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Spherical Functions for the Semisimple Symmetric Pair $\big(\mathrm{Sp}(2,\mathbb{R}),\mathrm{SL}(2,\mathbb{C})\big)$

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Abstract. Let π be an irreducible generalized principal series representation of $G = \mathrm{Sp}(2,\mathbb{R})$ induced from its Jacobi parabolic subgroup. We show that the space of algebraic intertwining operators from π to the representation induced from an irreducible admissible representation of $\mathrm{SL}(2,\mathbb{C})$ in G is at most one dimensional. Spherical functions in the title are the images of K-finite vectors by this intertwining operator. We obtain an integral expression of Mellin-Barnes type for the radial part of our spherical function.

0 Introduction

In our previous paper [Mo1], we study certain generalized spherical functions for the semisimple symmetric pair $\big(\mathrm{Sp}(2,\mathbb{R}),\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\big)$ in connection with automorphic forms. In this paper, we treat the same problem as [Mo1] for the pair $\big(\mathrm{Sp}(2,\mathbb{R}),\mathrm{SL}(2,\mathbb{C})\big)$.

Let us give the precise definition of our spherical functions and formulate our problem in a general setting. Let G be a linear reductive group with Lie algebra $\mathfrak{g}$ and R a (non-compact) reductive subgroup of G. For an irreducible continuous representation (η, V_η) of R, we form a C^∞-induced representation $C^\infty\text{-}\mathrm{Ind}_R^G(\eta)$ with representation space

$$C_\eta^\infty(R\setminus G) := \{F\colon G\to V_\eta \mid C^\infty\text{-class}, F(rg)=\eta(r)F(g)\ \forall(r,g)\in R\times G\},$$

on which G acts by right translation. For a continuous representation π of G, we denote its underlying $(\mathfrak{g},K)$-module by π^0. Here K is a maximal compact subgroup of G. For a standard representation π of G, we consider the space of algebraic intertwining operators $\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R\setminus G)^0\big)$. Let (τ, W_τ) be a K-type of π. For $\Phi\in\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R\setminus G)^0\big)$ and a specification of K-type $i\in\mathrm{Hom}_K(\tau,\pi)$, the composite $\Phi\cdot i$ can be considered as a $V_\eta\otimes W_\tau^*$-valued function on G. We call this function *a spherical function of type* (π,η,τ). Our concern is the following mutually related two problems:

(A) Is the dimension of the intertwining space $\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R\setminus G)^0\big)$ at most one?

(B) What kind of functions appear as the spherical functions of type (π,η,τ)?

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In this paper, we discuss these problems for the semisimple symmetric pair $(G,R)=\big(\mathrm{Sp}(2,\mathbb{R}),\mathrm{SL}(2,\mathbb{C})\big)$ and obtain the following:

***Main Results* (see Theorems 7.2 and 7.5 for details)** (1) Suppose that π is an irreducible generalized principal series representation of G induced from the maximal parabolic subgroup corresponding to the long simple root. Then, for an arbitrary irreducible admissible Hilbert representation (η,V_η) of R, we have an inequality

$$\dim_{\mathbb{C}}\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0,C_\eta^\infty(R\setminus G)^0\big)\le 1.$$

(2) Suppose that π and η be as in (1). Then the radial parts of spherical functions attached to (π,η) can be expressed in terms of integrals of the form

$$\int_{\sigma-\sqrt{-1}\infty}^{\sigma+\sqrt{-1}\infty}\prod_{k=1}^{4}\frac{\Gamma(c_k-s)}{\Gamma(1-a_k-s)}y^s\,ds.$$

Here the constants a_k (resp. c_k) depend on the defining parameters of η and π (resp. of π) and $\sigma\in\mathbb{R}$ is taken so that $\sigma<\mathrm{Re}(c_k)$ $(1\le k\le 4)$.

The integrals of Mellin-Barnes type in (2) are solutions of fourth-order generalized hypergeometric differential equations and known as Meijer's G-functions (see [Er, Ch. IV]). The appearance of such functions is quite natural from the viewpoint of automorphic L-functions. In fact, if the generalized principal series representation π is even (see Definition 2.6), our spherical functions are nothing but archimedean local Shintani functions on orthogonal groups in the work of Murase and Sugano [M-S] on automorphic L-functions. Our explicit formulae in (2) enable us to compute archimedean local components of the zeta integrals introduced by them. We shall deal with this in a future paper. We should also remark that Meijer's G-function and its generalization to several variables appear in explicit formulae of (generalized) Whittaker functions (*e.g.* [B, Ch. II], [H3], [I], [Mo2]).

Let us explain the contents of this paper. In Section 1 we introduce and fix some notation about Lie groups and Lie algebras. In Section 2 we recall some representation theory: the finite dimensional representations of $K\cong U(2)$; an infinitesimal description of the non-unitary principal series representations of $R\cong\mathrm{SL}(2,\mathbb{C})$; the multiplicity formulae for K-types of generalized principal series representations of G. In Section 3 the spherical functions of type (π,η,τ) are introduced. We discuss the restriction of spherical functions to a one-dimensional split torus A, which satisfies $G=RAK$. Our main results are proved in Section 7. The computations needed for our proof are done in Sections 4, 5 and 6. Firstly, in Section 4, we construct systems of differential equations satisfied by the spherical functions using two kinds of differential operators. One of them is *shift operators*, which are defined by means of the Schmid operator, and the other is the Casimir operator. Then, in Section 5 (resp. Section 6), we reduce the above system of differential equations for even (resp. odd) generalized principal series representations into fourth-order ordinary differential equations (Theorem 5.6 (resp. 6.8)). Although the computation in this process is rather long, the resulting differential equations are quite simple. The main results follow from these differential equations together with a result of Nörlund [N] on generalized hypergeometric functions.

Finally we mention several works treating the problems (A) and (B). Hirano investigates the case of $\big(\mathrm{GL}(2,F),\mathrm{GL}(1,F)\times\mathrm{GL}(1,F)\big)$ $(F=\mathbb{R},\mathbb{C})$ [H1], [H2]. For groups of real rank one, there is also a general study by Tsuzuki [T1], [T2] for the case $\big(U(n,1),U(1)\times U(n-1,1)\big)$. In both cases, there appear Gaussian hypergeometric functions as spherical functions in the above sense.

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1 Basic Notation

1.1 Lie Groups, Lie Algebras and a Root System

Let G be the real symplectic group $\mathrm{Sp}(2,\mathbb{R})$ of rank two, which is defined by

$$\mathrm{Sp}(2,\mathbb{R}) := \left\{ g\in M(4,\mathbb{R}) \;\middle|\; {}^tgJ_4g = J_4 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0\end{pmatrix}\right\}.$$

Here I_2 is the identity matrix of degree 2. A Cartan involution θ of G is given by $\theta(g)={}^tg^{-1}$ $(g\in G)$. The corresponding maximal compact subgroup K of G is

$$K=\left\{\begin{pmatrix} A & B\\ -B & A\end{pmatrix}\in\mathrm{Sp}(2,\mathbb{R}) \;\middle|\; A,B\in M(2,\mathbb{R})\right\}.$$

It is isomorphic to the unitary group $U(2):=\{g\in\mathrm{GL}(2,\mathbb{C})\mid {}^t\overline{g}g=I_2\}$. We fix an isomorphism $u\colon U(2)\to K$ by

$$u\colon U(2)\ni A+\sqrt{-1}B\mapsto\begin{pmatrix} A & B\\ -B & A\end{pmatrix}\in K,\quad \big(A,B\in M(2,\mathbb{R})\big).$$

We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. It is easily checked that

$$\mathfrak{k}=\left\{\begin{pmatrix} A & B\\ -B & A\end{pmatrix} \;\middle|\; A,B\in M(2,\mathbb{R}),{}^tA=-A,{}^tB=B\right\}.$$

The (-1)-eigenspace $\mathfrak{p}$ of θ is

$$\mathfrak{p}=\{X\in\mathfrak{g}\mid\theta(X)=-X\}=\left\{\begin{pmatrix} A & B\\ B & -A\end{pmatrix} \;\middle|\; A,B\in M(2,\mathbb{R}),{}^tA=A,{}^tB=B\right\},$$

which gives a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. The differential of the isomorphism u, which is also denoted by u, is given by

$$u\colon\mathfrak{u}(2)\ni A+\sqrt{-1}B\mapsto\begin{pmatrix} A & B\\ -B & A\end{pmatrix}\in\mathfrak{k},\quad\big(A,B\in M(2,\mathbb{R})\big).$$

For an arbitrary Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we denote its complexification $\mathfrak{l}\otimes\mathbb{C}$ by $\mathfrak{l}_\mathbb{C}$. We also write the dual space $\mathrm{Hom}_\mathbb{C}(\mathfrak{l}_\mathbb{C},\mathbb{C})$ of $\mathfrak{l}_\mathbb{C}$ by $\mathfrak{l}_\mathbb{C}^*$. A $\mathbb{C}$-basis of $\mathfrak{k}_\mathbb{C}$ is given by

$$Z := u\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right);\quad H' := u\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right);$$
$$X := u\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right);\quad Y := u\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right).$$

The simple Lie algebra $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{h} := \mathbb{R}T_1\oplus\mathbb{R}T_2$, where

$$T_1 := u\left(\begin{pmatrix}\sqrt{-1} & 0\\ 0 & 0\end{pmatrix}\right);\quad T_2 := u\left(\begin{pmatrix}0 & 0\\ 0 & \sqrt{-1}\end{pmatrix}\right).$$

Define a $\mathbb{C}$-basis $\{\beta_1,\beta_2\}$ of $\mathfrak{h}_\mathbb{C}^*$ by $\beta_i(T_j)=\sqrt{-1}\delta_{ij}$ $(1\le i,j\le 2)$. For each $\beta\in\mathfrak{h}_\mathbb{C}^*$, set

$$\mathfrak{g}_\beta := \{X\in\mathfrak{g}_\mathbb{C}\mid [H,X]=\beta(H)X,\forall H\in\mathfrak{h}_\mathbb{C}\}.$$

Then the root system $\Delta=\Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ for the pair $(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ is given by $\Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}) = \{\pm 2\beta_1,\pm 2\beta_2,\pm(\beta_1\pm\beta_2)\}$. We fix a positive system Δ^+ of Δ as $\Delta^+ := \{2\beta_1, \beta_1+\beta_2, 2\beta_2, \beta_1-\beta_2\}$. Denote by Δ_c the set of compact roots in Δ: $\Delta_c = \{\pm(\beta_1-\beta_2)\}$. Set $\Delta_n := \Delta\setminus\Delta_c$, $\Delta_c^+ := \Delta^+\cap\Delta_c$ and $\Delta_n^+ := \Delta^+\cap\Delta_n$. For each positive root $\beta = b_1\beta_1+b_2\beta_2=(b_1,b_2)$, the root space $\mathfrak{g}_\beta$ is spanned by the following root vector X_β:

$$X_{(2,0)} = \left(\begin{array}{cc|cc} 1 & 0 & \sqrt{-1} & 0\\ 0 & 0 & 0 & 0\\ \hline \sqrt{-1} & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right);$$

$$X_{(1,1)} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & \sqrt{-1}\\ 1 & 0 & \sqrt{-1} & 0\\ \hline 0 & \sqrt{-1} & 0 & -1\\ \sqrt{-1} & 0 & -1 & 0\end{array}\right);$$

$$X_{(0,2)} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & \sqrt{-1}\\ \hline 0 & 0 & 0 & 0\\ 0 & \sqrt{-1} & 0 & -1\end{array}\right);$$

$$X_{(1,-1)} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & -\sqrt{-1}\\ -1 & 0 & -\sqrt{-1} & 0\\ \hline 0 & \sqrt{-1} & 0 & 1\\ \sqrt{-1} & 0 & -1 & 0\end{array}\right).$$

For each negative root $-\beta$, the root space $\mathfrak{g}_{-\beta}$ is spanned by the root vector $X_{-\beta} := \overline{X}_\beta$. Set $\mathfrak{p}_\mathbb{C}^+ := \bigoplus_{\beta\in\Delta_n^+}\mathfrak{g}_\beta$ and $\mathfrak{p}_\mathbb{C}^- := \bigoplus_{\beta\in\Delta_n^+}\mathfrak{g}_{-\beta}$. Then $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_\mathbb{C}^+\oplus\mathfrak{p}_\mathbb{C}^-$. For each root $\beta=b_1\beta_1+b_2\beta_2=(b_1,b_2)$, we put $\|\beta\| = \sqrt{b_1^2+b_2^2}$.

