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CONVEX STANDARD FUNDAMENTAL DOMAIN FOR SUBGROUPS OF HECKE GROUPS

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Abstract

It is well known that if a convex hyperbolic polygon is constructed as a fundamental domain for a subgroup of SL(2, $\mathbb{R}$), then its translates by the group form a locally finite tessellation and its side-pairing transformations form a system of generators for the group. Such a hyperbolically convex fundamental domain for any discrete subgroup can be obtained by using Dirichlet's and Ford's polygon constructions. However, these two results are not well adapted for the actual construction of a hyperbolically convex fundamental domain due to their nature of construction. A third, and most important and practical, method of obtaining a fundamental domain is through the use of a right coset decomposition as described below. If Γ_2 is a subgroup of Γ_1 such that $\Gamma_1 = \Gamma_2 \cdot \{L_1, L_2, \ldots, L_m\}$ and $\mathbb{F}$ is the closure of a fundamental domain of the bigger group Γ_1, then the set

$$\mathcal{R} = \left(\bigcup_{k=1}^{m} L_k(\mathbb{F}) \right)^{o}$$

is a fundamental domain of Γ_2. One can ask at this juncture, is it possible to choose the right coset suitably so that the set $\mathcal{R}$ is a convex hyperbolic polygon? We will answer this question affirmatively for Hecke modular groups.

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1. Introduction

To study Dirichlet series satisfying some functional equations, Hecke [6] introduced a class of subgroups of SL(2, $\mathbb{R}$) (a group of orientation-preserving isometries of the upper half-plane $\mathbb{H}$) generated by two fractional linear transformations of the upper half-plane on itself,

$$S_\lambda : \tau \to \tau + \lambda \quad \text{and} \quad T : \tau \to \frac{-1}{\tau}$$

where λ is a positive real number. These groups are discrete if and only if $\lambda \geq 2$ or $\lambda = \lambda_q = 2\cos(\pi/q)$, where q is an integer greater than or equal to 3. These groups are called Hecke modular groups or Hecke triangle groups. In this article we restrict ourselves to the case $\lambda = \lambda_q = 2\cos(\pi/q)$, where $q \geq 3$. We denote these Hecke groups by $\mathcal{H}_q$. It is known (see [2]) that the Hecke modular groups are isomorphic to the free products of two finite cyclic groups of orders two and q; that is,

$$\mathcal{H}_q \cong \langle T \rangle \star \langle P_\lambda \rangle \cong \mathbb{Z}_2 \star \mathbb{Z}_q,$$

where $P_\lambda = TS_\lambda$ is an elliptic element of order q. The group $\mathcal{H}_q$ is a Fuchsian group of the first kind with signature $(2, q, \infty)$ (see [1, 4, 6]). The best known and most interesting Hecke group is the modular group $\mathcal{H}_3$, namely $\mathrm{SL}(2, \mathbb{Z})$, which is usually denoted by $\Gamma(1)$. Hence the Hecke modular groups can be thought of as natural generalizations of the modular group $\Gamma(1)$, thus validating the name. When q is 4 or 6, the Hecke groups are $G(\sqrt{2})$ and $G(\sqrt{3})$, respectively (see [20]). These two groups are the only Hecke groups, aside from the modular group, whose elements are completely known (see Remark 3.2). Also, $\mathcal{H}_q$ is commensurable with $\mathrm{SL}(2, \mathbb{Z})$ if and only if $q = 3$, 4 or 6 (see [14, 19, 20, 27]). We will consider the group $\mathcal{H}_4$ at the end of Section 3.

Let $G \subset \mathrm{SL}(2, \mathbb{R})$ be a Fuchsian group. The group G acts on the upper half-plane homeomorphically and the images of a single point under the group action form an orbit of the action. To illustrate the action of G, we usually picture a fundamental set, which, roughly speaking, is a subset of the upper half-plane containing exactly one point from each of these orbits of G. A fundamental set with desirable properties serves as a geometric realization for the abstract set of representatives of the orbits. A fundamental set with some topological niceties is called a fundamental domain. There are slightly different versions of fundamental domain in the literature but this causes little or no confusion. Here we follow Lehner's (see [13]) version of the fundamental domain of a subgroup G of $\mathrm{SL}(2, \mathbb{R})$.

DEFINITION 1.1. An open subset $\mathcal{F}$ of $\mathbb{H}$ is called a *fundamental domain* of G if:

(1) no two distinct points of $\mathcal{F}$ are equivalent under G (a packing of $\mathbb{H}$);
(2) every point of $\mathbb{H}$ is G-equivalent to a point of $\overline{\mathcal{F}}$ (a covering of $\mathbb{H}$).

