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SOME THEOREMS ON CONVEX POLYGONS

BY
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ABSTRACT. In this paper diagonals of various orders in a (strict) convex polygon P_n are considered. The sums of lengths of diagonals of the same order are studied. A relationship between the number of consecutive diagonals which do not intersect a given maximal diagonal and lie on one side of it and the order of the smallest diagonal among them is established. Finally a new proof of a conjecture of P. Erdos, considered already in [1], is given.

I. Notation and nomenclature. (1) A plane convex n-sided polygon will be denoted by

$$P_n = A_1A_2 \cdots A_n$$

(A_i are the vertices). Let $j \leq \left[\frac{n}{2}\right]$. A diagonal A_iA_{i+j} (where $i+j$ is taken mod n), i.e. a diagonal cutting off j sides of the polygon, is said to be a diagonal of order j; the sides of the polygon are diagonals of the 1-st order. Clearly, P_n contains diagonals of $\left[\frac{n}{2}\right]$ distinct orders.

(2) The sum of lengths of the diagonals of the j-th order will be denoted by

$$u_j = \sum_{i=1}^{n} A_iA_{i+j}.$$

For $n=2N$, the corresponding sum

$$u_N = \sum_{i=1}^{N} A_iA_{i+N}$$

includes every diagonal of the N-th order twice.

(3) The various lengths of the diagonals of P_n will be denoted by

$$d_1 > d_2 > \cdots$$

A diagonal of length d_x is said to be of the x-th degree.

II. THEOREM 1. *If* $0 < q < p \leq \left[\frac{n}{2}\right]$, *then*

$$u_q < u_p.$$

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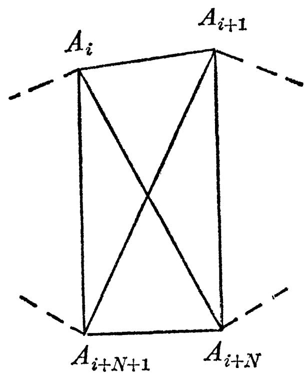

Figure 1.

Proof. (1) Let $N=\left[\frac{n}{2}\right]$. We have to prove that

$$u_{k+1} > u_k$$

for $k=1, 2, \ldots, N-1$.

(2) We first prove that $u_N > u_{N-1}$. Two cases will be distinguished:

(a) $\quad n$ *is even:* $n = 2N$,

Consider a convex quadrilateral (Fig. 1)

$$A_iA_{i+1}A_{i+N}A_{i+N+1}$$

In every convex quadrilateral, the sum of two opposite sides is smaller than the sum of the two diagonals. Hence:

$$\text{(A)} \qquad A_iA_{i+N}+A_{i+1}A_{i+N+1} > A_{i+N+1}A_i+A_{i+1}A_{i+N}$$

The diagonals of this quadrilateral are diagonals of the N-th order in P_n, while the sides appearing in the inequality (A) are diagonals of the $(N-1)$-th order in P_n. Summation of (A) for $i=1, \ldots, n$ yields

$$(\bar{\text{A}}) \qquad \begin{aligned} 2u_N &> 2u_{N-1} \\ u_N &> u_{N-1} \end{aligned}$$

(b) $\quad n$ *is odd:* $n = 2N+1$.

Consider a convex quadrilateral (Fig. 2)

$$A_iA_{i+1}A_{i+N}A_{i+N+1}$$

Here we have

$$\text{(B)} \qquad A_iA_{i+N}+A_{i+1}A_{i+N+1} > A_{i+N+1}A_i+A_{i+1}A_{i+N}$$

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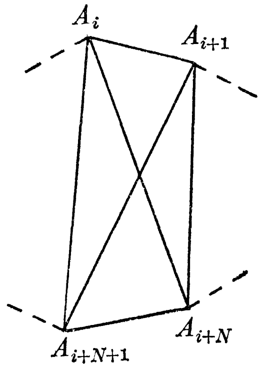

Figure 2.