We define an involution σ of G by

$$\sigma\colon G \ni g \mapsto g_0 g g_0^{-1} \in G, \quad \text{with} \quad g_0 := \left(\begin{array}{c|c} \tilde{g}_0 & \\ \hline & \tilde{g}_0^{-1} \end{array}\right), \quad \tilde{g}_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then σ commutes with the Cartan involution θ. We set

$$R := \{g \in G \mid \sigma(g) = g\}, \quad \mathfrak{r} := \mathrm{Lie}(R) \quad \text{and} \quad \mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}.$$

Then (G, R) is a semisimple symmetric pair. Since the multiplication by g_0 is a complex structure on the space $\mathbb{R}^4$ of four dimensional column vectors, the subgroup R is isomorphic to $\mathrm{Sp}(1, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})$. To avoid confusion, we write entries of the group $\mathrm{SL}(2, \mathbb{C})$ and its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ in the form $x + jy$ ($x, y \in \mathbb{R}$, $j^2 = -1$). Let $\varphi_0\colon \mathbb{C} \hookrightarrow M(2, \mathbb{R})$ be the ring homomorphism defined by $\varphi_0(x + jy) := \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$, ($x, y \in \mathbb{R}$). We fix an isomorphism φ from $\mathrm{SL}(2, \mathbb{C})$ to R by

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \left(\begin{array}{c|c} \varphi_0(a) & \varphi_0(b)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_0(c) & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_0(d)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}\right).$$

Set $K' := K \cap R$. Then K' is a maximal compact subgroup of R, whose inverse image by φ coincides with the special unitary group $\mathrm{SU}(2) := \{g \in \mathrm{SL}(2, \mathbb{C}) \mid {}^t\bar{g}g = I_2\}$. Multiplication by $j \in \mathbb{C}$ defines a complex structure on $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{r}$. This complex structure and its $\mathbb{C}$-linear extension to $\mathfrak{r}_{\mathbb{C}} := \mathfrak{r} \otimes \mathbb{C}$ are both denoted by J. We take a $\mathbb{R}$-basis of $\mathfrak{r}$ as follows:

$$\xi_0 := \varphi\left(\begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}\right); \quad \xi_1 := \varphi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right); \quad \xi_2 := \varphi\left(\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}\right);$$
$$J\xi_0 = \varphi\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right); \quad J\xi_1 = \varphi\left(\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}\right); \quad J\xi_2 = \varphi\left(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right).$$

Note that $\{\xi_0, \xi_1, \xi_2\}$ is a $\mathbb{R}$-basis of the Lie algebra of K'. We also define a $\mathbb{C}$-basis for $(\mathfrak{r} \cap \mathfrak{k}) \otimes \mathbb{C}$ by

$$h := \xi_1 \otimes \sqrt{-1}; \quad e^+ := (\xi_0 \otimes 1 - \xi_2 \otimes \sqrt{-1})/2; \quad e^- := (-\xi_0 \otimes 1 - \xi_2 \otimes \sqrt{-1})/2.$$

Then the set $\{h, e^+, e^-\}$ forms an $\mathfrak{sl}_2$-triple. A maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p} \cap \mathfrak{q}$ is given by

$$\mathfrak{a} := \mathbb{R}H_1 \quad \text{with} \quad H_1 := \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & -1 & \end{array}\right) \in \mathfrak{g}.$$

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Define a one-parameter subgroup a_t ($t \in \mathbb{R}$) of G by $a_t := \exp(tH_1)$ and set $A := \{a_t \mid t \in \mathbb{R}\}$. In order to state the next lemma (a generalized Cartan decomposition) we introduce some symbols.

Notation Define $\mathfrak{r}_{\mathbb{C}}$-valued functions $\mathcal{L}^{\pm}$, $\mathcal{M}^{\pm}$ and $\mathcal{N}$ on $\mathbb{R}^{\times}$ by

$$\mathcal{L}^{\pm} := \pm Je^{\mp} \mp \frac{e^{\mp}}{\sinh 2t}; \quad \mathcal{M}^{\pm} := \mp Je^{\pm} \mp \frac{e^{\pm}}{\sinh 2t}; \quad \mathcal{N} := \frac{-Jh}{\cosh 2t}.$$

Lemma 1.1 *For any $t \in \mathbb{R}^{\times}$, we have $\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}(a_{-t})\mathfrak{r}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$. To be more precise, each non-compact root vector X_β is decomposed as below:*

(i) $X_{(2,0)} = \mathrm{Ad}(a_{-t}) \cdot \mathcal{L}^{+} - \coth 2t \cdot X$;
(ii) $X_{(1,1)} = \mathrm{Ad}(a_{-t}) \cdot \mathcal{N} + H_1 - \tanh 2t \cdot Z$;
(iii) $X_{(0,2)} = \mathrm{Ad}(a_{-t}) \cdot \mathcal{M}^{+} - \coth 2t \cdot Y$;
(iv) $X_{(-2,0)} = \mathrm{Ad}(a_{-t}) \cdot \mathcal{L}^{-} + \coth 2t \cdot Y$;
(v) $X_{(-1,-1)} = \mathrm{Ad}(a_{-t}) \cdot (-\mathcal{N}) + H_1 + \tanh 2t \cdot Z$;
(vi) $X_{(0,-2)} = \mathrm{Ad}(a_{-t}) \cdot \mathcal{M}^{-} + \coth 2t \cdot X$.

The proof is direct computations.

1.2 The Jacobi Parabolic Subgroup

The semisimple Lie group $G = \mathrm{Sp}(2,\mathbb{R})$ has, up to conjugation, two maximal parabolic subgroups: one with abelian unipotent radical and the other with non-abelian unipotent radical. The latter is called the *Jacobi parabolic subgroup* of G and denoted by P_J. We fix the Jacobi parabolic subgroup P_J and its Langlands decomposition $P_J = M_J A_J N_J$ as follows:

$$P_J := \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ 0 & * & * & * \\ \hline 0 & 0 & * & 0 \\ 0 & * & * & * \end{array} \right) \in G \right\};$$

$$M_J := \left\{ \left(\begin{array}{cc|cc} \epsilon & & & \\ & a & & b \\ \hline & & \epsilon & \\ & c & & d \end{array} \right) \;\middle|\; \begin{array}{l} \epsilon \in \{\pm 1\}, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \end{array} \right\};$$

$$A_J := \{\mathrm{diag}(t, 1, t^{-1}, 1) \mid t \in \mathbb{R}_{>0}\}; \quad N_J := \left\{ \left(\begin{array}{cc|cc} 1 & * & * & * \\ 0 & 1 & * & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{array} \right) \in G \right\}.$$

Here $\mathrm{diag}(a_1, a_2, a_3, a_4)$ denotes the diagonal matrix whose (i,i)-components are given by a_i. Put $\mathfrak{a}_J := \mathrm{Lie}(A_J)$.

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2 Representations of K, R and G

In this section we collect some basic facts about representations of K, R and G. In Section 2.1 we parameterize the irreducible finite dimensional representations of K and describe the decompositions of their tensor products with $\mathfrak{p}_{\mathbb{C}}^-$ into irreducible factors. In Section 2.2 we introduce the non-unitary principal series representations of R and give their infinitesimal descriptions. In Section 2.3 we recall the generalized principal series representations of G induced from the Jacobi parabolic subgroup P_J and their K-type multiplicities.

2.1 Irreducible K-Modules

The irreducible finite dimensional representations of K are parameterized by the set of their highest weights relative to Δ_c^+:

$$\{\lambda = \lambda_1\beta_1 + \lambda_2\beta_2 = (\lambda_1, \lambda_2) \in \mathfrak{h}_{\mathbb{C}}^* \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2\}.$$

For each dominant integral weight $\lambda = (\lambda_1, \lambda_2)$, we set $d = d_\lambda = \lambda_1 - \lambda_2 (\geq 0)$. Then the degree of the representation $(\tau_\lambda, W_\lambda)$ associated to λ is $d + 1$. We can take a basis $\{w_k \mid 0 \leq k \leq d\}$ in W_λ so that the representation of $\mathfrak{k}_{\mathbb{C}}$ associated to τ_λ is given by

$$\tau_\lambda(Z)w_k = (\lambda_1 + \lambda_2)w_k; \quad \tau_\lambda(H')w_k = (-d + 2k)w_k;$$
$$\tau_\lambda(X)w_k = (k + 1)w_{k+1}; \quad \tau_\lambda(Y)w_k = (d + 1 - k)w_{k-1}.$$

We call this basis the *standard basis* of τ_λ. If we want to refer explicitly to the dominant weight λ, we denote w_k by w_k^λ.

The vector space $\mathfrak{p}_{\mathbb{C}}^-$ becomes a K-module via the adjoint representation of K. It is easily checked that $\mathfrak{p}_{\mathbb{C}}^- \cong W_{(0,-2)}$ and the correspondence of the bases is given by

$$(X_{(-2,0)}, X_{(-1,-1)}, X_{(0,-2)}) \mapsto (w_0, -w_1, w_2).$$

Let us consider the tensor products $W_\lambda \otimes \mathfrak{p}_{\mathbb{C}}^-$.

Lemma 2.1 *The tensor product $W_\lambda \otimes \mathfrak{p}_{\mathbb{C}}^-$ has the decomposition into irreducible factors as*

$$W_\lambda \otimes \mathfrak{p}_{\mathbb{C}}^- = \begin{cases} W_{(\lambda_1,\lambda_2-2)} \oplus W_{(\lambda_1-1,\lambda_2-1)} \oplus W_{(\lambda_1-2,\lambda_2)} & \text{if } \lambda_1 > \lambda_2; \\ W_{(\lambda_1,\lambda_2-2)} & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Here we understand $W_{(\lambda_1,\lambda_2)} = 0$ for $\lambda_1 < \lambda_2$.

Let P^{up}, P^{even} and P^{down} be the projectors from $W_\lambda \otimes \mathfrak{p}_{\mathbb{C}}^-$ to the factors $W_{(\lambda_1,\lambda_2-2)}$, $W_{(\lambda_1-1,\lambda_2-1)}$ and $W_{(\lambda_1-2,\lambda_2)}$, respectively. We shall write these projectors explicitly in the next lemma.

