular, if  $a_{i_0} g = \sum_{j=1}^q e_{i_0 j} g_j$ , for some  $a_{i_0}$  and  $g$ , then  $a_{i_0} \neq 0$ . □

Proposition 1.3 shows that  $\Phi_{H1} \in \mathbb{L}$ . We shall see that  $\Phi_{H1} = \mathbb{L}$  if  $\text{rank } R \geq \dim R + 1$ .

EXAMPLE 1.5. (1) Let  $p \leq n/2 - 1$  and let  $\{x_i\}_1^n$  be a basis of  $V$ . Suppose that  $R \subset V \otimes V$  has a basis

$$r_i = \sum_{j \leq n-2} x_j f_{ji} + x_{n-1} x_i + x_n x_{p+i}, \quad \forall i \leq p.$$

Let  $W = \sum_{i \geq 2p} kx_i$ . Then  $\Phi := \{W\}$  satisfies (H1) for  $i_0 = n$ . By Proposition 1.3,  $W \in \mathbb{L}$ . Using this fact it is easy to check that  $\Phi$  satisfies (H2) for  $i_0 = n$  and  $i_1 = n - 1$ .

(2) Suppose that  $V = W \oplus W'$  with  $W, W' \neq 0$  and  $0 \neq R \subset W' \otimes W$ . Pick a basis  $\{x_i\}_1^n$  of  $V$  such that  $\{x_i\}_1^n$  is a basis of  $W$  and  $\{x_i\}_{s+1}^n$  is a basis of  $W'$ . Set  $\Phi = \{W\}$ ; then  $\Phi$  clearly satisfies (H1). Similarly, for all  $i_0, i_1 > s$  and all  $a, b \in k$ ,  $(aP_{i_0} - bP_{i_1})(R') \subset W$ . By Proposition 1.3,  $W$  is in  $\mathbb{L}$  and hence every non-zero subspace of  $W$  is in  $\mathbb{L}$ .

If, moreover,  $\text{rank } R \geq 2$ , then, for every  $0 \neq R' \subset R$ , there are  $i_0, i_1 > s$  and  $i_0 \neq i_1$  such that, for all  $(0, 0) \neq (a, b)$ ,  $(aP_{i_0} - bP_{i_1})(R') \neq 0$ . Hence  $(aP_{i_0} - bP_{i_1})(R') \in \mathbb{L}$  and thus  $\Phi$  satisfies (H2). □

**2. Proof of Theorem 0.1.** For an integer  $l$  the shift module  $M(l)$  of a graded module  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  is defined by  $M(l)_i = M_{i+l}$ . If  $\dim M_i < \infty$  for all  $i$ , the Hilbert Series of  $M$  is defined to be  $H_M(t) = \sum_i \dim M_i t^i$ . The trivial  $A$ -module  $A/A_{>0}$  is denoted by  $k_A$ .

THEOREM 2.1. Let  $A$  be a quadratic algebra  $T(V)/(R)$  and suppose that there is a basis  $\{x_i\}_1^n$  such that, for every non-zero subspace  $R' \subset R$ ,  $P_i(R') \in \mathbb{L}$ , for some  $i$ . Then

(1)  $k_A$  has a minimal projective resolution

$$0 \rightarrow A(-2)^{\dim R} \rightarrow A(-1)^{\dim V} \rightarrow A \rightarrow k_A \rightarrow 0, \tag{E4}$$

(2)  $A$  is a Koszul algebra of global dimension 2,

(3)  $H_A(t) = (1 - \dim Vt + \dim Rt^2)^{-1}$ .

As a consequence, if there is a  $\Phi$  satisfying (H1), then (1), (2), (3) hold.