For any subgroup of $\mathcal{H}_q$, a fundamental domain can be constructed using Ford's isometric circle method [5] or Dirichlet's method [1]. Even though these methods yield hyperbolically convex fundamental domains, they are not well adapted to actual construction. A third and easier method of obtaining a fundamental domain for subgroups of the Hecke modular group is through the use of a right coset decomposition. Before we pursue this, we need to set the stage. It is well known that (see [4, 6]) the interior of the set

$$\mathbb{F}_q = \left\{\tau \in \mathbb{H} : 0 \le \mathrm{Re}(\tau) \le \frac{\lambda}{2}, \left|\tau - \frac{1}{\lambda}\right| \ge \frac{1}{\lambda}\right\}$$

is a fundamental domain for $\mathcal{H}_q$. Even though the point i (elliptic of order two) splits the positive imaginary axis into two sides of $\mathbb{F}_q$ with interior angle π, from here on we will not view i as a vertex of $\mathbb{F}_q$, since it plays no role in our discussion; that is, the h-line joining 0 and $i\infty$ is considered as a single side of $\mathbb{F}_q$. Thus, the vertices of $\mathbb{F}_q$

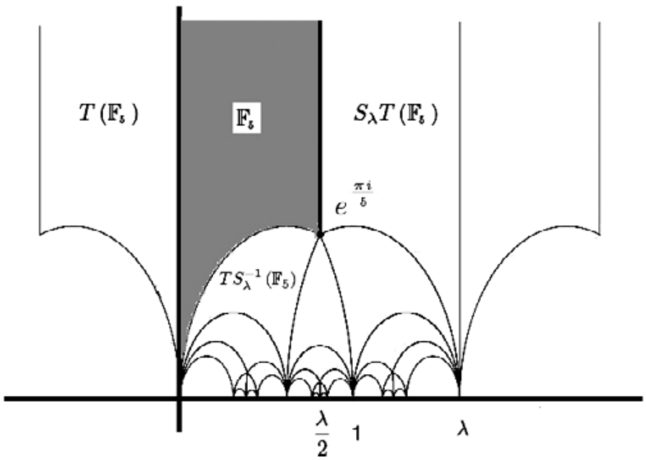

FIGURE 1. Tessellation $\mathcal{T}_5$.

are $i\infty$, 0, and $\eta_\lambda (= e^{\pi i/q})$. Also, note that the interior angle at the vertex η_λ (the only vertex of $\mathbb{F}_q$ in $\mathbb{H}$) is $2\pi/q$. The translates of $\mathbb{F}_q$ by members of the group $\mathcal{H}_q$, called tiles, define a tessellation or tiling $\mathcal{T}_q$ of $\mathbb{H}$ as shown in Figure 1 for $q = 5$. Moreover, exactly q tiles are joined at the elliptic point η_λ and the same is true at any other elliptic point of order q. Each edge or side of one tile is an edge of precisely one other tile. As seen in Figure 1, the tile $\mathbb{F}_5$ is adjacent to only three tiles, namely $T(\mathbb{F}_5)$, $S_\lambda T(\mathbb{F}_5)$, and $TS_\lambda^{-1}(\mathbb{F}_5)$. In general, it is not difficult to show that the tile $\mathbb{F}_q$ is adjacent only to the tiles $T(\mathbb{F}_q)$, $S_\lambda T(\mathbb{F}_q)$, and $TS_\lambda^{-1}(\mathbb{F}_q)$. Similarly, one can show that, for any $M \in \mathcal{H}_q$, the tile $M(\mathbb{F}_q)$ is adjacent only to the tiles $MT(\mathbb{F}_q)$, $MS_\lambda T(\mathbb{F}_q)$, and $MTS_\lambda^{-1}(\mathbb{F}_q)$.

Suppose that G is a subgroup of $\mathcal{H}_q$ of finite index. One way of obtaining a fundamental domain for G is through the use of a right coset decomposition. If

$$\mathcal{H}_q = G \cdot \mathscr{A}$$

is a right coset decomposition, then the interior of the set

$$\mathcal{R} = \bigcup_{M \in \mathscr{A}} M(\mathbb{F}_q)$$

is a fundamental domain for G. Fundamental domains obtained in this way are called *standard fundamental domains*. Both Rademacher [21] and Zagier [28] used this approach to construct an independent system of generators and to give an explicit formula for the Petersson norm of a cusp form of some congruence subgroups of the modular group, respectively. The standard fundamental domain $\mathcal{R}$ which, as just shown, is not difficult to construct, might not have desirable topological and geometric properties. It may not even be connected. However, the right coset system $\mathscr{A}$ of G in $\mathcal{H}_q$ can be chosen suitably to make $\mathcal{R}$ connected (see [23]).