$A_{i+N+1}A_i$, A_iA_{i+N} and $A_{i+1}A_{i+N+1}$ are diagonals of the N-th order in P_n, while $A_{i+1}A_{i+N}$ is a diagonal of the $(N-1)$-th order in P_n. Summation of (B) for $i=1,\ldots,n$ yields

$$(\bar{\text{B}})\qquad \begin{aligned} 2u_N &> u_N+u_{N-1} \\ u_N &> u_{N-1} \end{aligned}$$

(3) We make now an induction assumption that the theorem holds for k, i.e.

$$u_{k+1} > u_k$$

and prove that it holds for $k-1$, i.e.

$$u_k > u_{k-1}$$

To this end consider the convex quadrilateral (Fig. 3) $A_iA_{i+1}A_{i+k}A_{i+k+1}$. We have

$$(\text{C})\qquad A_iA_{i+k}+A_{i+1}A_{i+k+1} > A_iA_{i+k+1}+A_{i+1}A_{i+k}$$

$A_{i+1}A_{i+k+1}$ and A_iA_{i+k} are diagonals of the k-th order in P_n, A_iA_{i+k+1} is a diagonal of the $(k+1)$-th order, and $A_{i+1}A_{i+k}$ a diagonal of the $(k-1)$-th order. Summation

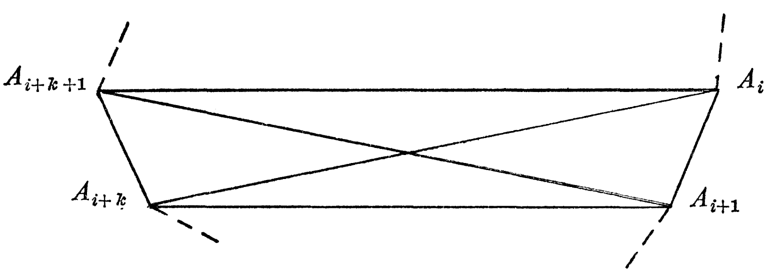

Figure 3.

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of (C) for $i=1,\ldots,n$ yields

$$2u_k > u_{k+1}+u_{k-1}. \tag{C}$$

By our assumption

$$u_{k+1} > u_k$$

hence

$$2u_k > u_k+u_{k-1}$$

$$u_k > u_{k-1}$$

and Theorem 1 is hereby proved.

(4) REMARK 1. By $(\bar{C})$:

$$2u_k > u_{k+1}+u_{k-1}$$

hence

$$u_k > \frac{u_{k+1}+u_{k-1}}{2} \qquad \left(1 < k < \left[\frac{n}{2}\right]\right),$$

i.e., the sequence $u_1, u_2, \ldots, u_N$ of the sums of the diagonals of consecutive orders increases at a slower rate than an arithmetic progression.

REMARK 2. By inscribing a convex polygon in P_n, several relations are obtainable between the sums u_j, in the same way as in the preceding proof. For example, by inscribing a triangle $A_iA_{i+k}A_{i+k+l}\left(k+l\leq\left[\frac{n}{2}\right]\right)$ we obtain

$$A_iA_{i+k}+A_{i+k}A_{i+k+l} > A_iA_{i+k+l}$$

and summation of this inequality for $i=1, 2, \ldots, n$ yields

$$u_k+u_l > u_{k+l} \qquad \left(k+l \leq \left[\frac{n}{2}\right]\right)$$

REMARK 3. Let $B_1B_2\cdots B_nB_{n+1}$ be a n-sided polygonal line, consisting of segments which are parallel and equal to the consecutive diagonals of the k-th order in P_n

$$B_iB_{i+1} \uparrow\uparrow A_iA_{i+k}$$

$$B_iB_{i+1} = A_iA_{i+k}$$

It is easily seen, by using vectors, that $B_1B_2\cdots B_nB_{n+1}$ is a closed polygon, i.e. $B_{n+1}=B_1$. It is called the k-th derivative of the polygon P_n and is denoted by $P_n^{(k)}$. This polygon is convex when P_n is convex. The l-th derivative of the polygon $P_n^{(k)}$ will be denoted by $P_n^{(kl)}$. The commutativity $P_n^{(kl)}=P_n^{(lk)}$ can be proven by using vectors. Examples of derivatives are shown in Fig. $4_{1,2,3}$.