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Lemma 2.2

(i) *Set* $\mu = (\lambda_1, \lambda_2 - 2)$. *Then, up to scalar multiple, the projector* P^{up} *is given by*

$$P^{\mathrm{up}}(w_k^{\lambda} \otimes w_2) = \frac{(k+2)(k+1)}{2} w_{k+2}^{\mu} \quad (0 \le k \le d);$$
$$P^{\mathrm{up}}(w_k^{\lambda} \otimes w_1) = (k+1)(d+1-k) w_{k+1}^{\mu} \quad (0 \le k \le d);$$
$$P^{\mathrm{up}}(w_k^{\lambda} \otimes w_0) = \frac{(d+1-k)(d+2-k)}{2} w_k^{\mu} \quad (0 \le k \le d).$$

(ii) *Set* $\nu = (\lambda_1 - 1, \lambda_2 - 1)$. *Then, up to scalar multiple, the projector* P^{even} *is given by*

$$P^{\mathrm{even}}(w_k^{\lambda} \otimes w_2) = (k+1) w_{k+1}^{\nu} \quad (0 \le k \le d);$$
$$P^{\mathrm{even}}(w_k^{\lambda} \otimes w_1) = (d-2k) w_k^{\nu} \quad (0 \le k \le d);$$
$$P^{\mathrm{even}}(w_k^{\lambda} \otimes w_0) = -(d+1-k) w_{k-1}^{\nu} \quad (0 \le k \le d).$$

(iii) *Set* $\pi = (\lambda_1 - 2, \lambda_2)$. *Then, up to scalar multiple, the projector* P^{down} *is given by*

$$P^{\mathrm{down}}(w_k^{\lambda} \otimes w_2) = w_k^{\pi} \quad (0 \le k \le d);$$
$$P^{\mathrm{down}}(w_k^{\lambda} \otimes w_1) = -2 w_{k-1}^{\pi} \quad (0 \le k \le d);$$
$$P^{\mathrm{down}}(w_k^{\lambda} \otimes w_0) = w_{k-2}^{\pi} \quad (0 \le k \le d).$$

Here we understand that w_k^{ν}, *or* w_k^{π} *is zero for* $k < 0$, *or* $k > d_{\nu}$ *or* $k > d_{\pi}$.

2.2 Non-Unitary Principal Series Representations of R

In this subsection, we introduce the non-unitary principal series representations of $\mathrm{SL}(2,\mathbb{C})$. It is well-known that every irreducible admissible representation of $\mathrm{SL}(2,\mathbb{C})$ is infinitesimally equivalent to a subrepresentation of some non-unitary principal series representations (see [Wa, 3.8.3., 5.7.]).

We fix a minimal parabolic subgroup P' of R and a Langlands decomposition $P' = M'A'N'$ of P' as

$$P' := \left\{ \varphi(p') \;\middle|\; p' = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \right\};$$
$$M' := \left\{ \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) \;\middle|\; |\alpha| = 1 \right\};$$
$$A' := \left\{ \varphi\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right) \;\middle|\; x \in \mathbb{R}, x > 0 \right\}; \quad N' := \left\{ \varphi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) \;\middle|\; y \in \mathbb{C} \right\}.$$

For $l \in \mathbb{Z}$ and $\nu \in \mathbb{C}$, we define a character χ_l of M' and a quasi-character a^ν of A' by

$$\chi_l(m') := \alpha^l, \quad m' = \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) \in M';$$
$$a^\nu(\tilde{x}) := x^\nu, \quad \tilde{x} = \varphi\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right) \in A'.$$

Then we understand by *a non-unitary principal representation of* R the induced representation

$$\eta(l,\nu) = L^2\text{-}\mathrm{Ind}_{P'}^R(\chi_l \otimes a^{\nu+2} \otimes 1_{N'})$$

from P' to R. The representation space $V_{l,\nu}$ of $\eta(l,\nu)$ is given by the completion of the pre-Hilbert space

$$\{f\colon R \xrightarrow{C^\infty} \mathbb{C} \mid f(m'a'n'r) = \chi_l(m')a^{\nu+2}(a')f(r),$$
$$\forall (m',a',n',r) \in M' \times A' \times N' \times R\}$$

with inner product

$$(f_1, f_2)_{K'} := \int_{K'} f_1(y)\overline{f_2(y)}\,dy.$$

Here dy is the Haar measure of K'. The action of R on this space is given by right translation.

We shall fix a $\mathbb{C}$-basis of the space ${V_{l,\nu}}^0$ of K'-finite vectors in $V_{l,\nu}$ and describe the action of Lie algebra of R on this basis. The representation space $V_{l,\nu}$ is a closed subspace of the Hilbert space $L^2(K')$. Thus, we start with constructing a complete orthonormal system for $L^2(K')$. For each non-negative integer m, there exists a unique, up to equivalence, irreducible $(m+1)$-dimensional representation of K', which we denote by $F^{(m)}$. The representation $F^{(1)}$ is the natural representation of $\mathrm{SU}(2)$ on the space of two-dimensional column vectors. That is, if we write $e_0 := {}^t(1,0)$ and $e_1 := {}^t(0,1)$, then we have

$$F^{(1)}(y)e_0 = \alpha e_0 - \overline{\beta}e_1; \quad F^{(1)}(y)e_1 = \beta e_0 + \overline{\alpha}e_1, \quad \forall y = \varphi\left(\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\right) \in K'.$$

The representation $F^{(m)}$ is realized in the space $\mathrm{Sym}^m(F^{(1)})$ of symmetric tensors of $F^{(1)}$ of degree m. As a basis of $F^{(m)}$, we take

$$v_p^{(m)} := \binom{m}{p} e_0^{m-p}e_1^p \in \mathrm{Sym}^m(F^{(1)}), \quad (0 \le p \le m).$$

Define matrix coefficients $f_{a,b}^{(m)}$ $(0 \le a, b \le m)$ of $F^{(m)}$ by

$$f_{a,b}^{(m)}(y) := \langle v_a^{*(m)}, F^{(m)}(y)v_b^{(m)} \rangle, \quad y \in K'.$$

Here $\{v_p^{*(m)} \mid 0 \le p \le m\}$ is the dual basis of $\{v_p^{(m)} \mid 0 \le p \le m\}$ and $\langle\ ,\ \rangle$ is the canonical pairing between $F^{(m)}$ and its dual vector space. By the classical theory of Peter-Weyl, we know that the set of functions $f_{a,b}^{(m)}/(f_{a,b}^{(m)},f_{a,b}^{(m)})_{K'}^{1/2}$ ($m\in\mathbb{Z}_{\geq 0}$, $0\leq a,b\leq m$) forms a complete orthonormal system of $L^2(K')$. We can easily check that

$$f_{a,b}^{(m)}(t_1yt_2)=\chi_{m-2a}(t_1)\chi_{m-2b}(t_2)f_{a,b}^{(m)}(y),\quad t_1,t_2\in M',y\in K'.$$

Hence, if we set $M(l):=\{m\in\mathbb{Z}\mid m\equiv l \pmod 2, m\geq |l|\}$, then we have the decomposition of ${V_{l,\nu}}^0$:

$$ {V_{l,\nu}}^0=\bigoplus_{m\in M(l)}\bigoplus_{p=0}^{m}\mathbb{C}f_{(m-l)/2,p}^{(m)}\cong\bigoplus_{m\in M(l)}F^{(m)}.$$

In what follows, we also write $f_p^{(m)}$ in place of $f_{(m-l)/2,p}^{(m)}$. We prefer to use another basis $\{g_p^{(m)}\mid m\in M(l), 0\leq p\leq m\}$ for ${V_{l,\nu}}^0$, which is given by

$$g_p^{(m)}:=\eta(c_R)f_p^{(m)},\quad c_R:=\varphi\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & j\\ j & 1\end{pmatrix}\right)\in K'.$$

We call this basis $\{g_p^{(m)}\}$ the *standard basis* of ${V_{l,\nu}}^0$.

Proposition 2.3 *Let* $\big(\eta=\eta(l,\nu),V_{l,\nu}\big)$ *be a non-unitary principal series representation of* R*. The action of* $\mathfrak{r}_{\mathbb{C}}$ *on the vectors* $g_p^{(m)}\in V_{l,\nu}$ *is given as follows:*

(i) $\eta(h)g_p^{(m)}=(2p-m)g_p^{(m)}$;
(ii) $\eta(e^+)g_p^{(m)}=(p+1)g_{p+1}^{(m)}$;
(iii) $\eta(e^-)g_p^{(m)}=(m-p+1)g_{p-1}^{(m)}$;
(iv)

$$\begin{aligned}\eta(Jh)g_p^{(m)}=\sqrt{-1}\{&-(\nu+m+2)(p+1)(m-p+1)A^{(m)}g_{p+1}^{(m+2)}\\&+(-\nu)(m-2p)B^{(m)}g_p^{(m)}+2(\nu-m)C^{(m)}g_{p-1}^{(m-2)}\};\end{aligned}$$

(v)

$$\begin{aligned}\eta(Je^+)g_p^{(m)}=\frac{\sqrt{-1}}{2}\{&(\nu+m+2)(p+1)(p+2)A^{(m)}g_{p+2}^{(m+2)}\\&+2\nu(p+1)B^{(m)}g_{p+1}^{(m)}+2(\nu-m)C^{(m)}g_p^{(m-2)}\};\end{aligned}$$

(vi)

$$\begin{aligned}\eta(Je^-)g_p^{(m)}=\frac{-\sqrt{-1}}{2}\{&(\nu+m+2)(m-p+2)(m-p+1)A^{(m)}g_p^{(m+2)}\\&+(-2\nu)(m-p+1)B^{(m)}g_{p-1}^{(m)}+2(\nu-m)C^{(m)}g_{p-2}^{(m-2)}\}.\end{aligned}$$

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Here we set for $m \in M(l)$

$$A^{(m)} := \binom{m+2}{2}^{-1}; \quad B^{(m)} := \begin{cases} l/\big(m(m+2)\big), & \textit{if } m > 0; \\ 0, & \textit{if } m = 0; \end{cases}$$

$$C^{(m)} := \begin{cases} (l^2 - m^2)/\big(4m(m+1)\big), & \textit{if } m > 0; \\ 0, & \textit{if } m = 0. \end{cases}$$

In these formulae, we understand $g_p^{(m)} = 0$ for $m \notin M(l)$ or $p < 0$ or $p > m$.