Proof. Consider the exact sequence

$$A(-2)^{\dim R} \rightarrow A(-1)^{\dim V} \rightarrow A \rightarrow k_A \rightarrow 0$$

constructed by using generators and relations. Pick a basis  $\{r_j\}_{j=1}^p$  of  $R$ . Then the boundary map  $\partial_2: A(-2)^{\dim R} \rightarrow A(-1)^{\dim V}$  can be represented as a left multiplication by a matrix  $B := (r_{ij})$ , where  $r_j = \sum_i x_i r_{ij}$ . We claim that  $\partial_2$  is injective; namely, if  $B(g_1, \dots, g_p)^t = 0$ , then  $g := (g_1, \dots, g_p) = 0$ . If  $g \neq 0$ , pick a basis for  $\{g_j\}_1^p$ , say  $\{g'_j\}_1^v$ , and write  $g^t = Ng'^t$  for a non-zero  $p \times v$  matrix  $N$  over  $k$ . Then  $BNg'^t = 0$ . Let  $R'$  be the subspace generated by the components of  $(r'_1, \dots, r'_v) := (r_1, \dots, r_p)N = (x_1, \dots, x_n)BN$ . Then  $R'$  is non-zero.

Write  $r'_j = \sum_{i=1}^n x_i r'_{ij}$ ; then  $\sum_{j=1}^v r'_{ij} g'_j = 0$  for all  $i$ . By the hypothesis there is  $i$  such that  $P_i(R') \in \mathbb{L}$ . By Lemma 1.2,  $r'_{ij} = 0$  for all  $j$ . This contradicts the fact that  $P_i(R')$  is non-zero. Therefore  $\partial_2$  is injective and hence (E4) is a minimal projective resolution of  $k_A$ .

The projective resolution (E4) is linear in the sense of [3, Definition 1.6], whence the algebra  $A$  is Koszul [3, Definition 5.4]. The global dimension of  $A$  is equal to the projective dimension of  $k_A$ , equal to 2. Since  $\dim$  is additive on exact sequences, it follows from (E4) that

$$H_A(t)(1 - \dim Vt + \dim Rt^2) = H_{k_A}(t) = 1.$$

Thus  $H_A(t) = (1 - \dim Vt + \dim Rt^2)^{-1}$ .

If there is a  $\Phi$  satisfying (H1) then, by Proposition 1.3, the hypothesis holds for  $i = i_0$  appearing in (C1). Therefore (1), (2), (3) hold. □

A simple case is when  $\dim V = 3$  and

$$\begin{aligned} r_1 &= ax_1^2 + bx_2x_3 + cx_3x_2, \\ r_2 &= cx_1x_3 + ax_2^2 + bx_3x_1 = ax_2^2 + bx_3x_1 + cx_1x_3, \end{aligned}$$

where  $a, b, c$  are non-zero scalars in  $k$ . By Example 1.1,  $\Phi = \{kx_1 + kx_2\}$  satisfies (H1) and, by Theorem 2.1,  $T(v)/(r_1, r_2)$  is a Koszul algebra of global dimension 2. Note that  $T(V)/(r_1, r_2)$  is isomorphic to  $k\langle x_1, x_2 \rangle[x_3, \sigma, \delta]$ , for some automorphism  $\sigma$  and some  $\sigma$ -derivation  $\delta$  of the free algebra  $k\langle x_1, x_2 \rangle$ , whence it is a domain. By adding another relation  $r_3 = ax_3^2 + bx_1x_2 + cx_2x_1 = 0$ , we obtain the type- $A$  algebra of global dimension 3 studied in [2]. For the algebra  $T(V)/(r_1, r_2, r_3)$ , there is no  $\Phi$  satisfying (H1) because its global dimension is not 2.

*Proof of Theorem 0.1(1).* Suppose that  $\text{rank } R \geq \dim R + 1 = p + 1$ . Let  $\Phi$  be the set of all non-zero subspaces of  $V$  of dimension at most  $p$ . Given a  $W \in \Phi$ , we pick a basis  $\{x_i\}_1^n$  of  $V$  such that  $\{x_i\}_1^s$  is a basis of  $W$ . For every  $0 \neq R' \subset R$ ,  $P_i(R')$  has dimension  $p$  or less. Since  $\text{rank } R' \geq \text{rank } R > p$ , there is  $i_0 > p \geq s$ ,  $P_{i_0}(R') \neq 0$ ; and hence  $P_{i_0}(R') \in \Phi$ . Thus  $\Phi$  satisfies (H1), and (a) and (b) of Theorem 0.1(1) follows from Theorem 2.1.