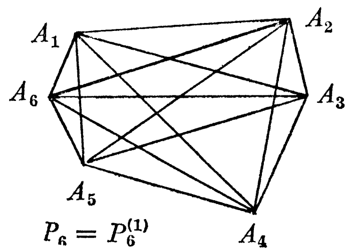

Figure 4_1

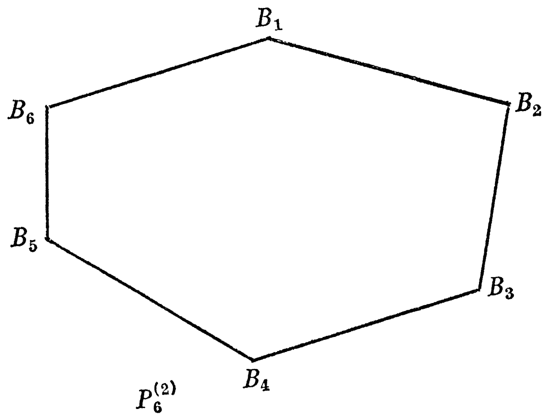

Figure 4_2

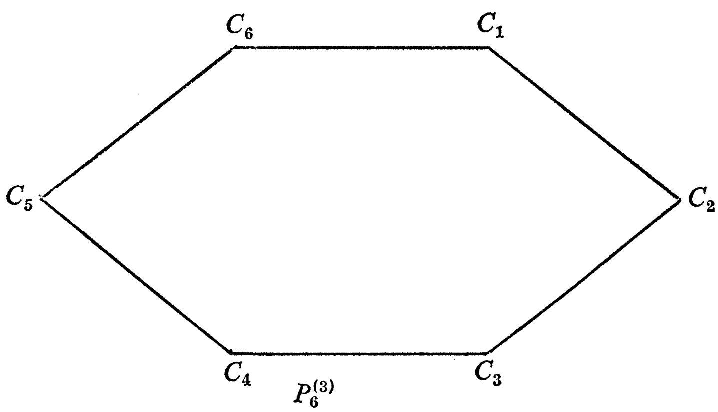

Figure 4_3

III. (1) Let $P_n = A_1A_2 \cdots A_n$ be a plane convex n-sided polygon, with A_1A_n as maximal diagonal (i.e. not smaller than any diagonal in P_n), in other words $\overline{A_1A_n} = d_1$ (see §I (3)).

Any diagonal A_kA_l, $1<k<l<n$, is said to be parallel to A_1A_n, or briefly a parallel.

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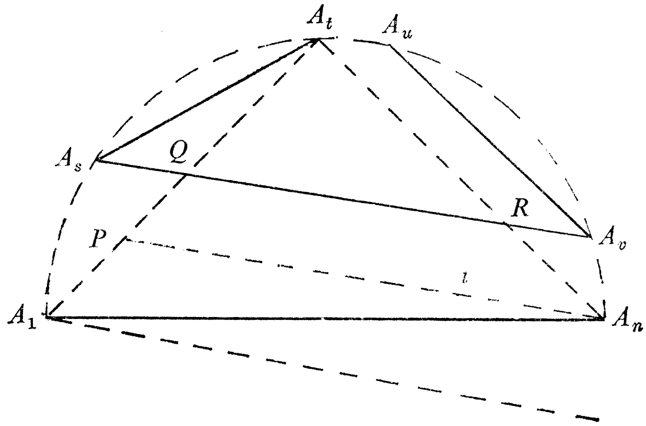

Figure 5.

Two parallels A_sA_t and A_uA_v are said to be consecutive if

$$s < t < v$$

$$s < u < v$$

(2) LEMMA 1(1). *If A_sA_t and A_uA_v, are two consecutive parallels, at least one of them is smaller than A_sA_v.*

Proof(2). Suppose l a support line of the convex hull of $A_1, A_2, \ldots, A_n$ (Fig. 5) which is parallel to A_sA_v and contains A_1. There is, then, a point P, on the segment A_1A_t such that line PA_n is parallel to l.

Let $A_1A_t \cap A_sA_v = Q$ and $A_tA_n \cap A_sA_v = R$. Since $\|A_sA_t\| \geq \|A_sA_v\|$ it follows from the triangle inequality that $\|A_tQ\| > \|QR\|$. From linearity we have $\|PA_t\| > \|PA_n\|$ and again the triangle inequality implies that $\|A_1A_t\| > \|A_1A_n\|$. This contradicts the fact that A_1A_n is a diameter of P_n. Clearly, there is no assertion to make about the relation of A_uA_v to A_sA_v.

The obtained contradiction proves the lemma.

A sequence of parallels

$$A_{s_1}A_{t_1}, A_{s_2}A_{t_2}, \ldots, A_{s_k}A_{t_k}$$

where for $i=1, \ldots, k-1$ the parallels $A_{s_i}A_{t_i}$ and $A_{s_{i+1}}A_{t_{i+1}}$ are consecutive, will be called a chain of consecutive parallels (Fig. 6).

(3) THEOREM 2. *Given a chain of f consecutive parallels in a strict convex polygon P_n. Let x be the degree of the smallest diagonal in the chain* (*see* [§I (3)]). *Then*

(1)(2) The author's original proof of Lemma 1 was based on separate case arguments ($u \lesseqgtr t$). The proof below proposed by the referee, makes these case arguments superfluous.