Proof Put

$$\tilde{h} := \mathrm{Ad}(c_R^{-1})h = \xi_0 \otimes \sqrt{-1}; \quad \tilde{e}^+ := \mathrm{Ad}(c_R^{-1})e^+ = (-\xi_1 \otimes 1 - \xi_2 \otimes \sqrt{-1})/2;$$
$$\tilde{e}^- := \mathrm{Ad}(c_R^{-1})e^- = (\xi_1 \otimes 1 - \xi_2 \otimes \sqrt{-1})/2.$$

Since

$$\eta(x)g_p^{(m)} = \eta(c_R)\eta\big(\mathrm{Ad}(c_R)^{-1}x\big)\, f_p^{(m)}, \quad \forall x \in \mathfrak{r}_{\mathbb{C}},$$

we have only to compute the action of $\tilde{h}, \tilde{e}^+, \tilde{e}^-$, $J\tilde{h}$, $J\tilde{e}^+$ and $J\tilde{e}^-$ on the vectors $f_p^{(m)}$. It is easy to see

$$\eta(\tilde{h}) f_p^{(m)} = (2p-m) f_p^{(m)}; \quad \eta(\tilde{e}^+) f_p^{(m)} = (p+1) f_{p+1}^{(m)}; \quad \eta(\tilde{e}^-) f_p^{(m)} = (m-p+1) f_{p-1}^{(m)}.$$

Thus the formulae from (i) to (iii) are proved. Next we compute $\eta(J\tilde{h}) f_p^{(m)}$. Since

$$J\xi_0 = \mathrm{Ad}(y^{-1}) \left\{ (|\alpha|^2 - |\beta|^2) \cdot J\xi_0 + \varphi\left(\begin{pmatrix} 0 & 4\alpha\beta \\ 0 & 0 \end{pmatrix} \right) \right\} + 2\alpha\overline{\beta} \cdot \tilde{e}^+ - 2\overline{\alpha}\beta \cdot \tilde{e}^-,$$

for any $y = \varphi\Big(\big(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix} \big) \Big) \in K'$, we have

$$[\eta(J\xi_0) f_{a,p}^{(m)}](y) = -(\nu+2) f_{1,1}^{(2)}(y) f_{a,p}^{(m)}(y) - 2(m+1-p) f_{1,2}^{(2)}(y) f_{a,p-1}^{(m)}(y)$$
$$- 2(p+1) f_{1,0}^{(2)}(y) f_{a,p+1}^{(m)}(y).$$

Here we use the following equalities:

$$f_{1,0}^{(2)}(y) = -\alpha\overline{\beta}; \quad f_{1,1}^{(2)}(y) = |\alpha|^2 - |\beta|^2; \quad f_{1,2}^{(2)}(y) = \overline{\alpha}\beta.$$

Our remaining task is to express the products $f_{1,1}^{(2)} \cdot f_{a,p}^{(m)}$, $f_{1,2}^{(2)} \cdot f_{a,p-1}^{(m)}$ and $f_{1,0}^{(2)} \cdot f_{a,p+1}^{(m)}$ of two matrix coefficients as linear combinations of $f_{a+1,p+1}^{(m+2)}$, $f_{a,p}^{(m)}$ and $f_{a-1,p-1}^{(m-2)}$, which we can know from Lemma 2.4 below. Inserting the formulae there, we arrive at

$$\eta(Jh) f_p^{(m)} = \sqrt{-1}\{-(\nu+m+2)(p+1)(m-p+1)A^{(m)} f_{p+1}^{(m+2)}$$
$$+ (-\nu)(m-2p)B^{(m)} f_p^{(m)} + 2(\nu-m)C^{(m)} f_{p-1}^{(m-2)}\}$$

with

$$A^{(m)} = A^{(m)}_{(m-l)/2}, \quad B^{(m)} = B^{(m)}_{(m-l)/2} \quad \text{and} \quad C^{(m)} = C^{(m)}_{(m-l)/2}.$$

Here $A^{(m)}_a$, $B^{(m)}_a$ and $C^{(m)}_a$ $(a = (m-l)/2)$ are given in Lemma 2.4. This proves the formula (iv). By virtue of the identities $Je^+ = (1/2)[Jh, e^+]$ and $Je^- = (-1/2)[Jh, e^-]$, the formulae (v) and (vi) are easily deduced from the others. ■

Lemma 2.4 *For a non-negative integer m and an integer a with $0 \le a \le m$, there exist rational numbers $A^{(m)}_a$, $B^{(m)}_a$ and $C^{(m)}_a$ satisfying the following equalities $(0)_p$, $(1)_p$ and $(2)_p$ $(0 \le p \le m)$ simultaneously:*

$$\begin{aligned}
(0)_p : \quad f^{(2)}_{1,0} \cdot f^{(m)}_{a,p} &= A^{(m)}_a \binom{m+2-p}{2} f^{(m+2)}_{a+1,p} + B^{(m)}_a(-m-1+p) f^{(m)}_{a,p-1} \\
&\quad + C^{(m)}_a f^{(m-2)}_{a-1,p-2}; \\
(1)_p : \quad f^{(2)}_{1,1} \cdot f^{(m)}_{a,p} &= A^{(m)}_a (p+1)(m+1-p) f^{(m+2)}_{a+1,p+1} + B^{(m)}_a (m-2p) f^{(m)}_{a,p} \\
&\quad + C^{(m)}_a (-2) f^{(m-2)}_{a-1,p-1}; \\
(2)_p : \quad f^{(2)}_{1,2} \cdot f^{(m)}_{a,p} &= A^{(m)}_a \binom{p+2}{2} f^{(m+2)}_{a+1,p+2} + B^{(m)}_a (p+1) f^{(m)}_{a,p+1} \\
&\quad + C^{(m)}_a f^{(m-2)}_{a-1,p}.
\end{aligned}$$

Here we understand that $f^{(m)}_{a,b}$ equals to zero unless $m \ge 0$ and $0 \le a, b \le m$. Moreover $A^{(m)}_a$, $B^{(m)}_a$ and $C^{(m)}_a$ are given by

$$A^{(m)}_a = \binom{m+2}{2}^{-1}; \quad B^{(m)}_a := \begin{cases} (m-2a)/\big(m(m+2)\big), & \text{if } m > 0; \\ 0, & \text{if } m = 0; \end{cases}$$

$$C^{(m)}_a := \begin{cases} (-a)(m-a)/\big(m(m+1)\big), & \text{if } m > 0; \\ 0, & \text{if } m = 0. \end{cases}$$

Proof For brevity, we give proofs for the case $m \ge 2$ and $a \ge 1$. The proofs for the other cases are the same in principle. Multiplication of functions on K' defines a K'-homomorphism from

$$\Big(\bigoplus_{q=0}^{2} \mathbb{C} f^{(2)}_{1,q}\Big) \otimes \Big(\bigoplus_{p=0}^{m} \mathbb{C} f^{(m)}_{a,p}\Big) \cong F^{(2)} \otimes F^{(m)}$$

to

$$\Big(\bigoplus_{p=0}^{m+2} \mathbb{C} f^{(m+2)}_{a+1,p}\Big) \oplus \Big(\bigoplus_{p=0}^{m} \mathbb{C} f^{(m)}_{a,p}\Big) \oplus \Big(\bigoplus_{p=0}^{m-2} \mathbb{C} f^{(m-2)}_{a-1,p}\Big) \cong F^{(m+2)} \oplus F^{(m)} \oplus F^{(m-2)}.$$

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On the other hand, the pull-back of an irreducible $(m+1)$-dimensional representation $\tau_{(\lambda_1,\lambda_2)}$ $(\lambda_1-\lambda_2=m)$ via the injective homomorphism

$$K' \ni \varphi\left(\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\right) \mapsto u\left(\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\right) \in K$$

is equivalent to $F^{(m)}$ and the correspondence of bases is given by $v_p^{(m)} \mapsto w_p$. Therefore, the existence of $A_a^{(m)}, B_a^{(m)}$ and $C_a^{(m)}$ follows from Lemma 2.2. To determine these constants, we use $(2)_{p=0}$. Direct computation shows

$$\begin{aligned} f_{a+1,2}^{(m+2)}(y) &= \binom{m+2}{a+1}^{-1}\binom{m+2}{2}\alpha^{m-a-1}(-\overline{\beta})^{a-1} \\ &\quad\times\left\{\binom{m+2}{a+1}|\alpha|^4 - 2\binom{m+1}{a+1}|\alpha|^2 + \binom{m}{a+1}\right\}; \\ f_{a,1}^{(m)}(y) &= \alpha^{m-a-1}(-\overline{\beta})^{a-1}\{m|\alpha|^2-(m-a)\}; \\ f_{a-1,0}^{(m-2)}(y) &= \alpha^{m-a-1}(-\overline{\beta})^{a-1} \end{aligned}$$

for any $y = \varphi\left(\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)\right) \in K'$. Inserting these to $(2)_{p=0}$, we get the desired formulae. ■

2.3 The Generalized Principal Series Representations of G

A discrete series representation (σ, V_σ) of the semisimple part M_J of P_J is of the form $\sigma = \epsilon \boxtimes D_\lambda (|\lambda| \geq 2)$, where $\epsilon\colon \{\pm 1\} \to \mathbb{C}^*$ is a character, D_λ is the discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ with Blattner parameter λ, that is, the extreme weight vector $v \in D_\lambda$ satisfies

$$D_\lambda\left(\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\right) v = e^{\sqrt{-1}\lambda x} v, \quad x \in \mathbb{R}.$$

For an element $\nu_J \in \mathfrak{a}_{J,\mathbb{C}}^*$, let $A_J \ni a_J \mapsto a_J^{\nu_J} \in \mathbb{C}^*$ be the corresponding quasi-character of A_J. We also identify $\nu_J \in \mathfrak{a}_{J,\mathbb{C}}^*$ with its values at $\mathrm{diag}(1,0,-1,0) \in \mathfrak{a}_J$. Define a representation $\sigma \otimes \nu_J$ of P_J by

$$\sigma \otimes \nu_J(p_J) = \sigma(m_J) a_J^{\nu_J}, \quad \text{for } p_J = m_J a_J n_J \in P_J = M_J A_J N_J.$$

Then the generalized principal representation $\pi(\sigma, \nu_J)$ of G is defined as the induced representation $C^\infty\text{-}\mathrm{Ind}_{P_J}^G\big(\sigma \otimes (\nu_J + \rho_J)\big)$ of G with representation space

$$\begin{aligned} \{F\colon G \longrightarrow V_\sigma \mid C^\infty\text{-class}, F(m_J a_J n_J g) = \sigma(m_J) a_J^{\nu_J+\rho_J} F(g), \\ \forall (m_J, a_J, n_J, g) \in M_J \times A_J \times N_J \times G\}, \end{aligned}$$

on which G acts by right translation. Here $\rho_J \in \mathfrak{a}_{J,\mathbb{C}}^*$ is defined by $\rho_J(H) = \mathrm{trace}\big(\mathrm{ad}(H) \mid \mathrm{Lie}(N_J)\big)/2$ for $H \in \mathfrak{a}_J$.

We describe the K-types of the generalized principal series representation $\pi(\sigma, \nu_J)$.

Proposition 2.5 *Let $\pi(\sigma,\nu_J)$ be a generalized principal series representation of G with $\sigma=\epsilon\boxtimes D_\lambda$ and $\nu_J\in\mathfrak{a}^*_{J,\mathbb{C}}$. Then for a dominant integral weight $q=(q_1,q_2)\in\mathbb{Z}^{\oplus 2}(q_1\geq q_2)$, the irreducible representation $\tau_{(q_1,q_2)}$ of K occurs in $\pi(\sigma,\nu_J)$ with multiplicity*

$$\sharp\{m\in\mathbb{Z}\mid m\equiv\lambda\pmod 2,\operatorname{sgn}(\lambda)m\geq|\lambda|,(-1)^{q_1+q_2-m}=\epsilon,q_2\leq m\leq q_1\},$$

where $\operatorname{sgn}(\lambda)$ *stands for* $\lambda/|\lambda|\in\{\pm1\}$. *In particular,*

(1) *when $\epsilon=(-1)^\lambda$ and $\lambda\geq 2$, then each of $\tau_{(q,q)}$ ($q\in\mathbb{Z}$, $q\equiv\lambda\pmod 2$, $q\geq\lambda$) or $\tau_{(\lambda,q)}$ ($q\in\mathbb{Z}$, $q\equiv\lambda\pmod 2$, $\lambda\geq q$) occurs in $\pi(\sigma,\nu_J)$ with multiplicity one;*
(2) *when $\epsilon=(-1)^{\lambda+1}$ and $\lambda\geq 2$, then each of $\tau_{(q,q-1)}$ ($q\in\mathbb{Z}$, $q\geq\lambda$) or $\tau_{(\lambda,q-1)}$ ($q\in\mathbb{Z}$, $q\equiv\lambda\pmod 2$, $\lambda\geq q-1$) occurs in $\pi(\sigma,\nu_J)$ with multiplicity one.*

Proof This follows from Frobenius reciprocity for compact groups. See [Mo1, Proposition (2.4)] for example. (In [Mo1], we mistakenly impose an unnecessary condition $q\equiv\lambda\pmod 2$ for $\tau_{(q,q-1)}$ occurring in $\pi\big((-1)^{\lambda+1}\boxtimes D_\lambda,\nu_J\big)$ with multiplicity one.) ∎

Thanks to Remark 3.1 below, we may and do assume $\lambda\geq 2$.