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Kulkarni [9] (for $q = 3$) and Lang [11] (for $q > 3$ and q prime) have shown that every subgroup of $\mathcal{H}_q$ of finite index admits a fundamental domain which is a convex hyperbolic polygon with special properties. These polygons are called special polygons. One of the objectives of this article is to extend their results to any subgroup of the Hecke modular group $\mathcal{H}_q$ for any $q \geq 3$ by using right coset decomposition. Another objective of this article is to provide an elementary and algorithmic proof that will enable us choose a right coset system $\mathscr{A}$ suitably to make $\mathcal{R}$ a special polygon (convex hyperbolic polygon with special properties). We would like to point out that Yayenie has shown in [26] that some h-convex standard fundamental domains cannot be obtained using either Ford's and/or Dirichlet's method; that is, the three methods mentioned above are independent.

In Section 2 we introduce the necessary facts concerning hyperbolic geometry and prove a proposition about standard fundamental domains. In Section 3 we prove the main result, Theorem 3.1, and conclude with a few examples and remarks.

2. Preliminaries

A polygon in hyperbolic geometry on the upper half-plane is the interior of a closed Jordan curve

$$[\tau_1, \tau_2] \cup [\tau_2, \tau_3] \cup \cdots \cup [\tau_{n-1}, \tau_n] \cup [\tau_n, \tau_1].$$

The interior angle at the vertex τ_k is denoted by θ_k for $k = 1, 2, \ldots, n$. We allow the vertices to lie on the closure of the real axis $\overline{\mathbb{R}}$. If τ_j is such a vertex, then $\theta_j = 0$. A subset $\mathcal{S}$ of $\overline{\mathbb{H}}$ is h-convex, we will also say *convex*, if the h-line segment of any two points in $\mathcal{S}$ is also contained in $\mathcal{S}$. We say that a set D is locally convex if for each point $\tau \in D$ there exists an open set U containing τ such that the set $D \cap U$ is convex. The notions of convexity and local convexity are meaningful in both Euclidean and hyperbolic spaces, and they extend in an obvious way to the closed hyperbolic plane. The following result concerning convex polygons is used in this article very often. It is a necessary and sufficient condition for a polygon to be convex. The first part is a theorem due to Tietze [24].

PROPOSITION 2.1.

(a) *Let E be the Euclidean plane or the closed hyperbolic plane. A closed subset D of E is convex if and only if it is connected and locally convex.*

(b) *Let P be a polygon with interior angles*

$$\theta_1, \theta_2, \theta_3, \ldots, \theta_n$$

and vertices

$$\tau_1, \tau_2, \tau_3, \ldots, \tau_n.$$

Then P is convex if and only if each θ_k satisfies $0 \leq \theta_k \leq \pi$.

The proof of this proposition can be found in [1, 24].

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PROPOSITION 2.2. *Let G be a subgroup of $\mathcal{H}_q$ with $[\mathcal{H}_q : G] = \mu < \infty$ and suppose that $\mathscr{A}$ is a finite set consisting of inequivalent elements of $\mathcal{H}_q$ modulo G. If*

$$\mathcal{R} = \bigcup_{M \in \mathscr{A}} M(\mathbb{F}_q)$$

is connected and if every tile adjacent to $\mathcal{R}$ is equivalent to a tile contained in $\mathcal{R}$ modulo G, then $|\mathscr{A}| = \mu$, that is,

$$\mathcal{H}_q = G \cdot \mathscr{A}.$$

PROOF. Since $\mathcal{R}$ has a finite number of sides, $M(\mathcal{R})$, where $M \in G$, has a finite number of sides as well, and at each side of $M(\mathcal{R})$ there exists a tile which is adjacent to it at that side. Moreover, if $M_1(\mathbb{F}_q)$ is adjacent to $M(\mathcal{R})$ for some $M \in G$, then $M^{-1}M_1(\mathbb{F}_q)$ is adjacent to $\mathcal{R}$. So $M^{-1}M_1 \in G \cdot \mathscr{A}$ and $M_1 \in G \cdot \mathscr{A}$, since $M \in G$. Thus, if a tile, say $\mathbb{F}$, is adjacent to $M(\mathcal{R})$ and equivalent to one of the tiles contained in $M(\mathcal{R})$, then every tile adjacent to $\mathbb{F}$ is also contained in $N(\mathcal{R})$ for some $N \in G$. Let