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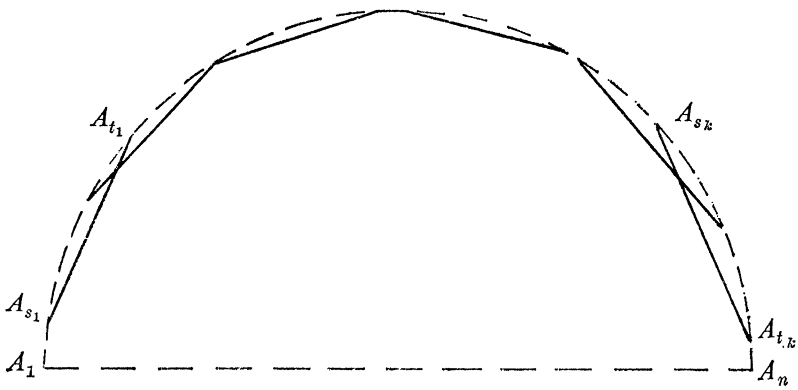

Figure 6.

(a) *If A_1A_n is the only diagonal of the* 1*-st degree in P_n, then*

$$f \leq x-2$$

(b) *If A_1A_n is not the only diagonal of the* 1*-st degree in P_n, then*

$$f \leq x-1$$

Thus there are no parallels of the 1-st degree, no two consecutive parallels of the 2-nd degree, no three consecutive parallels of the 3-rd degree, etc. In case (a), there are no parallels of the 2-nd degree, no two consecutive parallels of the 3-rd degree, etc.

Proof by induction on f.

(4) **Proof for** $f=1$. We have to prove that there is no parallel of the 1-st degree, and that in case (a) there is even no parallel of the 2-nd degree.

Let A_iA_j be a parallel (Fig. 7) and let x be its degree. Consider the convex quadrilateral

$$A_1A_iA_jA_n.$$

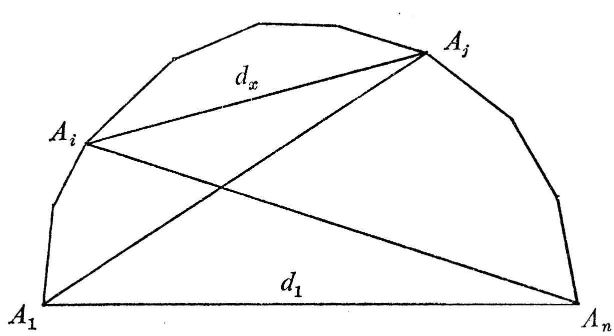

Figure 7.

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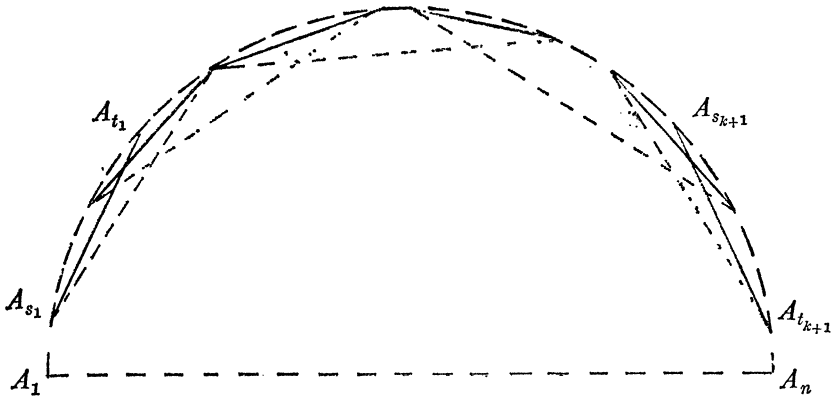

Figure 8.

The sum of two opposite sides is smaller than that of the diagonals, i.e.

$$A_1A_j+A_iA_n > A_1A_n+A_iA_j = d_1+d_x$$

so that either A_1A_j or A_iA_n must have a length d_y exceeding d_x. Hence $x \geq 2$. In case (a), A_1A_n is the only diagonal of length d_1, hence

$$d_x < d_y < d_1$$

i.e.,

$$x > y > 1$$

or $x \geq 3$. The theorem is hereby proved for $f=1$.