Definition 2.6

(i) We say that a generalized principal series representation $\pi=\pi(\epsilon\boxtimes D_\lambda,\nu_J)$ with $\lambda\geq 2$ is *even* (resp. *odd*) if $\epsilon=(-1)^\lambda$ (resp. $\epsilon=(-1)^{\lambda+1}$).
(ii) For an even (resp. odd) generalized principal series representation π, we call its K-type $\tau:=\tau_{(\lambda,\lambda)}$ (resp. $\tau_{(\lambda,\lambda-1)}$) the *corner K-type* of π.

3 Spherical Functions

In this section, we introduce the putative spherical functions and discuss their restrictions to the subgroup $A=\exp\mathfrak{a}$.

3.1 Definition of Spherical Functions

For a continuous representation (η,V_η) of R, we define a C^∞-induced module $C^\infty\text{-}\operatorname{Ind}_R^G(\eta)$ with representation space

$$C^\infty_\eta(R\setminus G):=\{F\colon G\to V_\eta\mid C^\infty\text{-class},F(rg)=\eta(r)F(g)\;\forall(r,g)\in R\times G\},$$

on which G acts by right translation. Note that for each $F\in C^\infty_\eta(R\setminus G)$ and $g\in G$, $F(g)$ is a smooth vector in V_η. For a finite dimensional K-module (τ,W_τ), we denote by $C^\infty_{\eta,\tau}(R\setminus G/K)$ the space of C^∞-functions $F\colon G\to V_\eta\otimes W^*_\tau$ with the property

$$F(rgk)=\big(\eta(r)\otimes\tau^*(k)^{-1}\big)F(g),\quad(r,g,k)\in R\times G\times K,$$

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where (τ^*, W_τ^*) stands for the contragredient representation of (τ, W_τ). For an admissible representation π of G and a K-equivariant map $i\colon \tau \longrightarrow \pi|_K$, we define a $\mathbb{C}$-linear map

$$i^*\colon \mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R\setminus G)^0\big) \longrightarrow \mathrm{Hom}_K\big(\tau, C_\eta^\infty(R\setminus G)\big) \cong C_{\eta,\tau}^\infty(R\setminus G/K)$$

by the pullback of i. Here π^0 and $C_\eta^\infty(R\setminus G)^0$ stand for the underlying $(\mathfrak{g},K)$-module of π and $C_\eta^\infty(R\setminus G)$, respectively. We call the image of i^* *a spherical functions of type* (π,η,τ). Our main objective in this paper is to give an explicit integral expression of the A-radial part of the spherical functions when π is a generalized principal series representation. We should note that if π is irreducible, then the above map i^* is injective.

Remark 3.1 Set $g_1 := \mathrm{diag}(1,1,-1,-1) \in \mathrm{SL}(4,\mathbb{R})$. The conjugation by g_1 defines an involution σ_1 of G. It is known that σ_1 generates the outer automorphism group of G. For any representation $(\pi,\mathcal{H}_\pi)$ of G, we have a twisted representation $\pi\circ\sigma_1$ of π by letting $g\in G$ act on $\mathcal{H}_\pi$ via $\pi\big(\sigma_1(g)\big)$. It is easily seen that $\pi(\epsilon\boxtimes D_\lambda,\nu_J)\circ\sigma_1$ is equivalent to $\pi(\epsilon\boxtimes D_{-\lambda},\nu_J)$. Moreover, the subgroup R is stable under σ_1. In fact, the restriction of σ_1 to R is an inner automorphism. Hence, σ_1 naturally induces an isomorphism

$$\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi(\epsilon\boxtimes D_\lambda,\nu_J)^0, C_\eta^\infty(R\setminus G)^0\big) \cong \mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi(\epsilon\boxtimes D_{-\lambda},\nu_J)^0, C_\eta^\infty(R\setminus G)^0\big).$$

3.2 Radial Parts of Spherical Functions

We recall some structure theory of semisimple symmetric spaces, by which we can regard a spherical function of type (π,η,τ) as a C^∞-functions of one real variable. Let R, A and K be as in Section 1. Set

$$N_{R\cap K}(\mathfrak{a}) := \{g\in R\cap K \mid \mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}\};$$
$$Z_{R\cap K}(\mathfrak{a}) := \{g\in R\cap K \mid \mathrm{Ad}(g)t = t, \forall t\in\mathfrak{a}\};$$
$$W_{R\cap K}(\mathfrak{a}) := N_{R\cap K}(\mathfrak{a})/Z_{R\cap K}(\mathfrak{a}).$$

Proposition 3.2

(i) *The multiplication map* $\Phi\colon R\times A\times K \ni (r,a,k)\mapsto rak\in G$ *is a* C^∞*-surjection and regular at* (r,a,k) *if and only if* $a\neq 1$.
(ii) *The fiber of* Φ *above* $g = rak$ *is given by as follows:*

$$\Phi^{-1}(g) = \begin{cases} \{(rx^{-1},1,xk) \mid x\in R\cap K\}, & \text{if } a = 1;\\ \{(rx^{-1},xax^{-1},xk) \mid x\in N_{R\cap K}(\mathfrak{a})\}, & \text{if } a\neq 1.\end{cases}$$

Proof See [R, Theorems 9 and 10]. ∎

Let $C_{\eta,\tau}^\infty(A)$ be the space of $V_\eta\otimes W_\tau^*$-valued C^∞-functions on A satisfying the following conditions (a), (b) and (c):

(a) $\eta(m)\otimes\tau^*(m)\phi(a_t)=\phi(a_t)$ for any $m\in Z_{R\cap K}(\mathfrak{a})$;
(b) $\eta(n_0)\otimes\tau^*(n_0)\phi(a_t)=\phi(a_{-t})$ for a representative n_0 of the non-trivial element in $W_{R\cap K}(\mathfrak{a})\cong\{\pm1\}$;
(c) $\eta(r)\otimes\tau^*(r)\phi(e)=\phi(e)$ for any $r\in R\cap K$.

Proposition 3.3

(1) *The restriction map*

$$\mathrm{res}\,|_A\colon C^\infty_{\eta,\tau}(R\setminus G/K)\to C^\infty_{\eta,\tau}(A)$$

is a linear injection.

(2) *Suppose that τ is an irreducible $(d+1)$-dimensional representation of K and that (η,V_η) is a non-unitary principal series representation $\big(\eta(l,\nu),V_{l,\nu}\big)$ of R. Then any element $\phi\in C^\infty_{\eta,\tau}(A)$ can be written as*

$$\phi(t)=\sum_{k=0}^{d}\sum_{m\in M(l)}\phi_k^{(m)}(t)g^{(m)}_{p(m,k)}\otimes w_k,\qquad p(m,k):=\frac{m-d}{2}+k.$$

Here $\{w_k\}$ or $\{g_p^{(m)}\}$ is the standard basis of (τ^*,W^*_τ) or (η,V_η), respectively and $\{\phi_k^{(m)}\mid m\in M(l),0\le k\le d\}$ is a family of $\mathbb{C}$-valued C^∞-functions satisfying

$$\phi_k^{(m)}(-t)=(-1)^{(m-d)/2}\phi^{(m)}_{d-k}(t)\quad(t\in\mathbb{R}).$$

Proof The assertion (1) follows from Proposition 3.2. In order to prove (2), we write $\phi(a_t)\in C^\infty_{\eta,\tau}(A)$ in the form

$$\phi(a_t)=\sum_{k=0}^{d}\sum_{m\in M(l)}\sum_{p=0}^{m}\phi^{(m)}_{p,k}(t)g_p^{(m)}\otimes w_k$$

with a family $\{\phi^{(m)}_{p,k}(t)\}$ of C^∞-functions on $\mathbb{R}$. It is easily checked that

$$Z_{R\cap K}(\mathfrak{a})=\left\{m_\theta:=\varphi\left(\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\right)=u\left(\begin{pmatrix}e^{\sqrt{-1}\theta}&0\\0&e^{-\sqrt{-1}\theta}\end{pmatrix}\right)\;\middle|\;\theta\in\mathbb{R}\right\}.$$

The condition (a) in the definition of $C^\infty_{\eta,\tau}(A)$ implies that

$$\exp\big(\sqrt{-1}(m-2p+2k-d)\theta\big)\,\phi^{(m)}_{p,k}(t)=\phi^{(m)}_{p,k}(t)$$

for all $\theta\in\mathbb{R}$, m, p and k. Thus $\phi^{(m)}_{p,k}(t)$ is identically zero unless $p=p(m,k)$. Similarly the condition (b) in the definition of $C^\infty_{\eta,\tau}(A)$ implies

$$\phi(a_{-t})=\eta(n_0)\otimes\tau^*(n_0)\phi(a_t)=\sum_{k=0}^{d}\sum_{m\in M(l)}(-1)^{(m-d)/2}\phi^{(m)}_{p(m,k),d-k}(t)g^{(m)}_{p(m,k)}\otimes w_k.$$

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This proves our assertion. ■

Owing to Proposition 3.3, a spherical function $F(g)$ of type (π, η, τ) is uniquely determined by its restriction $\phi = \operatorname{res}|_A(F)$ to A. From now on, we allow ourselves to use the term *a spherical function of type* (π, η, τ) for $\phi = \operatorname{res}|_A(F)$, too. We frequently write $\phi(t)$ instead of $\phi(a_t)$. For any $\mathbb{C}$-linear map

$$\mathcal{A}\colon C^\infty_{\eta,\tau}(R \setminus G/K) \to C^\infty_{\eta,\tau'}(R \setminus G/K),$$

there exists a $\mathbb{C}$-linear map $\rho(\mathcal{A})\colon C^\infty_{\eta,\tau}(A) \to C^\infty_{\eta,\tau'}(A)$ such that $\operatorname{res}|_A \circ \mathcal{A} = \rho(\mathcal{A}) \circ \operatorname{res}|_A$, and call $\rho(\mathcal{A})$ the A-radial part of $\mathcal{A}$.

If $C^\infty_{\eta,\tau}(A) = 0$, there are no non-zero spherical functions.

Assumption 3.4 From now on, we (tacitly) assume that $C^\infty_{\eta,\tau}(A) \neq \{0\}$.

4 Differential Operators and Differential Equations

Throughout this paper, we assume that $\pi = \pi(\epsilon \boxtimes D_\lambda, \nu_J)$ is a generalized principal series representation of G with $\lambda \geq 2$. We denote the corner K-type $\tau_{(\lambda,\lambda)}$ or $\tau_{(\lambda,\lambda-1)}$ of $\pi = \pi(\epsilon \boxtimes D_\lambda, \nu_J)$ by τ (see Definition 2.6). In this section, we construct systems of differential equations for the spherical functions and calculate their A-radial parts.

4.1 Differential Operators

Here we introduce two kinds of differential operators, that is, shift operators and the Casimir operator.