$$A = \Big(\bigcup_{M \in G} M(\mathcal{R})\Big)^o = \Big(\bigcup_{M \in G \cdot \mathscr{A}} M(\mathbb{F}_q)\Big)^o$$

and

$$B = \Big(\bigcup_{M \in \mathcal{H}_q - G \cdot \mathscr{A}} M(\mathbb{F}_q)\Big)^o.$$

From the definition of the two sets and the above observation we can easily verify that:

$$\text{(i) } A \cap B = \emptyset, \quad \text{(ii) } \mathbb{H} = A \cup B, \quad \text{and} \quad \text{(iii) } A \cap \mathbb{H} \neq \emptyset.$$

The connectedness of $\mathbb{H}$ implies that $B = \emptyset$. Hence $\mathcal{H}_q - G \cdot \mathscr{A} = \emptyset$. Therefore, $|\mathscr{A}| = \mu$. □

To proceed we shall recall the definition of the stabilizer of a point in the upper half-plane with respect to the group of Möbius transformations. Suppose that τ is any point of $\overline{\mathbb{H}}$ and G is a subgroup of $\mathcal{H}_q$. The stabilizer of τ modulo G is defined to be the subset G_τ of G consisting of all $M \in G$ for which $M(\tau) = \tau$. Clearly, G_τ is a subgroup of G and if $M \in \mathcal{H}_q$, then

$$(M^{-1}GM)_\tau = M^{-1}G_{M(\tau)}M$$

and

$$G_\tau \subset (\mathcal{H}_q)_\tau \quad \text{and} \quad (\mathcal{H}_q)_{\eta_q} = \langle TS_\lambda^{-1} \rangle.$$

It can be shown that if $G \leqq \mathcal{H}_q$ and $B \in \mathcal{H}_q$, then there exists a smallest positive integer d with $d \mid q$ such that

$$G_{B(\eta_\lambda)} = \langle B(TS_\lambda^{-1})^d B^{-1} \rangle.$$

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3. Convex standard fundamental domains

In this section we show that any finite index subgroup of $\mathcal{H}_q$ admits a convex standard fundamental domain. The approach we take is algorithmic and at each step in the process we construct a convex polygon which will be contained in the standard fundamental domain obtained at the end of the algorithm.

THEOREM 3.1. *If G is a subgroup of $\mathcal{H}_q$ and $[\mathcal{H}_q : G] = \mu < \infty$, then there exist a finite number of elements $M_1, M_2, \ldots, M_m$ in $\mathcal{H}_q$ and m disjoint sets*

$$S_j := \{M_j, M_j(TS_\lambda^{-1}), \ldots, M_j(TS_\lambda^{-1})^{\delta_j - 1}\} \quad \forall j = 1, 2, \ldots, m$$

such that:

(1) $\mu = \delta_1 + \delta_2 + \cdots + \delta_m$ *and* $\delta_j \mid q$ $\forall j = 1, \ldots, m$;
(2) $\mathcal{H}_q = G \cdot \Sigma$, *where* $\Sigma = \bigcup_{j=1}^m S_j$;
(3) $\mathcal{R} = \{\bigcup_{M \in \Sigma} M(\mathbb{F}_q)\}^0$ *is a convex hyperbolic polygon. Moreover, $\mathcal{R}$ is a standard fundamental domain for G.*

PROOF. We will construct a sequence of sets contained in $\mathcal{H}_q$ such that:

(a) $\Sigma_1 \subset \Sigma_2 \subset \Sigma_3 \subset \cdots \subset \Sigma_m = \Sigma$;
(b) Σ_k contains elements of $\mathcal{H}_q$ that are inequivalent under the group G for all $k = 1, 2, \ldots, m$;
(c) $\mathcal{R}_k = \bigcup_{M \in \Sigma_k} M(\mathbb{F}_q)$ is a convex hyperbolic polygon for all $k = 1, 2, \ldots, m$.

At step k we adjoin at least the smallest divisor of q and at most q elements of $\mathcal{H}_q$ to Σ_{k-1}. We terminate the process at step m if either $|\Sigma_m| = \mu$ or every tile adjacent to $\mathcal{R}_m$ is equivalent to a tile contained in $\mathcal{R}_m$. By Proposition 2.2 these two conditions are equivalent.