(5) Now assume that the theorem holds for a chain of k consecutive parallels. Let a chain C of $k+1$ parallels:

$$A_{s_1}A_{t_1}, A_{s_2}A_{t_2}, \ldots, A_{s_{k+1}}A_{t_{k+1}}$$

be given, and let x be the degree of the smallest diagonal in C. We inscribe in C a chain C' of k consecutive parallels (Fig. 8) by connecting the origin of every diagonal of C (except the last), to the end of the next one. The chain C' will thus consist of the parallels

$$A_{s_1}A_{t_2}, A_{s_2}A_{t_3}, \ldots, A_{s_k}A_{t_{k+1}}$$

which are consecutive, as is easily shown.

By Lemma 1, the diagonal $A_{s_i}A_{t_{i+1}}$ of C' exceeds one of the diagonals $A_{s_i}A_{t_i}$, $A_{s_{i+1}}A_{t_{i+1}}$ of C. The length of any diagonal of C' thus exceeds d_x, hence the degree of the smallest diagonal in C' is at most $x-1$. By the assumption that the theorem holds for $f=k$, we have: In case (a): $k \leq (x-1)-2 = x-3$. Hence

$$k+1 \leq x-2$$

In case (b):

$$k \leq (x-1)-1 = x-2$$

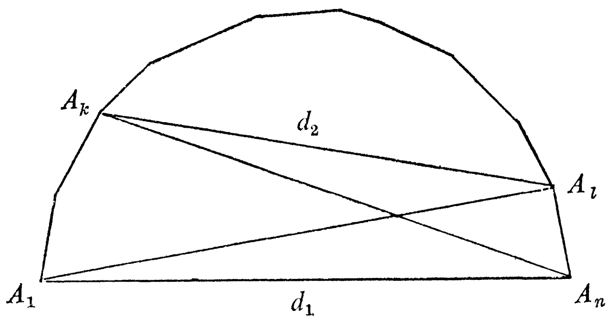

Figure 9.

hence

$$k+1 \leq x-1.$$

So the theorem holds for $f=k+1$ as well. Theorem 2 is hereby proved.

(6) REMARK. Existence of a parallel A_kA_l of the 2-nd degree implies that (Fig. 9):

$$A_1A_l+A_kA_n > A_kA_l+A_1A_n = d_1+d_2.$$

This is possible only if

$$A_1A_l = A_kA_n = d_1.$$

Thus existence of a parallel of the 2-nd degree is possible only if another diagonal of length d_1 originates from each end point of A_1A_n.

By the same induction as in (5), we conclude that the existence of a chain of consecutive parallels, satisfying

$$f = x-1$$

is possible only if another diagonal of length d_1 originates from each end point of A_1A_n.

(7) COROLLARY TO THEOREM 2. *If a chain* $A_iA_{i+1}\cdots A_{i+f}$ *of* f *consecutive sides of a plane convex n-sided polygon* $P_n=A_1A_2\cdots A_n$ *is not cut by a maximal diagonal* A_kA_l *of* P_n (Fig. 10), *and the degree of the smallest side of the chain is* x, *then by Theorem* 2 *it follows that:*

(a) *If the maximal diagonal is the only diagonal of length* d_1, *then*

$$f \leq x-2.$$

(b) *If there are other diagonals of length* d_1, *then*

$$f \leq x-1.$$

Moreover, $f=x-1$ *is possible only if another diagonal of length* d_1 *originates from each end point of the maximal diagonal.*

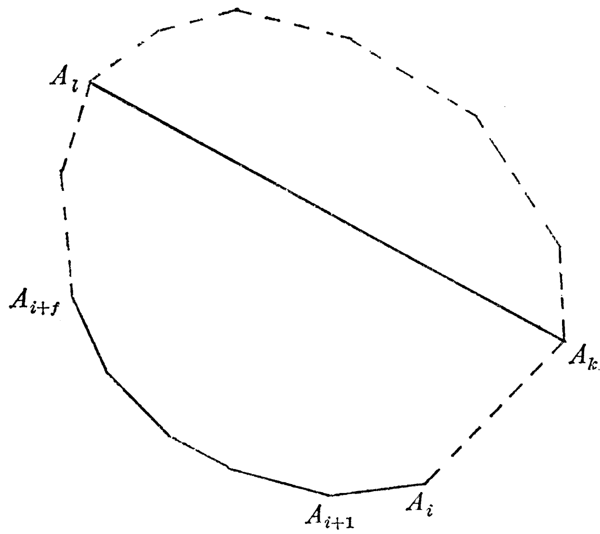

Figure 10.