4.1.1 Shift Operators

Before introducing shift operators, we recall the definition of the Schmid operator. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ in Section 1 and $\operatorname{Ad} = \operatorname{Ad}|_{\mathfrak{p}_\mathbb{C}}$ the adjoint representation of K on $\mathfrak{p}_\mathbb{C}$. Then, for each continuous representation (η, V_η) of R, we have a differential operator ∇_τ from $C^\infty_{\eta,\tau}(R \setminus G/K)$ to $C^\infty_{\eta,\tau\otimes\operatorname{Ad}^*}(R \setminus G/K)$:

$$\nabla_\tau F = \sum_i R_{X_i} F \otimes X_i, \quad F \in C^\infty_{\eta,\tau}(R \setminus G/K).$$

Here $(X_i)_i$ is any fixed orthonormal basis of $\mathfrak{p}$ with respect to the Killing form B of $\mathfrak{g}$ and we set $R_X F(g) := \frac{d}{dt}\Big(F\big(g\exp(tX)\big)\Big)|_{t=0}$ $(g \in G, X \in \mathfrak{g})$. We call this differential operator ∇_τ the Schmid operator.

Let $P_{\tau'}\colon W^*_\tau \otimes \mathfrak{p}_\mathbb{C} \to W^*_{\tau'}$ be the projection to an irreducible component $W^*_{\tau'}$ of the K-module $W^*_\tau \otimes \mathfrak{p}_\mathbb{C}$. We define a $V_\eta \otimes W^*_{\tau'}$-valued function $F' \in C^\infty_{\eta,\tau'}(R \setminus G/K)$ by $F' := P_{\tau'}(\nabla_\tau F)$. We state the following key proposition, a proof of which we refer [Mo1, Proposition (4.1)], for example.

Proposition 4.1 *For a spherical function F of type (π, η, τ), the $V_\eta \otimes W^*_{\tau'}$-valued function F' is also a spherical function of type (π, η, τ').*

If we suitably normalize the Killing form of $\mathfrak{g}$, then we have $\nabla_\tau = \nabla_\tau^+ + \nabla_\tau^-$ with

$$\nabla_\tau^+ F = \sum_{\beta\in\Delta_n^+} \|\beta\|^2 R_{X_\beta} F \otimes X_{-\beta}, \quad \nabla_\tau^- F = \sum_{\beta\in\Delta_n^+} \|\beta\|^2 R_{X_{-\beta}} F \otimes X_\beta.$$

Define *shift operators* D_λ^- and E_λ^- by

$$D_\lambda^- := P^{\mathrm{down}} \circ \nabla^-_{\tau_{(\lambda,\lambda-2)}} \circ \nabla^-_{\tau_{(\lambda,\lambda)}} : C^\infty_{\eta,\tau_{(\lambda,\lambda)}}(R\setminus G/K) \to C^\infty_{\eta,\tau_{(\lambda-2,\lambda-2)}}(R\setminus G/K);$$
$$E_\lambda^- := P^{\mathrm{even}} \circ \nabla^-_{(\lambda,\lambda-1)} : C^\infty_{\eta,\tau_{(\lambda,\lambda-1)}}(R\setminus G/K) \to C^\infty_{\eta,\tau_{(\lambda-1,\lambda-2)}}(R\setminus G/K).$$

4.1.2 The Casimir Operator

The Casimir element L of $\mathfrak{g}_\mathbb{C}$ is up to constant given by

$$L = Z^2 + {H'}^2 - 6Z - 2H' + 2X_{(2,0)}\cdot X_{(-2,0)} + 2X_{(0,2)}\cdot X_{(0,-2)} + X_{(1,1)}\cdot X_{(-1,-1)} - X_{(1,-1)}\cdot X_{(-1,1)}.$$

We extend the action R_Y ($Y\in\mathfrak{g}_\mathbb{C}$) of $\mathfrak{g}_\mathbb{C}$ on $C^\infty_{\eta,\tau}(R\setminus G/K)$ to the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}$. In particular, the Casimir operator is defined to be the action R_L of the Casimir element L.

4.2 Differential Equations

We use the differential operators introduced above to construct systems of differential equations satisfied by the spherical functions:

Proposition 4.2 *Let $\pi = \pi(\epsilon\boxtimes D_\lambda, \nu_J)$ be a generalized principal series representation of G with $\lambda \geq 2$.*

(i) *If π is even (see Definition 2.6), then a spherical function F of type $(\pi,\eta,\tau_{(\lambda,\lambda)})$ satisfies*

(a-1) $$D_\lambda^- F = 0;$$

(a-2) $$R_L F = \{2\nu_J^2 + 2(\lambda-1)^2 - 10\}F.$$

(ii) *If π is odd (see Definition 2.6), then a spherical function F of type $(\pi,\eta,\tau_{(\lambda,\lambda-1)})$ satisfies*

(b-1) $$E_\lambda^- F = 0;$$

(b-2) $$R_L F = \{2\nu_J^2 + 2(\lambda-1)^2 - 10\}F.$$

Proof From the irreducible decomposition of $\pi|_K$ as a K-module (Proposition 2.5) and Proposition 4.1, we have the equations (a-1) and (b-1). We shall prove (a-2) and (b-2). It is easy to see $R_L F = \chi_\pi(L)F$. Here χ_π is the infinitesimal character of π. The value $\chi_\pi(L)$ of χ_π at L is equal to $2\nu_J^2 + 2(\lambda-1)^2 - 10$ (see [M-O, Section 6]). This proves (a-2) and (b-2). ∎

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4.3 The Radial Part of the Schmid Operator

We begin with calculating the A-radial parts $\rho(R_S)$ of the actions R_S of $S \in \mathfrak{p}_{\mathbb{C}}$ on $C^\infty_{\eta,\tau}(R \setminus G/K)$.

Proposition 4.3 *For $\phi \in C^\infty_{\eta,\tau}(A)$, we have*

$$\begin{aligned}
\left(\rho(R_{X_{(2,0)}})\phi\right)(t) &= \{\eta(\mathcal{L}^+) + \coth 2t \cdot \tau^*(X)\}\phi(t);\\
\left(\rho(R_{X_{(1,1)}})\phi\right)(t) &= \left\{\frac{d}{dt} + \eta(\mathcal{N}) + \tanh 2t \cdot \tau^*(Z)\right\}\phi(t);\\
\left(\rho(R_{X_{(0,2)}})\phi\right)(t) &= \{\eta(\mathcal{M}^+) + \coth 2t \cdot \tau^*(Y)\}\phi(t);\\
\left(\rho(R_{X_{(-2,0)}})\phi\right)(t) &= \{\eta(\mathcal{L}^-) - \coth 2t \cdot \tau^*(Y)\}\phi(t);\\
\left(\rho(R_{X_{(-1,-1)}})\phi\right)(t) &= \left\{\frac{d}{dt} - \eta(\mathcal{N}) - \tanh 2t \cdot \tau^*(Z)\right\}\phi(t);\\
\left(\rho(R_{X_{(0,-2)}})\phi\right)(t) &= \{\eta(\mathcal{M}^-) - \coth 2t \cdot \tau^*(X)\}\phi(t).
\end{aligned}$$

We can deduce these formulae from the generalized Cartan decomposition (Lemma 1.1) and the following lemma.

Lemma 4.4 *Let $U = \mathrm{Ad}(a_{-t})(X_1 \cdot X_2 \cdots X_l) \cdot H_1^m \cdot Y_1 \cdot Y_2 \cdots Y_n$ be an element of $U(\mathfrak{g}_{\mathbb{C}})$, where $X_i \in \mathfrak{r}_{\mathbb{C}}$, $m \in \mathbb{Z}_{\geq 0}$ and $Y_i \in \mathfrak{k}_{\mathbb{C}}$. Then for $F \in C^\infty_{\eta,\tau}(R \setminus G/K)$ we have*

$$\begin{aligned}
[R_U F](a_t) = \eta(X_1) \circ \eta(X_2) \circ \cdots \circ \eta(X_l) \circ \left(\frac{d}{dt}\right)^m \circ \left(-\tau^*(Y_n)\right)\\
\circ \cdots \circ \left(-\tau^*(Y_2)\right) \circ \left(-\tau^*(Y_1)\right) F(a_t).
\end{aligned}$$

This lemma can be proved by direct computation.

Proposition 4.5 *The A-radial part of ∇^-_τ is given by as follows:*

$$\begin{aligned}
&\rho(\nabla^-_\tau)\phi(t)\\
&\quad = 4\{\eta(\mathcal{L}^-) - \coth 2t(\tau^* \otimes \mathrm{Ad})(Y)\}\phi(t) \otimes X_{(2,0)}\\
&\qquad + 2\left\{\frac{d}{dt} - \eta(\mathcal{N}) - \tanh 2t\left((\tau^* \otimes \mathrm{Ad})(Z) - 2\right) + 4\coth 2t\right\}\phi(t) \otimes X_{(1,1)}\\
&\qquad + 4\{\eta(\mathcal{M}^-) - \coth 2t(\tau^* \otimes \mathrm{Ad})(X)\}\phi(t) \otimes X_{(0,2)}.
\end{aligned}$$

Proof It is easy to compute $\rho(\nabla^-_\tau)$ by using Proposition 4.3. ■

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4.4 The Radial Part of the Shift Operator

Firstly we suppose that $\pi = \pi\big((-1)^{\lambda}\boxtimes D_{\lambda}, \nu_J\big)$ is an even generalized principal series representation of G with $\lambda \geq 2$. We represent a spherical function $\phi \in C^{\infty}_{\eta,\tau}(A)$ of type $(\pi, \eta, \tau_{(\lambda,\lambda)})$ as

$$\phi(a_t) = \phi_0(t) w_0$$

with a non-zero vector $w_0 \in W^*_{(\lambda,\lambda)}$ and a V_η-valued C^∞-function $\phi_0(t)$ on $\mathbb{R}$. We are going to write down explicitly the differential equation (a-1) arising from the shift operators in terms of the coefficient function $\phi_0(t)$.