Step 1. We take M_1 to be the identity element I, but one can choose any element of G as M_1. There exists a smallest positive integer d_1 such that $d_1 \mid q$ and

$$G_{M_1(\eta_\lambda)} = \langle M_1(TS_\lambda^{-1})^{d_1} M_1^{-1} \rangle.$$

Now we let

$$\Sigma_1 = \{M_1, M_1(TS_\lambda^{-1}), \ldots, M_1(TS_\lambda^{-1})^{d_1 - 1}\}.$$

If $d_1 = q$, then Σ_1 contains q elements, otherwise Σ_1 contains at most $q/2$ elements, since $d_1 \mid q$ and $d_1 \neq q$. Moreover, the set

$$\mathcal{R}_1 = \bigcup_{A \in \Sigma_1} A(\mathbb{F}_q)$$

is a closed connected set. Next we will show that $\mathcal{R}_1$ is a convex hyperbolic polygon. If $d_1 = q$, then $\mathcal{R}_1$ has no vertex in $\mathbb{H}$ and hence it is locally convex. Therefore, by Proposition 2.1, $\mathcal{R}_1$ is convex. If $d_1 < q$, then the only vertex of $\mathcal{R}_1$ in $\mathbb{H}$ is $v_1 = M_1(\eta_\lambda)$ and the interior angle at v_1 is $(2\pi/q)d_1$, which is at most $(2\pi/q)q/2 = \pi$.

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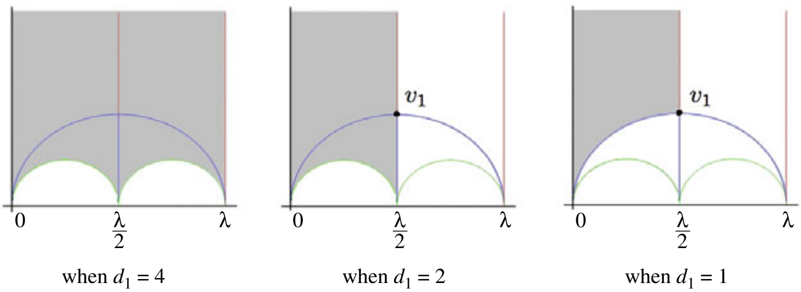

FIGURE 2. The different possibilities of $\mathcal{R}_1$ for $q = 4$.

Therefore, $\mathcal{R}_1$ is locally convex and hence convex by Proposition 2.1. See Figure 2 for different possibilities for $\mathcal{R}_1$ in the case $q = 4$.

To simplify things we will use the notation $M \overset{G}{\sim} \mathscr{A}$ if $M \in G \cdot \mathscr{A}$ and $M \overset{G}{\nsim} \mathscr{A}$ otherwise. Next we will show that, for any $A \in \Sigma_1$,

$$A(TS_\lambda^{-1})^t \overset{G}{\sim} \Sigma_1 \quad \forall t \in \mathbb{Z}.$$

Since $A \in \Sigma_1$, there exists an integer r, $0 \le r < d_1$, such that

$$\begin{aligned} A(TS_\lambda^{-1})^t &= M_1(TS_\lambda^{-1})^{t+r} \\ &= M_1(TS_\lambda^{-1})^{ad_1+t_0} \\ &= \underbrace{M_1(TS_\lambda^{-1})^{d_1 a} M_1^{-1}}_{\in G} \underbrace{M_1(TS_\lambda^{-1})^{t_0}}_{\in \Sigma} \end{aligned}$$

where $t + r = ad_1 + t_0$ and $0 \le t_0 \le d_1 - 1$. Hence

$$A(TS_\lambda^{-1})^t \overset{G}{\sim} \Sigma_1 \quad \forall t \in \mathbb{Z}.$$

Terminate the process if either $[\mathcal{H}_q : G] = |\Sigma_1|$ or there is no $B \in \mathcal{H}_q$ such that $B(\mathbb{F}_q)$ is adjacent to $\mathcal{R}_1$ and $B \overset{G}{\nsim} \Sigma_1$. Otherwise go to the next step.

