Bull. Aust. Math. Soc. **90** (2014), 250–256
doi:10.1017/S0004972714000495

SOLVABILITY OF FINITE GROUPS WITH FOUR CONJUGACY CLASS SIZES OF CERTAIN ELEMENTS

QINHUI JIANG and CHANGGUO SHAO✉

(Received 3 February 2014; accepted 9 March 2014; first published online 21 July 2014)

Abstract

Assume that m and n are two positive integers which do not divide each other. If the set of conjugacy class sizes of primary and biprimary elements of a group G is $\{1, m, n, mn\}$, we show that up to central factors G is a $\{p, q\}$-group for two distinct primes p and q.

2010 *Mathematics subject classification*: primary 20E45; secondary 20D10.

Keywords and phrases: conjugacy class sizes, primary and biprimary elements, finite groups.

1. Introduction

Throughout this paper all groups considered are finite and G always denotes a group. For an element x of a group G, we denote by x^G the conjugacy class of x and by $|x^G|$ the conjugacy class size of x. We say that x is a primary element if its order is a prime power and x is a biprimary element if its order has exactly two distinct prime divisors. All unexplained notation and terminology are standard, as in [11].

A classical problem in group theory is to study how the set of its conjugacy class sizes controls the solvability of a group. For instance, groups with two class sizes are nilpotent and groups with three class sizes are solvable. However, if a group has four conjugacy class sizes, it may be simple, such as $PSL_2(5)$. Beltrán and Felipe studied groups G whose set of conjugacy class sizes is $\{1, m, n, mn\}$, where m and n are two coprime positive integers. They claimed in [3, 4] that G is nilpotent with m and n two prime powers. Further, they proved in [5] that G is solvable if m and n are two arbitrary numbers which do not divide each other.

On the other hand, many authors considered the influence of conjugacy class sizes of certain elements in a group. This seems still to keep control of the structure of a group. For example, we showed in [13] that a solvable group is nilpotent if the set of the conjugacy class sizes of its primary and biprimary elements is $\{1, m, n, mn\}$ with m and n two coprime integers. In [9, Theorem C], Kong and Liu proved that

This paper has been supported by the research project NNSF of PR China (Grant Nos. 11101258, 11201401 and 11301218) and University of Jinan Research Funds for Doctors (XBS1335 and XBS1336).

a p-solvable group G is solvable if the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, p^a, n, p^a n\}$, where p divides the positive integer n but p^a does not divide n. Then G is, up to central factors, a $\{p, q\}$-group with p and q two distinct primes. In this present paper, we first prove the following theorem, which generalises the result above without considering the p-solvability of G.

Theorem A. *Let G be a group and a and n be integers. If the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, p^a, n, p^a n\}$ with prime p and integer n such that p divides n while p^a does not divide n, then G is solvable. In particular, up to central factors, G is a $\{p, q\}$-group.*

There are errors in Cases 1 and 2 of the proof in [9]. The error in Case 2 was corrected in [10]; our method of the proof of Theorem A corrects the error in Case 1. Furthermore, we prove a more general result.

Theorem B. *Let G be a group and m and n be integers. If the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, m, n, mn\}$ such that m and n do not divide each other, then G is solvable. In particular, up to central factors, G is a $\{p, q\}$-group with distinct primes p and q.*

2. Preliminaries

We collect some results which will be used in the sequel.

Lemma 2.1 [8, Lemma 2.4]. *Let G be a group. A prime p does not divide the conjugacy class size of any primary element of G if and only if G has a central Sylow p-subgroup.*

Remark 2.2. This is an immediate corollary of the result in [12].

Lemma 2.3 [6, Corollary B]. *Let N be a normal subgroup of a group G and p a fixed prime. Suppose that $|x^G| = 1$ or m for every q-element in N and for every prime $q \neq p$. Then N has nilpotent p-complements.*

Lemma 2.4. *Let G be a group. If each primary p'-element of G has conjugacy class size 1 or m, then $m = p^a q^b$, where a, b are two integers and q is a prime distinct from p. Moreover, $G = PQ \times A$, where P is a Sylow p-subgroup of G, Q is a Sylow q-subgroup of G and $A \leq Z(G)$. In particular, if $b = 0$, then G has abelian p-complements; if $a = 0$, then $G = P \times Q \times A$.*

Proof. By Lemma 2.3, G has a nilpotent p-complement, say H. Write $G = PH$. Then G is solvable as it is a product of two nilpotent groups. If $H \leq Z(G)$, there is nothing to prove. Suppose that $H \nleq Z(G)$ and $v \in H$ is a noncentral q-element for some prime $q \neq p$. It is easy to see that the conjugacy class size of v is a $\{p, q\}$-number, yielding that m is a $\{p, q\}$-number. Write $m = p^a q^b$ with $a, b \geq 0$.