Proposition 4.6 *The equation* (a-1) *is equivalent to*

$$\text{(A-1)}\quad \Big\{ \Big(\frac{d}{dt} - \eta(\mathcal{N}) + (2\lambda-2)\tanh 2t + 4\coth 2t\Big)\Big(\frac{d}{dt} - \eta(\mathcal{N}) + 2\lambda\tanh 2t\Big) - 2\eta(\mathcal{L}^- \cdot \mathcal{M}^- + \mathcal{M}^- \cdot \mathcal{L}^-)\Big\}\phi_0(t) = 0.$$

Proof We put $\phi^{\langle 1\rangle} := \rho(\nabla^-_{\tau_{(\lambda,\lambda)}})\phi$, $\phi^{\langle 2\rangle} := P^{\mathrm{down}}\big(\rho(\nabla^-_{\tau_{(\lambda,\lambda-2)}})\phi^{\langle 1\rangle}\big) = \rho(D^-_{\lambda})\phi$. We define V_η-valued C^∞-functions $\phi^{\langle 1\rangle}_k$ $(k = 0,1,2)$ and $\phi^{\langle 2\rangle}_0$ on A by

$$\phi^{\langle 1\rangle}(t) = \sum_{0\leq k\leq 2} \phi^{\langle 1\rangle}_k(t) w^{\langle 1\rangle}_k, \quad \phi^{\langle 2\rangle}(t) = \phi^{\langle 2\rangle}_0(t) w^{\langle 2\rangle}_0,$$

where $\{w^{\langle 1\rangle}_k, 0 \leq k \leq 2\}$ or $\{w^{\langle 2\rangle}_0\}$ is the standard basis of $W_{(2-\lambda,-\lambda)}$ or $W_{(2-\lambda,2-\lambda)}$. By Lemma 2.2 (iii), we have

$$\phi(t)\otimes X_{(2,0)} = \phi_0(t) w^{\langle 1\rangle}_2; \quad \phi(t)\otimes X_{(1,1)} = \phi_0(t) w^{\langle 1\rangle}_1; \quad \phi(t)\otimes X_{(0,2)} = \phi_0(t) w^{\langle 1\rangle}_0.$$

From these equations and Proposition 4.5, we obtain

$$\phi^{\langle 1\rangle}_2(t) = 4\eta(\mathcal{L}^-)\phi_0(t); \quad \phi^{\langle 1\rangle}_1(t) = 2\Big(\frac{d}{dt} - \eta(\mathcal{N}) + 2\lambda\tanh 2t\Big)\phi_0(t);$$
$$\phi^{\langle 1\rangle}_0(t) = 4\eta(\mathcal{M}^-)\phi_0(t).$$

Using these formulae and Lemma 2.2 (iii), we conclude that

$$\phi^{\langle 2\rangle}_0(t) = (-8)\Big\{\Big(\frac{d}{dt} - \eta(\mathcal{N}) + (2\lambda-2)\tanh 2t + 4\coth 2t\Big) \cdot \Big(\frac{d}{dt} - \eta(\mathcal{N}) + 2\lambda\tanh 2t\Big) - 2\eta(\mathcal{L}^- \cdot \mathcal{M}^- + \mathcal{M}^- \cdot \mathcal{L}^-)\Big\}\phi_0(t).$$

Thus the proposition follows. ■

Next we suppose that $\pi = \pi\big((-1)^{\lambda+1} \boxtimes D_\lambda, \nu_J\big)$ is an odd generalized principal series representation of G with $\lambda \geq 2$. Again, we represent a spherical function $\phi \in C^\infty_{\eta,\tau}(A)$ of type $(\pi, \eta, \tau_{(\lambda,\lambda-1)})$ as

$$\phi(a_t) = \sum_{k=0,1} \phi_k(t) w_k$$

with the standard basis $\{w_k \mid k = 0, 1\}$ of $W_{(1-\lambda,-\lambda)}$ (see Section 2.1) and V_η-valued C^∞-functions $\phi_k(t)$ on $\mathbb{R}$.

Proposition 4.7 *The equation* (b-1) *is equivalent to the system:*

$$\text{(B-1)}\qquad \left(\frac{d}{dt} - \eta(\mathcal{N}) + (2\lambda-1)\tanh 2t + 2\coth 2t\right)\phi_0(t) - 2\eta(\mathcal{M}^-)\phi_1(t) = 0;$$

$$\text{(B-2)}\qquad 2\eta(\mathcal{L}^-)\phi_0(t) - \left(\frac{d}{dt} - \eta(\mathcal{N}) + (2\lambda-1)\tanh 2t + 2\coth 2t\right)\phi_1(t) = 0.$$

Proof We can prove this in the same manner as Proposition 4.6. ■

4.5 The Radial Part of the Casimir Operator

We write down the differential equations arising from the Casimir operator in terms of the coefficient functions $\phi_k(t)$.

Proposition 4.8

(i) *The equation* (a-2) *is equivalent to* (A-2) *below:*

$$\text{(A-2)}\qquad \begin{aligned}\Big\{&\left(\frac{d}{dt} + \eta(\mathcal{N}) - (2\lambda-2)\tanh 2t + 4\coth 2t\right)\left(\frac{d}{dt} - \eta(\mathcal{N}) + 2\lambda\tanh 2t\right)\\ &+ 2\eta(\mathcal{L}^+\cdot\mathcal{L}^- + \mathcal{M}^+\cdot\mathcal{M}^-) + 4\lambda^2 - 12\lambda\Big\}\phi_0(t) = \chi_\pi(L)\phi_0(t).\end{aligned}$$

(ii) *The equation* (b-2) *is equivalent to* (B-3) *and* (B-4) *below:*

$$\text{(B-3)}\qquad \begin{aligned}\Big\{&\left(\frac{d}{dt} + \eta(\mathcal{N}) - (2\lambda-3)\tanh 2t + 2\coth 2t\right)\\ &\cdot\left(\frac{d}{dt} - \eta(\mathcal{N}) + (2\lambda-1)\tanh 2t + 2\coth 2t\right)\\ &+ 2\eta(\mathcal{L}^+\cdot\mathcal{L}^- + \mathcal{M}^-\cdot\mathcal{M}^+) + 4\lambda^2 - 8\lambda - 6\Big\}\phi_0(t)\\ &+ 4\frac{\coth 2t}{\sinh 2t}\cdot\eta(e^-)\phi_1(t) = \chi_\pi(L)\phi_0(t);\end{aligned}$$

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(B-4)
$$\begin{aligned}&\Big\{\Big(\frac{d}{dt}+\eta(\mathcal{N})-(2\lambda-3)\tanh 2t+2\coth 2t\Big)\\&\quad\Big(\frac{d}{dt}-\eta(\mathcal{N})+(2\lambda-1)\tanh 2t+2\coth 2t\Big)\\&\quad+2\eta(\mathcal{L}^-\cdot\mathcal{L}^++\mathcal{M}^+\cdot\mathcal{M}^-)+4\lambda^2-8\lambda-6\Big\}\phi_1(t)\\&\quad+4\frac{\coth 2t}{\sinh 2t}\cdot\eta(e^+)\phi_0(t)=\chi_\pi(L)\phi_1(t).\end{aligned}$$

Proof Using Lemma 1.1 (the generalized Cartan decomposition) repeatedly, we can express the Casimir element as a linear combination of such elements as $\mathrm{Ad}(a_{-t})(U_1)\cdot H_1^m\cdot U_2$ ($U_1\in U(\mathfrak{r}_\mathbb{C})$, $m\in\mathbb{Z}_{\geq 0}$, $U_2\in U(\mathfrak{k}_\mathbb{C})$):

$$\begin{aligned}L={}&Z^2+H'^2-2H'-2\coth^2 2t\cdot(X\cdot Y+Y\cdot Y)-\tanh^2 2t\cdot Z^2\\&+4X\cdot Y+\mathrm{Ad}(a_{-t})(2\mathcal{L}^+\cdot\mathcal{L}^-+2\mathcal{M}^+\cdot\mathcal{M}^--\mathcal{N}^2-4\coth 2t\cdot\mathcal{N})\\&-4\frac{\coth 2t}{\sinh 2t}\Big\{\big(\mathrm{Ad}(a_{-t})e^-\big)\cdot Y+\big(\mathrm{Ad}(a_{-t})e^+\big)\cdot X\Big\}\\&+2\tanh 2t\big(\mathrm{Ad}(a_{-t})\mathcal{N}\big)\cdot Z+H_1^2+(2\tanh 2t+4\coth 2t)H_1.\end{aligned}$$

Now Lemma 4.4 allows us to compute the A-radial parts of the Casimir operator. ∎

5 Reduction of Differential Equations (The Even Case)

Throughout this section, we assume that $\pi=\pi\big((-1)^\lambda\boxtimes D_\lambda,\nu_J\big)$ $(\lambda\geq 2)$ is an even generalized principal series representation of G and that (η,V_η) is a non-unitary principal series representation $\big(\eta(l,\nu),V_{l,\nu}\big)$ of R. Then $\tau=\tau_{(\lambda,\lambda)}$ is the corner K-type of π (see Definition 2.6). Recall that any spherical function $\phi\in C^\infty_{\eta,\tau}(A)$ of type (π,η,τ) can be written as

$$\phi(t)=\phi_0(t)w_0=\sum_{m\in M(l)}\phi^{(m)}(t)g^{(m)}_{m/2}\otimes w_0,$$

where each $\phi^{(m)}(t)$ is a C^∞-function on $\mathbb{R}$ and w_0 is a non-zero vector in $W_{(-\lambda,-\lambda)}$ (Proposition 3.3). Our main objective in this section is to derive a single differential equation satisfied by $\phi^{(|l|)}(t)$ from the differential equations (A-1) and (A-2). Further, we show that this differential equation is essentially a generalized hypergeometric differential equation (Theorem 5.6).

5.1 Difference-Differential Equations

Proposition 5.1 *The system of differential equations* (A-1), (A-2) *in the previous sec-*

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tion is equivalent to the system:

$$\text{(A-3)}\quad \Big[\Big(\frac{d}{dt}+4\coth 2t\Big)\Big(\frac{d}{dt}+2\lambda\tanh 2t\Big)+\frac{-2}{\sinh^2 2t}\eta(e^+\cdot e^-+e^-\cdot e^+)+\frac{\chi_0}{2} - \Big(\frac{d}{dt}\cdot\eta(\mathcal{N})+\frac{2}{\sinh 2t}\eta(Je^-\cdot e^+-Je^+\cdot e^-)\Big)\Big]\phi_0(t)=0;$$

$$\text{(A-4)}\quad \Big[(2\lambda-2)\tanh 2t\Big(\frac{d}{dt}+2\lambda\tanh 2t\Big)-\frac{\chi_0}{2} - \Big\{\Big(\frac{d}{dt}+4\lambda\tanh 2t\Big)\eta(\mathcal{N})+\frac{2}{\sinh 2t}\eta(Je^-\cdot e^+-Je^+\cdot e^-)\Big\} + \eta(\mathcal{N}^2+2Je^+\cdot Je^-+2Je^-\cdot Je^+)\Big]\phi_0(t)=0$$

with

$$\chi_0:=4\lambda^2-12\lambda-\chi_\pi(L)=2(\lambda-2)^2-2\nu_J^2.$$

Proof To get (A-3) or (A-4) from (A-1) and (A-2), it is enough to compute $\big((\text{A-1})+(\text{A-2})\big)/2$ or $\big((\text{A-1})-(\text{A-2})\big)/2$, respectively. ■

We derive difference-differential equations for the coefficient functions $\phi^{(m)}(t)$ ($m\in M(l)$) from (A-3) and (A-4):

Proposition 5.2 *Suppose that $\phi\in C^\infty_{\eta,\tau}(A)$ satisfies the differential equations* (A-3) *and* (A-4). *Then the family $\{\phi^{(m)}(t)\mid m\in M(l)\}$ of C^∞-functions on $\mathbb{R}$ satisfies the following system of difference-differential equations* $(\text{A-3})_{m\in M(l)}$ *and* $(\text{A-4})_{m\in M(l)}$*:*

$(\text{A-3})_m$

$$\frac{f(m-2,\nu)}{\cosh 2t}\Big(\frac{d}{dt}-2\tanh 2t-(m-2)\coth 2t\Big)\phi^{(m-2)}(t) + \Big\{\Big(\frac{d}{dt}+4\coth 2t\Big)\Big(\frac{d}{dt}+2\lambda\tanh 2t\Big)-\frac{m(m+2)}{\sinh^2 2t}+\frac{\chi_0}{2}\Big\}\phi^{(m)}(t) + \frac{e(m+2,\nu)}{\cosh 2t}\Big(\frac{d}{dt}-2\tanh 2t+(m+4)\coth 2t\Big)\phi^{(m+2)}(t)=0;$$