Step 2. If there exists $B \in \mathcal{H}_q$ such that $B(\mathbb{F}_q)$ is adjacent to $\mathcal{R}_1$ and $B \overset{G}{\nsim} \Sigma_1$, then there exists $A \in \Sigma_1$ such that $B(\mathbb{F}_q)$ is adjacent to $A(\mathbb{F}_q)$ and from this adjacency we conclude that

$$B = AT, \quad B = ATS_\lambda^{-1}, \quad \text{or} \quad B = A(TS_\lambda^{-1})^{q-1}.$$

Since $ATS_\lambda^{-1} \overset{G}{\sim} \Sigma_1$ and $A(TS_\lambda^{-1})^{q-1} \overset{G}{\sim} \Sigma_1$, we have $B = AT$. That means $B(\mathbb{F}_q)$ and $A(\mathbb{F}_q)$ share the common side $B(\mathcal{I})$, where $\mathcal{I}$ is the positive imaginary axis. There exists d_2, the smallest positive integer such that $d_2 \mid q$ and

$$G_{B(\eta_\lambda)} = \langle B(TS_\lambda^{-1})^{d_2} B^{-1} \rangle.$$

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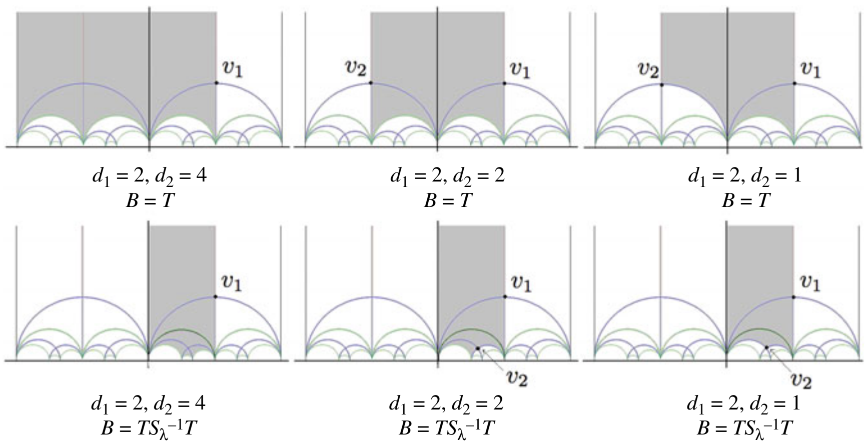

FIGURE 3. The six possibilities for $\mathcal{R}_2$ in the case $q = 4$, $M_1 = I$, and $d_1 = 2$.

We want to show that $B(TS_\lambda^{-1})^t \overset{G}{\not\sim} \Sigma_1$ for all $0 \le t \le d_2 - 1$. Suppose that there exists $0 \le t \le d_2 - 1$ such that $B(TS_\lambda^{-1})^t \overset{G}{\sim} \Sigma_1$. Then there exists $M \in G$ and $0 \le a \le d_1 - 1$ such that

$$B(TS_\lambda^{-1})^t = MM_1(TS_\lambda^{-1})^a$$

$$B = (MM_1)(TS_\lambda^{-1})^{a-t} = \underbrace{(MM_1)(TS_\lambda^{-1})^{bd_1}M_1^{-1}}_{\in G}\,\underbrace{M_1(TS_\lambda^{-1})^{t_0}}_{\in \Sigma_1},$$

where $a - t = bd_1 + t_0$, for $0 \le t_0 \le d_1 - 1$. This contradicts the fact that $B \overset{G}{\not\sim} \Sigma_1$. Thus, if we let

$$\Sigma_2 = \Sigma_1 \cup \{B, B(TS_\lambda^{-1}), \ldots, B(TS_\lambda^{-1})^{d_2-1}\} \quad \text{and} \quad \mathcal{R}_2 = \bigcup_{A \in \Sigma_2} A(\mathbb{F}_q),$$

then:

(1) any two elements of Σ_2 are inequivalent under the group G;
(2) $v_2 = B(\eta_\lambda) \neq v_1 = M_1(\eta_\lambda)$;
(3) a vertex of $\mathcal{R}_1$ which lies in $\mathbb{H}$ is also a vertex of $\mathcal{R}_2$ with the same interior angle;
(4) if $d_2 = q$, then v_2 is an interior point of $\mathcal{R}_2$;
(5) if $d_2 < q$, then v_2 is the only additional vertex of $\mathcal{R}_2$ which lies in $\mathbb{H}$ and the interior angle at vertex v_2 is $(2\pi/q)d_2$ (which is at most π);
(6) $\mathcal{R}_2$ is closed and connected.