To prove (c) we let  $W$  be a subspace of  $V$  such that  $R \cap W \otimes V = 0$ . Pick a basis  $\{x_i\}_1^n$  such that  $\{x_i\}_1^s$  is a basis of  $W$ . Assume that  $\sum_{i=s} x_i t_i = 0$ . The notation being as in (E1-3), we obtain, in a similar way, that

$$\sum_{i=s} x_i t_i = \sum f_w^u r^u g_w^u + \sum_{j=1}^q r_j g_j$$

and that

$$\begin{aligned} t_i &= \sum f_{i_w}^u r^u g_w^u + \sum_{j=1}^q r_{ij} g_j, & \forall i \leq s, \\ 0 &= \sum f_{i_w}^u r^u g_w^u + \sum_{j=1}^q r_{ij} g_j, & \forall i > s. \end{aligned} \tag{E5}$$

We pick  $q$  as small as possible. If  $q > 0$ , then  $q \leq p$  and  $\{g_1, \dots, g_q\}$  are linearly independent in  $A$ . Since  $R \cap W \otimes V = 0$ , there is  $i_0 > s$  such that  $W' := P_{i_0}(\sum_{j=1}^q kr_j) \neq 0$ . Hence  $W'$  is in  $\Phi$  and, by Proposition 1.3,  $W'$  is in  $\mathbb{L}$ . Passing to  $A$ , (E5) implies that

$0 = \sum_j r_{i0j}g_j$ . This contradicts Lemma 1.2. Hence  $q = 0$  and so  $t_i = 0$ , for all  $i \leq s$ . Therefore  $W \in \mathbb{L}$ , as desired. □

By the proof of (c) above we also see that if  $\text{rank } R \geq \dim R + 1$  then

$$\Phi_{H1} = \mathbb{L} = \{W \mid 0 \neq W \subset V \text{ and } R \cap W \otimes V = 0\}.$$

**THEOREM 2.2.** *Let  $A$  be a quadratic algebra  $T(V)/(R)$  with a  $\Phi$  satisfying (H2).*

- (1)  $A$  is a domain.
- (2) Let  $f, g \in A_{>0} - \{0\}$ . If  $fg \in WA$ , for some  $W \in \Phi$ , then  $f \in WA$ .

*Proof.* By Proposition 1.3,  $WA \cong W \otimes A$ , for all  $W \in \Phi$ . Without loss of generality, we may assume that  $\Phi$  is closed under the operation of taking non-zero subspaces, whence we can use (C1') instead of (C1). Since  $A$  is  $\mathbb{N}$ -graded, it suffices to prove the following statement:

- (#) Let  $f$  and  $g$  be homogeneous elements in  $A_{>0} - \{0\}$  and let  $W \in \Phi$ . Then
  - (#a)  $fg \neq 0$ , and
  - (#b) if  $fg \in WA$ , then  $f \in WA$ .

We prove (#) by induction on  $m := \deg f + \deg g$ .

*Initial step:  $m = 2$ .*

(#a) Assume on the contrary that  $fg = 0$ . Then  $fg = *r' \in R$ . Since  $f, g \neq *0$  in  $T(V)$ ,  $r' \neq 0$  and  $\text{rank } r' = \text{rank } fg = 1$ . This contradicts (H2). Therefore  $fg \neq 0$  in  $A$ .

(#b) If  $fg \in WA$ , then  $fg = *r' + \sum_{i \leq s} x_i h_i$  in the free algebra  $T(V)$ , where  $\{x_i\}_1^s$  is a basis of  $W$ . Suppose that  $r' \neq 0$ . Write  $f = * \sum_i x_i a_i$  and  $r' = * \sum_i x_i r'_i$ ; then we have  $a_i g = *r'_i + h_i$ , for all  $i \leq s$ , and  $a_i g = *r'_i$ , for all  $i > s$ . Thus  $\dim \sum_{i > s} kP_i(r') \leq \dim kg \leq 1$ . This contradicts (H2). Therefore  $r' = 0$  and  $f \in \sum_{i \leq s} x_i T(V)$ , whence  $f \in WA$ .