Let $r \neq p, q$ be a prime and $u \in H \setminus Z(G)$ be an r-element. Since H is nilpotent, we have that $|u^G| = m$ is a $\{p, r\}$-number, forcing $|u^G| = 1$ and thus $u \in Z(G)$. This contradiction forces $G = PQ \times A$, where P is a Sylow p-subgroup of G, Q is a Sylow q-subgroup of G and $A \leq Z(G)$.

If $b = 0$, then each q-element has a p-number conjugacy class size, yielding that Q is abelian, so that G has abelian p-complements and the conclusion holds. Assume then $a = 0$. Then each q-element has a q-number conjugacy class size, yielding that $G = P \times Q \times A$, which completes the proof of this lemma. □

Remark 2.5. This is a generalisation of the main results in [1, 2].

Lemma 2.6. *Let G be a group with a subgroup A. Assume that every noncentral primary element $x \in A$ has centraliser A and $\pi := \pi(A/A \cap Z(G))$ such that $|\pi| > 1$. Then either:*

(i) *$N_G(A)/A$ is a π'-group; or*

(ii) *$|N_G(A)/A| = p$ for some $p \in \pi$.*

Proof. Let $v \in A \setminus Z(G)$ be an arbitrary element, which exists as $|\pi(A/A \cap Z(G))| > 1$. Consider the primary decomposition of $v = v_1, \ldots, v_n$, where the orders of $v_1, \ldots, v_n$ are powers of distinct primes and $v_1, \ldots, v_n$ commute pairwise. Since $C_G(v_i) = A$ or G for all $i \in \{1, \ldots, n\}$, we obtain that $C_G(v) = A$ as $v \notin Z(G)$. Then the lemma holds by [7, Proposition 1]. □

3. Proof of Theorem A

Proof. According to Lemma 2.1, we may assume that G is a $\pi(n)$-group, as p divides n. Clearly, if n is a power of p, the conclusion holds. In the following, we assume that $|\pi(n)| \geq 2$ and split the proof into two cases.

Case 1. There exists no p-elements of conjugacy class size p^a.

Let x be an element of conjugacy class size p^a. By considering its primary decomposition, we may assume that x is a q-element for some prime $q \neq p$.

For an arbitrary primary q'-element y of $C_G(x)$, we see that the conjugacy class size of y in $C_G(x)$ must be 1 or n, since p^a does not divide n.

If the conjugacy class size of y in $C_G(x)$ is n, it follows by Lemma 2.4 that $C_G(x)$ has a nilpotent q-complement H, yielding that $C_G(x)$ is solvable. Moreover, $n = p^r q^t$ with $r > 0$. On the other hand, Lemma 2.1 shows that $G = PQ \times A$ with a Sylow p-subgroup P, a Sylow q-subgroup Q and $A \leq Z(G)$.

If the conjugacy class size of y in $C_G(x)$ is 1, then $C_G(x) = Q_0 \times H$, where Q_0 is a Sylow q-subgroup of $C_G(x)$. If $H \leq Z(G)$, then the proof is finished. Now consider the case that $H \nleq Z(G)$. Note that p divides the order of $C_G(x)$. Then we may take some noncentral p-element $z \in C_G(x)$, which exists as $p < p^a n_p \leq |G : Z(G)|_p$. In this case, $z \in H$ is of conjugacy class size p^a, which is a contradiction.

Case 2. There is a p-element of conjugacy class size p^a.

A similar argument as in [10] will complete the proof. □

https://doi.org/10.1017/S0004972714000495 Published online by Cambridge University Press

4. Proof of Theorem B

Proof. According to Lemma 2.1, we may assume that G is a $(\pi(m) \cup \pi(n))$-group. Suppose that x is a primary or biprimary element of conjugacy class size m. By considering its primary decomposition, x can be assumed to be a primary element. In the following, we fix x as a p-element for some prime $p \in \pi(G)$.

Step 1. Either $C_G(x)$ is abelian or $C_G(x) = P_xQ_x \times T_x$, where P_x is a Sylow p-subgroup of $C_G(x)$, Q_x is a Sylow q-subgroup of $C_G(x)$ and $T_x \leq Z(C_G(x))$ is a Hall $\{p, q\}'$-subgroup of $C_G(x)$. In particular, if $C_G(x)$ is nonabelian, then $n = p^a q^b$ for some prime q distinct from p with positive integers a and b.

Symmetrically, if y is a primary element of conjugacy class size n, then either $C_G(y)$ is abelian or m is a product of two distinct primes.

Proof of Step 1. It is not difficult to see that each primary p'-element y of $C_G(x)$ has conjugacy class size 1 or n, as $|x^G| = m$ and m, n do not divide each other. If the conjugacy class size of y is n, it follows by Lemma 2.4 that the second statement holds; if the conjugacy class size of y is 1, then $C_G(x) = P_x \times H_x$, where H_x is an abelian Hall p'-subgroup of $C_G(x)$.