Therefore, $\mathcal{R}_2$ is closed and locally convex and hence, by Proposition 2.1, $\mathcal{R}_2$ is convex. Figure 3 shows some of the possibilities for $\mathcal{R}_2$ in the case $q = 4$ and $d_1 = 2$.

Terminate the process if either $[\mathcal{H}_q : G] = |\Sigma_2|$ or there is no $B \in \mathcal{H}_q$ such that $B(\mathbb{F}_q)$ is adjacent to $\mathcal{R}_2$ and $B \overset{G}{\not\sim} \Sigma_2$. Otherwise go to the next step.

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Step k (The inductive step). If there exists $B \in \mathcal{H}_q$ such that $B(\mathbb{F}_q)$ is adjacent to $\mathcal{R}_{k-1}$ and $B \overset{G}{\nsim} \Sigma_{k-1}$, then there exists $A \in \Sigma_{k-1}$ such that $B(\mathbb{F}_q)$ is adjacent to $A(\mathbb{F}_q)$ and because of adjacency

$$B = AT, \quad B = ATS_\lambda^{-1} \quad \text{or} \quad B = A(TS_\lambda^{-1})^{q-1}.$$

From the definition of Σ_{k-1} we can easily see that

$$ATS_\lambda^{-1} \overset{G}{\sim} \Sigma_{k-1} \quad \text{and} \quad A(TS_\lambda^{-1})^{q-1} \overset{G}{\sim} \Sigma_{k-1}.$$

Therefore, $B = AT$. That means $B(\mathbb{F}_q)$ and $A(\mathbb{F}_q)$ share the common side $B(\mathcal{I})$. There exists a smallest positive integer d_k such that $d_k \mid q$ and

$$G_{B(\eta_\lambda)} = \langle B(TS_\lambda^{-1})^{d_k} B^{-1} \rangle.$$

Using a similar reasoning as in the previous steps we can show that $B(TS_\lambda^{-1})^t \overset{G}{\nsim} \Sigma_{k-1}$ for any $0 \le t \le d_k - 1$. Thus, if we let

$$\Sigma_k = \Sigma_{k-1} \cup \{B, B(TS_\lambda^{-1}), \ldots, B(TS_\lambda^{-1})^{d_k-1}\}$$

and

$$\mathcal{R}_k = \bigcup_{A \in \Sigma_k} A(\mathbb{F}_q),$$

then:

(1) any two elements of Σ_k are inequivalent under the group G;
(2) every vertex of $\mathcal{R}_{k-1}$ which is in $\mathbb{H}$ is a vertex of $\mathcal{R}_k$;
(3) $\mathcal{R}_k$ contains one additional vertex $v_k = B(\eta_\lambda)$ in $\mathbb{H}$, if $d_k < q$;
(4) if $v_r \in \mathbb{H}$ is a vertex of $\mathcal{R}_k$, for some $r = 1, 2, \ldots, k$, and θ_r is its interior angle, then $\theta_r = (2\pi/q)d_r \le \pi$;
(5) $\mathcal{R}_k$ is closed and connected.

Therefore, $\mathcal{R}_k$ is closed and locally convex and hence, by Proposition 2.1, $\mathcal{R}_k$ is convex.

Terminate the process if either $[\mathcal{H}_q : G] = |\Sigma_k|$ or there is no $B \in \mathcal{H}_q$ such that $B(\mathbb{F}_q)$ is adjacent to $\mathcal{R}_k$ and $B \overset{G}{\nsim} \Sigma_k$. Otherwise go to the next step.

Since the index $[\mathcal{H}_q : G] = \mu < \infty$, the process has to terminate, say it terminates at step m. The process terminates if either $|\Sigma_m| = \mu$ or every tile which is adjacent to $\mathcal{R}_m$ is equivalent to some tile contained in $\mathcal{R}_m$. By Proposition 2.2, these two statements are equivalent. Therefore, $(\mathcal{R}_m)^o$ is a convex standard fundamental domain for G. $\square$

REMARK 3.2. The convex standard fundamental domain $\mathcal{R}$ obtained in Theorem 3.1 is a special polygon as defined by Kulkarni [9] and Lang [11] with minor modification for the case $q > 3$ and q not a prime. When $q > 3$ and q is not a prime, we allow the interior angle at the intersection of elliptic vertices of order q to be $(2\pi/q)d$ (which is $\le \pi$), where d is a proper positive divisor of q.