*Inductive step:  $m > 2$ .*

(#b) For any  $W \in \Phi$ , pick a basis  $\{x_i\}_1^n$  of  $V$  as in (H2). In particular,  $\{x_i\}_1^s$  is a basis of  $W$ . If  $fg \in WA$  then, similarly to (E1),  $fg = * \sum f_w^u r_w^u g_w^u + \sum_{j=1}^q r_j g_j + \sum_{i \leq s} x_i h_i$ , for some  $f_w^u, g_j, h_i \in T(V)_{>0}$ . We choose a preimage  $f$  in  $T(V)$  such that  $q$  is as small as possible. Write  $f = * \sum_i x_i a_i, f_w^u = * \sum_i x_i f_{iw}^u$  and  $r_j = * \sum_i x_i r_{ij}$ . Then

$$a_i g = * \sum f_{iw}^u r_w^u g_w^u + \sum_{j=1}^q r_{ij} g_j + h_i, \quad \forall i \leq s,$$

$$a_i g = * \sum f_{iw}^u r_w^u g_w^u + \sum_{j=1}^q r_{ij} g_j, \quad \forall i > s.$$

Passing to  $A$ , we have  $a_i g = \sum_{j=1}^q r_{ij} g_j$ , for all  $i > s$ . Assume that  $q > 0$ . Then  $\{g_j\}_1^q$  are

linearly independent in  $A$  and  $\{r_i\}_{i=1}^q$  are linearly independent in  $R$ . By Corollary 1.4,  $a_{i_0} \neq 0$  for  $i_0$  as in (C1).

If  $a_{i_0}$  has degree 0 (or  $\deg f = 1$ ) then, for  $i_0$  and  $i_1$  as in (C2),

$$0 = a_{i_0}a_{i_1}(g - g) = a_{i_1}(a_{i_0}g) - a_{i_0}(a_{i_1}g) = \sum (a_{i_1}r_{i_0j} - a_{i_0}r_{i_1j})g_j. \tag{E6}$$

By (C2),  $(a_{i_1}P_{i_0} - a_{i_0}P_{i_1})\left(\sum_{j=1}^q kr_j\right) \in \mathbb{L}$ . By Lemma 1.2, (E6) implies that  $a_{i_1}r_{i_0j} - a_{i_0}r_{i_1j} = 0$ , for all  $j$ , which contradicts that  $(a_{i_1}P_{i_0} - a_{i_0}P_{i_1})\left(\sum_{j=1}^q kr_j\right) \neq 0$ , as stated in (C2). Next we consider the case when the degree of  $a_{i_0}$  is positive. By (C1'),  $W' := \sum_{j=1}^q kr_{i_0j}$  is in  $\Phi$ . Without loss of generality, we suppose that  $\{r_{i_0j}\}_1^l$  is a basis of  $W'$ , for some  $l \leq q$ . Hence  $a_{i_0}g \in W'A \cong W' \otimes A$ . By the induction hypothesis (#b),  $a_{i_0} = \sum_{j=1}^l r_{i_0j}w_j$ , for some  $w_j \in A$ . Replacing  $f$  by  $f - \sum_{j=1}^l r_jw_j$  (which is equal to  $f$  in  $A$ ), we obtain  $a_{i_0} = 0$  for this new  $f$ . (Note that replacing  $f$  by  $f - \sum_{j=1}^l r_jw_j$  will not change  $\{r_j\}_1^q$  because  $q$  is the smallest possible, but will change  $\{g_j\}_1^q$  obviously.) This contradicts  $a_{i_0} \neq 0$ , proved in the end of the last paragraph. Therefore we must reject the assumption that  $q > 0$ . Hence  $q = 0$ . As a consequence  $a_{i_0}g = 0$  in  $A$  and so  $a_i = 0$ , for all  $i > s$ , by the induction hypothesis (#a). Therefore  $f = \sum_{i \leq s} x_i a_i \in WA$ , as required.