Suppose that $H_x \leq Z(G)$. Then $|C_G(x)/Z(G)|$ is a p-power, yielding that $|G/Z(G)| = mp^\alpha$ for some positive integer α. Notice that mn divides $|G/Z(G)|$. Then n is a power of p, and the proof is finished according to Theorem A.

Consequently, we may assume that $H_x \nleq Z(G)$. Take a noncentral q-element $z \in H_x$. Clearly, $C_G(x) = C_G(z)$, as $C_G(x)$ is maximal in G. Further, each primary q'-element y has conjugacy class size 1 or n in $C_G(z)$. If the conjugacy class size of y is n in $C_G(x)$, then it follows that each p-element of $C_G(x)$ has conjugacy class size 1 or n in $C_G(x)$, yielding that n is a power of p; then theorem holds by Theorem A. If the conjugacy class size of y is 1 in $C_G(x)$, then $C_G(z) = Q_z \times H_z$ with an abelian Hall q'-subgroup H_z of $C_G(z)$ and a Sylow q-subgroup Q_z of $C_G(z)$. As $C_G(x) = C_G(z)$, we obtain that $C_G(x)$ is abelian, and the claim holds.

Step 2. There exists at least one primary element of conjugacy class size m or n whose centraliser is nonabelian.

Proof of Step 2. Suppose that the centraliser of each primary element of conjugacy class size m or n is always abelian. Then, for each primary element $u \in C_G(x) \cap C_G(y)$, we see that $C_G(x) \leq C_G(u)$ and $C_G(y) \leq C_G(u)$. As a result, $u \in Z(G)$, implying $C_G(x) \cap C_G(y) = Z(G)$. Consequently,

$$|G : Z(G)| = |G : C_G(x) \cap C_G(y)| = |G : C_G(x)||C_G(x) : C_G(x) \cap C_G(y)| \leq mn,$$

which is a contradiction to the fact that there is an element of conjugacy class size mn in G.

Step 3. By the symmetry of m and n, we may assume that $C_G(x)$ is nonabelian. Then $T_x \leq Z(G)$, where T_x is as in Step 1. In particular, $|G : Z(G)|_{\{p,q\}'} = m_{\{p,q\}'}$.

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Proof of Step 3. By Step 1, we have $C_G(x) = P_xQ_x \times T_x$, where P_x is a Sylow p-subgroup of $C_G(x)$, Q_x is a Sylow q-subgroup of $C_G(x)$ and $T_x \leq Z(C_G(x))$ is a Hall $\{p, q\}'$-subgroup of $C_G(x)$.

We show that T_x is central. Otherwise, there exists a noncentral s-element $v \in T_x$ for some prime s. Easily, $C_G(x) \leq C_G(v)$ and thus $C_G(x) = C_G(v)$, as $C_G(x)$ is maximal in G. Further, each primary s'-element of $C_G(v)$ has conjugacy class size 1 or n in $C_G(v)$. Again by the argument in Step 1, we see that $C_G(v) = R_vS_v \times T_v$, where R_v is a Sylow r-subgroup of $C_G(v)$, S_v is a Sylow s-subgroup of $C_G(v)$ and $T_v \leq Z(C_G(v))$. Since $C_G(v) = C_G(x)$ is nonabelian, we see that $r = p$. Furthermore, $C_G(x)$ is nilpotent, yielding that n is a power of p or q; then the theorem holds by Theorem A. Hence, we may assume that $T_x \leq Z(G)$.

Step 4. If z is a primary or biprimary element of conjugacy class size mn, then $C_G(z) = P_zQ_z \times H_z$, where P_z is a noncentral Sylow p-subgroup of $C_G(z)$, Q_z is a noncentral Sylow q-subgroup of $C_G(z)$ and $H_z \leq Z(G)$ is a Hall $\{p, q\}'$-subgroup of $C_G(z)$, respectively.

Proof of Step 4. Let z be a primary or biprimary element of conjugacy class size mn. Suppose that there exists a prime r dividing the order of $C_G(z)/Z(G)$ such that $r \neq p, q$. Since $|G : Z(G)| = |G : C_G(z)||C_G(z) : Z(G)|$, we obtain that $|G/Z(G)|_r > (mn)_r \geq m_r$, contrary to the fact that $|G/Z(G)|_r = m_r$ by Step 3. As a consequence, we may write $C_G(z) = P_zQ_z \times H_z$, where P_z is a Sylow p-subgroup of $C_G(z)$, Q_z is a Sylow q-subgroup of $C_G(z)$ and H_z is a Hall $\{p, q\}'$-subgroup of $C_G(z)$.