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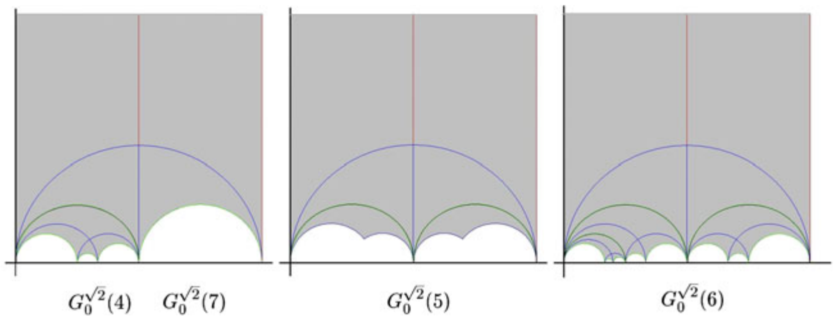

FIGURE 4. Standard fundamental domains for $G_0^{\sqrt{2}}(n)$, where $n = 4$, 5, 6, and 7.

REMARK 3.3. The previous theorem can be implemented on a computer using Maple to generate a convex hyperbolic standard fundamental domain for any congruence subgroup of Hecke groups. A few examples are given below. It is well known [3, 25] that the group $\mathcal{H}_4$ (also denoted by $G(\sqrt{2})$) consists of the mappings of all of the following types:

(i) $$N(\tau) = \frac{a\tau + b\sqrt{2}}{c\sqrt{2}\tau + d}, \quad a, b, c, d \in \mathbb{Z}, ad - 2bc = 1,$$

(ii) $$N(\tau) = \frac{a\sqrt{2}\tau + b}{c\tau + d\sqrt{2}}, \quad a, b, c, d \in \mathbb{Z}, 2ad - bc = 1.$$

It is also known that [3, 12, 25] the congruence subgroup

$$G_0^{\sqrt{2}}(n) = \{M \in G(\sqrt{2}) : c \equiv 0 \bmod n\}$$

of $\mathcal{H}_4$ satisfies

$$[\mathcal{H}_4 : G_0^{\sqrt{2}}(n)] = \begin{cases} n \prod_{p|n} \left(1 + \frac{1}{p}\right) & \text{if } (2, n) = 1 \\ 2n \prod_{p|n, p\neq 2} \left(1 + \frac{1}{p}\right) & \text{if } 2 \mid n. \end{cases}$$

Figures 4 and 5 show hyperbolically convex standard fundamental domains for the subgroups $G_0^{\sqrt{2}}(n)$ for $4 \leq n \leq 11$.

REMARK 3.4. The number of vertices (in $\mathbb{H}$) of $\overline{\mathcal{R}}$ (of Theorem 3.1) is exactly the same as the number of elliptic cycles of order q, which in turn is the same as the number of inequivalent (in G) elliptic points of order q under G in the class $\langle e^{\pi i/q}\rangle$. For the examples given above, only $G_0^{\sqrt{2}}(5)$ has exactly two inequivalent elliptic points of order four; the rest have zero elliptic points of order four, which is consistent with the results of Lang [12].

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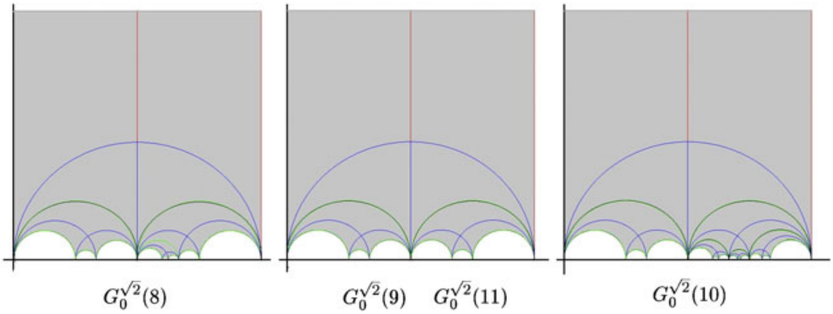

FIGURE 5. Standard fundamental domains for $G_0^{\sqrt{2}}(n)$, where $n = 8, 9, 10$, and 11.

REMARK 3.5. Theorem 3.1 does not hold as shown in [26] if we replace $\mathbb{F}_q$ by

$$\mathcal{F}_q = \left\{\tau : |\mathrm{Re}(\tau)| \le \frac{\lambda}{2},\ |\tau| \ge 1\right\}.$$

Note that the interiors of both sets are fundamental domains of $\mathcal{H}_q$.

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