(#a) Assume on the contrary that  $fg = 0$ . Then  $fg \in WA$  for any  $W = \Phi$ . By (#b) proved above,  $f \in WA$ ; i.e.,  $f = \sum_{i=1}^s x_i a_i$ . Hence  $0 = fg = \sum_{i=1}^s x_i a_i g$ . Since  $WA \cong W \otimes A$ ,  $a_i g = 0$ , for all  $i \leq s$ . By the induction hypothesis (#a),  $a_i = 0$  for all  $i \leq s$ , whence  $f = \sum_{i=1}^s x_i a_i = 0$ . This contradicts  $f \neq 0$ . Therefore  $fg \neq 0$ . □

**EXAMPLE 2.3.** Let  $r = x_1 y_1 + x_2 y_2 \in V \otimes V$  be an element of rank 2 and  $R$  the 1-dimensional subspace  $kr$ . Hence  $\text{rank } R = \dim R + 1$ . By Theorem 0.1(1c), every 1-dimensional subspace of  $V$  is in  $\mathbb{L}$ . Suppose that  $ky_1 + ky_2 \neq kx_1 + kx_2 := W_x$ . Then either  $y_1 \notin W_x$  or  $y_2 \notin W_x$ . Let  $\Phi$  be the set of all 1-dimensional subspaces of  $V$  not contained in  $W_x$ . It is easy to check that  $\Phi$  satisfies (H2). By Theorem 2.2,  $T(V)/(r)$  is a domain. □

**PROPOSITION 2.4.** *Let  $A$  be a quadratic algebra  $T(V)/(R)$  and  $\dim V > 1$ . Suppose that there is a right regular element  $x \in V$  such that if  $fg \in xA$ , for some  $f, g \in A_{>0} - \{0\}$ , then  $f \in xA$ . Then  $A$  has no non-trivial normal elements.*

*Proof.* Assume the contrary and let  $g$  be a non-trivial normal element. Since  $A$  is graded we may assume that  $g$  is homogeneous of position degree. Since  $g$  is normal, there is  $y \in V$  such that  $gy = xg \in xA$ . By the hypothesis,  $g = xg_0$  and hence  $xg_0y = xxg_0$ . Therefore  $g_0y = xg_0$  by cancelling the right regular element  $x$ . It follows by induction that  $y = x$  and  $g = lx^n$ , for some nonzero scalar  $l$  and some positive integer  $n$ . Hence  $x^n$  is normal. Pick an element  $h \in V - kx$ . By the hypothesis,  $hx^n \notin xA$ , which contradicts the fact that  $x^n$  is normal. Therefore  $A$  has no non-trivial normal element. □

*Proof of Theorem 0.1(2).* Suppose that  $\text{rank } R \geq \dim R + 2 = p + 2$ . Let  $\Phi$  be the set of all non-zero subspaces of  $V$  of dimension at most  $p$ . We shall prove that  $\Phi$  satisfies (H2). By the proof of Theorem 0.1(1),  $\Phi$  satisfies (H1) and Proposition 1.3,  $\Phi \subset \mathbb{L}$ . It remains to check (C2).

For every  $W \in \Phi$ , pick a basis  $\{x_i\}_{i=1}^n$  of  $V$  such that  $\{x_i\}_{i=1}^s$  is a basis of  $W$ . For every  $i_0, i_1 > s, i_0 \neq i_1$  and every  $(a, b) \in k^2, (aP_{i_0} - bP_{i_1})(R)$  is a subspace of dimension at most  $p$ . It suffices to show that for every  $0 \neq r \in R$ , there exist  $i_0, i_1 > s$  such that for every  $(0, 0) \neq (a, b) \in k^2, (aP_{i_0} - bP_{i_1})(r) \neq 0$ . This follows immediately from the fact that  $\text{rank } r \geq \text{rank } R \geq p + 2$ . Hence  $\Phi$  satisfies (H2). By Theorem 2.2(1),  $A$  is a domain. By Theorem 2.2(2), the hypothesis of Proposition 2.4 holds for every nonzero element  $x$  in  $V$ . Therefore  $A$  has no non-trivial normal element.  $\square$