IV. A new solution to a problem by P. Erdos. In [1], the author proved the following conjecture by P. Erdos:

THEOREM 3. *In every plane strictly convex n-sided polygon P_n there are at least $\left[\frac{n}{2}\right]$ different distances between various pairs of vertices.*[3]

Here a proof of this theorem will be given based on Corollary (7).

Proof. Two cases will be distinguished:

(a) There are no two maximal diagonals with a common end point (Fig. 11): Let A_lA_m be a maximal diagonal. It must cut off at least $\left[\frac{n}{2}\right]$ sides of P_n; hence there is a chain of $\left[\frac{n}{2}\right]-2$ consecutive sides of P_n, which is not cut by A_lA_m. Let x be the degree of the smallest side of this chain. There is no other maximal diagonal originating from either end point of A_lA_m. Hence, by Corollary (7).

$$x-2 \geq \left[\frac{n}{2}\right]-2$$

$$x \geq \left[\frac{n}{2}\right].$$

(3) Define the distance set of the vertex set $\{P_1, \ldots, P_n\}$ of points in a real normed linear space by: $\{\|P_iP_j\| \; 1 \leq i < j \leq n\}$.

With the referee's proof of Lemma 2, Theorem 3 can read: The distance set of the vertex set of a plane strictly convex polygon of n sides in a strictly convex real normed linear space consists of at least $\left[\frac{n}{2}\right]$ positive numbers.

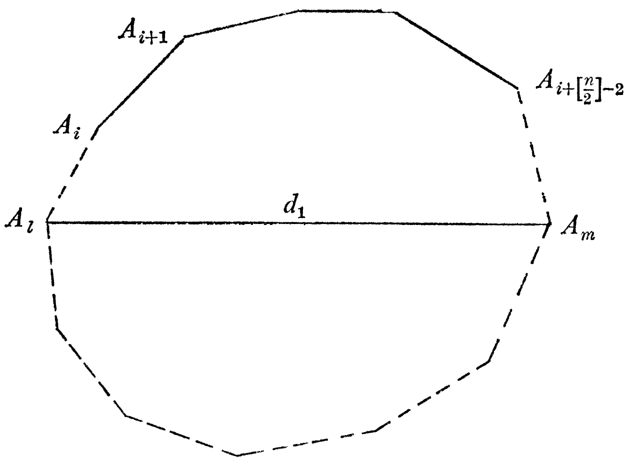

Figure 11.

The polygon has, therefore, a diagonal whose degree is not less than $\left[\frac{n}{2}\right]$.

(b) There are two maximal diagonals with a common end point (Fig. 12). Clearly, one of them (denote by A_lA_m) must cut off at least $\left[\frac{n}{2}\right]+1$ sides; hence there is a chain of $\left[\frac{n}{2}\right]-1$ sides of P_n, which is not cut by A_lA_m. For this chain

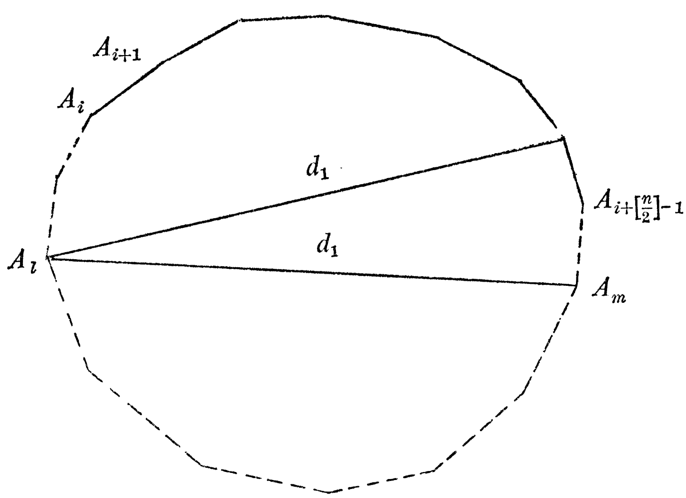

Figure 12.

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we have, by Corollary (7),

$$x-1 \geq \left[\frac{n}{2}\right]-1$$

$$x \geq \left[\frac{n}{2}\right].$$

Hence the polygon comprises at least $\left[\frac{n}{2}\right]$ different distances. The conjecture is hereby proved.

ACKNOWLEDGMENT. The author wishes to thank the referee for his elegant proof of the key Lemma 1 and for other important remarks.

REFERENCE

1. F. Altman, *On a Problem by P. Erdos*, Amer. Math. Monthly, **70** (1963), 148–157.

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