We assert that neither P_z nor Q_z is central. Suppose, to the contrary, that Q_z is central. We claim that q divides the order of $C_G(x)/Z(G)$. Otherwise, we see that $C_G(x) = P_x \times Q_x \times T_x$ and thus $|C_G(x) : Z(G)|$ is a p-number. On the other hand, mn divides $|G : Z(G)|$ and $|G : Z(G)| = |G : C_G(x)||C_G(x) : Z(G)| = m|C_G(x) : Z(G)|$, whence n is a power of p. In this case, the theorem holds by Theorem A.

Consequently, there exists a q-element $v \in C_G(x) \setminus Z(G)$. If vx has conjugacy class size mn, then $|C_G(vx)/Z(G)| = |C_G(z)/Z(G)|$ is a p-power, as Q_z is central, which is impossible, as $v \in C_G(vx)$. Consequently, vx has conjugacy class size m, forcing $C_G(vx) = C_G(x) = C_G(v)$. Therefore, $C_G(x) = P_x \times Q_x \times T_x$ with an abelian Sylow q-subgroup Q_x by an argument similar to that in the last paragraph in Step 1.

Analogously, if we take a noncentral p-element $w \in C_G(v)$, by applying an argument similar to that above, we conclude that $C_G(wv) = C_G(v) = C_G(w)$, yielding that $C_G(x)$ has an abelian Sylow p-subgroup. Therefore, $C_G(x)$ is abelian, contradicting our assumption prior to Step 3.

Similarly, P_z is also noncentral.

Step 5. G is a $\{p, q\}$-group.

Proof of Step 5. As m and n are not coprime, without loss of generality, we assume that p is a common divisor of m and n.

Recall that y is a primary element of conjugacy class size n in G. Now we prove that $C_G(y)$ is nonabelian.

Suppose $C_G(y)$ is abelian. Since mn divides $|G : Z(G)| = |G : C_G(y)||C_G(y) : Z(G)|$, we see that p divides $|C_G(y) : Z(G)|$. Hence, there exists a noncentral p-element $v \in C_G(y)$, yielding that $C_G(v) = C_G(y)$.

Assume that $N_G(C_G(y)) = C_G(y)$. Let P be a Sylow p-subgroup of G containing v. Notice that p is a common divisor of m and n. Then P is nonabelian. Then we may take an element $\overline{z} \in Z(P/Z(P))$. We show that $P \subseteq N_G(C_G(z))$. In fact, for each $a \in P \setminus Z(P)$, we see that $[\overline{a}, \overline{z}] = \overline{1}$, yielding that $t_0 := [a, z] \in Z(P) \leq Z(G)$. Moreover, $z = z^{a^{-1}} t_0$, which implies that $C_G(z) = C_G(z^{a^{-1}} t_0) = z^{a^{-1}}$. Consequently, $a \in N_G(C_G(z))$, as desired.

Note that $C_G(y) = C_G(z)$, as $C_G(y)$ is abelian. Then $P \leq N_G(C_G(z)) = N_G(C_G(y)) = C_G(y)$, which is a contradiction. Therefore, $N_G(C_G(y)) \gneq C_G(y)$. Further, by Lemma 2.6 $|N_G(C_G(y))/C_G(y)| = p$. Then we may choose a p-element $z \in N_G(C_G(y)) \setminus C_G(y)$.

By Step 4, we see that both p and q divide the order of $C_G(y)/Z(G)$. Let Q_y be a Sylow q-subgroup of $C_G(y)$ and Q be a Sylow q-subgroup of G containing Q_y. As p divides n, we see that $Q_y \lneq Q$. For an arbitrary element $w \in N_Q(Q_y) \setminus Q_y$, we see that $Q_y \subseteq C_G(y) \cap C_G(y^w)$. As $C_G(y)$ is abelian, we conclude that $C_G(y) = C_G(y^w) = C_G(y)^w$. This shows that w is a q-element in $N_G(C_G(y)) \setminus C_G(y)$, which is a contradiction. Consequently, $C_G(y)$ is nonabelian.

Further, we conclude that $m = p^c q^d$ for positive integers c and d. Therefore, both m and n are $\{p, q\}$-numbers, showing that G is a $\{p, q\}$-group, up to central factors. This completes the proof. □

Acknowledgements

The authors are grateful to the referee for his/her valuable suggestions. The second author would like to thank Professor A. Beltrán for his hospitality when the second author was visiting Universitat Jaume I. It should be said that we could not have polished the final version of this paper well without their outstanding efforts.

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QINHUI JIANG, School of Mathematical Sciences,
University of Jinan, Shandong, 250022, PR China
e-mail: syjqh2001@163.com

CHANGGUO SHAO, School of Mathematical Sciences,
University of Jinan, Shandong, 250022, PR China
e-mail: shaoguozi@163.com