If  $\text{rank } R \geq \dim R + 2$ , then

$$\Phi_{H2} = \{W \mid 0 \neq W \subset V \text{ and } R \cap (W + kx) \otimes V = 0 \text{ for all } x \in V\}.$$

**3. Some remarks.**

REMARK 3.1. The rank of a single element of  $V \otimes V$  is preserved under field extension, but the rank of a subspace of  $V \otimes V$  is not.

Let  $V$  be a 4-dimensional space over the real numbers  $\mathbb{R}$  with a basis  $\{x_1, x_2, x_3, x_4\}$  and let  $r_1 = x_1x_4 + x_2x_3$  and  $r_2 = x_1x_3 - x_2x_4$ . Then  $R := \mathbb{R}r_1 + \mathbb{R}r_2$  has rank 2 and  $R \subset W' \otimes W$  with  $W = \mathbb{R}x_1 + \mathbb{R}x_2$  and  $W' = \mathbb{R}x_3 + \mathbb{R}x_4$ . By Example 1.5(2),  $\Phi = \{W\}$  satisfies (H2) and, by Theorem 2.2(1),  $T(V)/(R)$  is a domain. After extending  $\mathbb{R}$  to complex numbers  $\mathbb{C}$ , the rank of the relations  $R \otimes \mathbb{C}$  becomes 1 because

$$(x_1 + ix_2)(x_3 + ix_4) = x_1x_3 - x_2x_4 + i(x_1x_4 + x_2x_3) \in R \otimes \mathbb{C}.$$

Hence  $A \otimes \mathbb{C}$  is not a domain. In particular, there are no  $\Phi$  satisfying (H2). Therefore extending the base field does not preserve the hypothesis (H2) generally.

REMARK 3.2. Suppose that  $\Phi$  satisfies (H1) for  $T(V)/(R)$ . Let  $R'$  be a non-zero subspace of  $R$ . Then it is easy to see that  $\Phi$  satisfies (H1) for  $T(V)/(R')$ . As a consequence,  $T(V)/(R')$  is Koszul of global dimension 2, for all  $R'$  with  $0 \neq R' \subset R$ .

REMARK 3.3. If  $k$  is an infinite field and  $R$  is a generic subspace of  $V \otimes V$  of dimension less than  $n := \dim V$ , then  $T(V)/(R)$  is Koszul of global dimension 2, as we will now show. The Koszul property and global dimension are preserved by extending the field, because the minimal projective resolution of the trivial module  $k_A$  remains easily the same. Hence, since  $k$  is infinite, we can replace it by its algebraic closure. By Theorem 2.1, it suffices to show that there exists a  $\Phi$  satisfying (H1). By Remark 3.2, we only need consider the case when  $R$  has dimension  $n - 1$ . Pick  $n - 1$  generic elements in  $V \otimes V$  as basis elements of  $R$

$$r_s = \sum_{ij} a_{sij}x_ix_j \quad (s = 1, \dots, n - 1).$$

Since  $k$  is algebraically closed, after changing a basis for  $V$  we may assume that  $a_{smn} = 0$ , for all  $s$ . Hence  $R$  has a basis consisting of elements

$$r_s = \sum_{i < n} a_{sij}x_ix_j + x_nx_s \quad (s = 1, \dots, n - 1).$$

By Example 1.1, there is a  $\Phi$  satisfying (H1) as desired.



REMARK 3.4. Let  $k$  be an infinite field and  $R$  a  $d$ -dimensional subspace of  $V \otimes V$  with  $1 \leq d < n$ . Then  $T(V)/(R)$  is generically a domain. We sketch a proof next and details are left to the interested reader. If  $A \otimes K$  is a domain with field extension  $K$  over the base field  $k$ , then  $A$  itself is a domain. By the argument of Remark 3.3, we may assume  $k$  is algebraically closed. After changing the basis  $\{x_i\}_1^n$  of  $V$  we may assume that  $R$  has a basis  $\{r_s\}_1^d$  and that

$$r_s = \sum_{i < n} a_{sij} x_i x_j + x_n x_s \quad (s = 1, \dots, d).$$

By Remark 3.3 and Theorem 2.1,  $A = T(V)/(R)$  is Koszul of global dimension 2 and the Hilbert series of  $A$  is  $(1 - \dim Vt + \dim Rt^2)^{-1}$ . In particular,  $A$  has a  $k$ -linear monomial basis

$$\{x_{i_1} \dots x_{i_l} \mid \text{for all } (i_1, \dots, i_l), \text{ and for } 1 \leq p < l, (i_p, i_{p+1}) \neq (n, s) \text{ for any } s \leq d\}. \quad (E7)$$

Consider the  $\mathbb{N}$ -filtration of  $A$  induced by  $\deg x_i = 1$ , for all  $n > i > d$ ,  $\deg x_i = 2$ , for all  $i \leq d$  and  $\deg x_n = 3$ , and let  $gr(A)$  be the associated graded ring. By definition  $gr(A)$  is a factor ring of  $C := k(x_i)/(\bar{r}_1, \dots, \bar{r}_d)$ , where  $\bar{r}_s = \sum_{i \leq d} a_{sin} x_i x_n + x_n x_s$ . It is easy to see that  $C$  is an extension of  $k(x_1, \dots, x_d, x_n)/(\bar{r}_s)$  by adding  $x_{d+1}, \dots, x_{n-1}$  freely. Hence  $C$  has a monomial basis (E7). Therefore  $gr(A)$  is isomorphic to  $C$ . Since  $k(x_1, \dots, x_d, x_n)/(\bar{r}_s)$  is isomorphic to  $k(x_1, \dots, x_d)[x_n, \sigma]$ , for some automorphism  $\sigma$  (generically), it is a domain. Therefore  $C$  is a domain and hence  $A$  is a domain.

REMARK 3.5. Let  $k$  be an infinite field and  $1 \leq d < n = \dim V$ . By linear algebra, a generic  $d$ -dimensional subspace  $R \subset V \otimes V$  has rank at least  $n - \lfloor \sqrt{d-1} \rfloor$ . If  $k$  is algebraically closed, then the rank of a generic  $d$ -dimensional subspace  $R$  is  $n - \lfloor \sqrt{d-1} \rfloor$  (proved by S. P. Smith). As a consequence, if  $k$  is an infinite field and  $d \leq (\sqrt{n - \frac{3}{4}} - \frac{1}{2})^2$ , then a generic  $d$ -dimensional vector space  $R$  has rank at least  $d + 2$ . By Theorem 0.1(2),  $A = T(V)/(R)$  is a domain. (This is another way to prove the result in Remark 3.4 when  $d \leq (\sqrt{n - \frac{3}{4}} - \frac{1}{2})^2$ .)

REMARK 3.6. S. P. Smith and the author [4] and [5] proved the following result for one-relator quadratic algebras, some of which was based on this note. Parts (1) and (2) are well known.

Let  $A$  be a one-relator quadratic algebra  $T(V)/(r)$  for some  $r$  with  $0 \neq r \in V \otimes V$ . Then

- (1)  $A$  is Koszul;
- (2)  $gl. \dim A = \begin{cases} \infty & \text{if } r = x^2, \text{ for some } x \in V, \\ 2 & \text{otherwise;} \end{cases}$
- (3)  $A$  is noetherian if and only if  $\text{rank } r = \dim V = 2$ ;
- (4)  $A$  is a domain if and only if  $\text{rank } r \geq 2$ ;
- (5) if  $A$  has a non-trivial normal element, then  $\dim V = \text{rank } r = 2$ ;
- (6) if  $W \subset V$  and  $r \notin W \otimes V$ , then the multiplication  $W \otimes A \rightarrow WA$  induces an  $A$ -module isomorphism, and hence the subalgebra  $k[W]$  is free.

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