$(\text{A-4})_m$

$$f(m-2,\nu)f(m-4,\nu)(-\tanh^2 2t)\phi^{(m-4)}(t) + \frac{f(m-2,\nu)}{\cosh 2t}\Big(\frac{d}{dt}+(4\lambda-2)\tanh 2t-(m-2)\coth 2t\Big)\phi^{(m-2)}(t) + \Big\{(2\lambda-2)\tanh 2t\frac{d}{dt}+2\lambda(2\lambda-2)\tanh^2 2t+\chi_1(m,\nu)+\frac{\chi_2(m,\nu)}{\cosh^2 2t}\Big\}\cdot\phi^{(m)}(t) + \frac{e(m+2,\nu)}{\cosh 2t}\Big(\frac{d}{dt}+(4\lambda-2)\tanh 2t+(m+4)\coth 2t\Big)\phi^{(m+2)}(t) + e(m+2,\nu)e(m+4,\nu)(-\tanh^2 2t)\phi^{(m+4)}(t)=0.$$

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Here the constant χ_0 is as in Proposition 5.1 and

$$f(m,\nu) := -\sqrt{-1}(\nu+m+2)(m/2+1)^2 A^{(m)};$$
$$e(m,\nu) := 2\sqrt{-1}(\nu-m)C^{(m)};$$
$$\begin{aligned}\chi_1(m,\nu) := {} & 2\big(\nu^2-(m+2)^2\big)(m/2+1)(m/2+2)A^{(m)}C^{(m+2)}\\ & -4\nu^2(m/2+1)(m/2){B^{(m)}}^2\\ & +2(\nu^2-m^2)(m/2)(m/2-1)A^{(m-2)}C^{(m)}-\chi_0/2;\end{aligned}$$
$$\chi_2(m,\nu) := f(m,\nu)e(m+2,\nu)+f(m-2,\nu)e(m,\nu).$$

In these formula we understand that $\phi^{(m)}(t)=0$ and $f(m,\nu)=0$ unless $m \in M(l)$.

Proof These are easily established by using Proposition 2.3. ■

Up to this step, we know that $\phi^{(|l|)}(t)$, $\phi^{(|l|+2)}(t)$ and $\phi^{(|l|+4)}(t)$ satisfy $(\text{A-3})_{|l|}$, $(\text{A-3})_{|l|+2}$ and $(\text{A-4})_{|l|}$ and that the other coefficient functions $\phi^{(m)}(t)$ are determined recursively by these three functions as long as $e(m,\nu)\neq 0$. We make a change of variable from t to $y=y(t)=(\cosh 2t)^{-2}$. Until the end of this section, we concentrate our attention on the solutions of the system for $t>0$. A C^∞-function $f(t)$ on $\mathbb{R}_{>0}$ can be considered as a C^∞-function in y on the interval $(0,1)$, which we denote by $f(y)$ by a slight abuse of notation.

Lemma 5.3 *For any C^∞-function $f(t)$ on $\mathbb{R}_{>0}$, we have*

(i) $\frac{df}{dt}(t) = -4y(t)\sqrt{1-y(t)}\frac{df}{dy}\big(y(t)\big)$;
(ii) $\frac{d^2f}{dt^2}(t) = -16y(t)^2\big(y(t)-1\big)\frac{d^2f}{dy^2}\big(y(t)\big) + \big(-24y(t)^2+16y(t)\big)\frac{df}{dy}\big(y(t)\big)$.

Using this lemma, we have:

Proposition 5.4 *Suppose that $\phi^{(|l|)}(t)$, $\phi^{(|l|+2)}(t)$ and $\phi^{(|l|+4)}(t)$ are C^∞-functions on the half line $\mathbb{R}_{>0}$. Put*

$$\psi^{(|l|+2)}(t) := (\sinh 2t)^{-1}\phi^{(|l|+2)}(t).$$

Then the differential equation $(\text{A-3})_{|l|}$ (resp. $(\text{A-3})_{|l|+2}$, $(\text{A-4})_{|l|}$) for $\phi^{(|l|)}(t)$, $\phi^{(|l|+2)}(t)$ and $\phi^{(|l|+4)}(t)$ is equivalent to the following differential equation (A-5) (resp. (A-6), (A-7)) for $\phi^{(|l|)}(y)$, $\psi^{(|l|+2)}(y)$ and $\phi^{(|l|+4)}(y)$:

$$\text{(A-5)}\qquad \begin{aligned} & 4y\left(\frac{d^2}{dy^2}+p_1(y)\frac{d}{dy}+p_2(y)\right)\phi^{(|l|)}(y)\\ & \quad - e(|l|+2,\nu)\left(\frac{d}{dy}+p_3(y)\right)\psi^{(|l|+2)}(y)=0;\end{aligned}$$

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$$\text{(A-6)}\quad \begin{aligned} &f(|l|,\nu)\left(\frac{d}{dy}+q_1(y)\right)\phi^{(|l|)}(y)\\ &\qquad +4(y-1)\left(\frac{d^2}{dy^2}+q_2(y)\frac{d}{dy}+q_3(y)\right)\psi^{(|l|+2)}(y)\\ &\qquad +e(|l|+4,\nu)\left(\frac{d}{dy}+q_4(y)\right)\phi^{(|l|+4)}(y)=0;\end{aligned}$$

$$\text{(A-7)}\quad \begin{aligned} &8y\left((\lambda-1)\frac{d}{dy}+r_1(y)\right)\phi^{(|l|)}(y)\\ &\qquad +e(|l|+2,\nu)4y\left(\frac{d}{dy}+r_2(y)\right)\psi^{(|l|+2)}(y)\\ &\qquad +e(|l|+2,\nu)e(|l|+4,\nu)\phi^{(|l|+4)}(y)=0,\end{aligned}$$

where we set

$$p_1(y):=\frac{(3-\lambda)y+\lambda}{2y(y-1)};$$

$$p_2(y):=\frac{-8\lambda y^2+(-8\lambda-\chi_0-2|l|^2-4|l|)y+16\lambda+\chi_0}{32y^2(y-1)^2};$$

$$p_3(y):=\frac{2y+|l|+4}{4y(y-1)};\quad q_1(y):=\frac{2y-|l|-2}{4y(y-1)};\quad q_2(y):=\frac{(3-\lambda)y+\lambda+2}{2y(y-1)};$$

$$q_3(y):=\frac{-8\lambda y^2-\big(8+16\lambda+\chi_0+2(|l|+2)(|l|+4)\big)y+24+24\lambda+\chi_0}{32y^2(y-1)^2};$$

$$q_4(y):=\frac{2y+|l|+4}{4y(y-1)};\quad r_1(y):=\frac{(-4\lambda^2+4\lambda+\chi_2)y+4\lambda^2-4\lambda+\chi_1}{8y(y-1)};$$

$$r_2(y):=\frac{(-4\lambda+2)y+4\lambda+|l|+4}{4y(y-1)}.$$

Here the constant χ_0 is as in Proposition 5.1; the constants χ_1 and χ_2 are given by

$$\chi_1:=\chi_1(|l|,\nu)=\frac{|l|+2}{|l|+3}(-\nu^2+|l|+4)-(\lambda-2)^2+\nu_J^2;$$

$$\chi_2:=\chi_2(|l|,\nu)=e(|l|+2,\nu)f(|l|,\nu)=\frac{-\nu^2+(|l|+2)^2}{|l|+3}.$$

5.2 Elimination of $\phi^{(|l|+4)}(y)$

In order to eliminate the terms involving $\phi^{(|l|+4)}(y)$ from (A-6) and (A-7), we compute

$$e(|l|+2,\nu)\cdot\text{(A-6)}-\left(\frac{d}{dy}+q_4(y)\right)\cdot\text{(A-7)}.$$

Proposition 5.5 *Let* $\{\phi^{(|l|)}(y), \psi^{(|l|+2)}(y), \phi^{(|l|+4)}(y)\}$ *be a set of* C^∞*-functions on the interval* $(0,1)$ *satisfying the differential equations* (A-6) *and* (A-7)*. Then we have*

$$\begin{aligned}(\text{A-8})\qquad & 2y\left((\lambda-1)\frac{d^2}{dy^2}+s_1(y)\frac{d}{dy}+s_2(y)\right)\phi^{(|l|)}(y)\\ &\quad + e(|l|+2,\nu)\left(\frac{d^2}{dy^2}+s_3(y)\frac{d}{dy}+s_4(y)\right)\psi^{(|l|+2)}(y)=0\end{aligned}$$

with

$$s_1(y)=\{\chi_1+\chi_2+2(|l|+2\lambda)(\lambda-1)+(-12+16\lambda-4\lambda^2)y\}/\big(8y(y-1)\big);$$

$$\begin{aligned}s_2(y)=\{&4\chi_1-2\chi_2+\chi_1|l|-\chi_2|l|-16\lambda-4|l|\lambda+16\lambda^2+4|l|\lambda^2\\ &+(-2\chi_1+4\chi_2+2\chi_2|l|+8\lambda+4|l|\lambda-8\lambda^2-4|l|\lambda^2)y\\ &+8(\lambda-\lambda^2)y^2\}/\big(32y^2(y-1)^2\big);\end{aligned}$$

$$s_3(y)=\{2+\lambda+(3+|l|)y+(1-\lambda)y^2\}/\big(2y(y-1)\big);$$

$$\begin{aligned}s_4(y)=\{&24+\chi_0+24\lambda+(-16-2\chi_0+4|l|-8\lambda+8|l|\lambda)y\\ &+(8+\chi_0+12|l|+2|l|^2-8\lambda-8|l|\lambda)y^2+(8-8\lambda)y^3\}/\big(32y^2(y-1)^2\big).\end{aligned}$$

Here the constants χ_0 *(resp.* χ_1 *and* χ_2*) are as in Proposition 5.1 (resp. 5.4).*

5.3 A Single Differential Equation for $\phi^{(|l|)}(y)$

Finally we eliminate the terms involving $\phi^{(|l|+2)}(y)$ from (A-5) and (A-8) and derive a generalized hypergeometric differential equation.

Theorem 5.6 *Suppose that a set* $\{\phi^{(|l|)}(y), \psi^{(|l|+2)}(y)\}$ *of* C^∞*-functions on the interval* $(0,1)$ *satisfies the differential equations* (A-5) *and* (A-8)*. We set*

$$\psi^{(|l|)}(y):=(1-y)^{-\rho_0}\phi^{(|l|)}(y),\qquad \rho_0:=\frac{-|l|-2}{4}.$$

Then we have

$$(\text{A-9})\qquad \left\{y\prod_{k=1}^{4}(\delta_y+\alpha_k)-\prod_{k=1}^{4}(\delta_y-\gamma_k)\right\}\psi^{(|l|)}(y)=0,$$

with the constants

$$\begin{aligned}&\alpha_1:=(-|l|)/4; && \alpha_2:=(-|l|-2)/4;\\ &\alpha_3:=(-2\lambda+\nu)/4; && \alpha_4:=(-2\lambda-\nu)/4;\\ &\gamma_1:=(\nu_J+\lambda+4)/4; && \gamma_2:=(\nu_J+\lambda+2)/4;\\ &\gamma_3:=(-\nu_J+\lambda+4)/4; && \gamma_4:=(-\nu_J+\lambda+2)/4.\end{aligned}$$

Here δ_y *stands for the Euler operator* $y\frac{d}{dy}$.

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Proof Set

$$\tilde{p}_3(y) := p_3(y) + \frac{2}{y}; \quad \tilde{s}_3(y) := s_3(y) + \frac{2}{y};$$
$$\tilde{s}_4(y) := s_4(y) + \big(p_3(y) - s_3(y)\big)\left(\frac{-2}{y}\right) + \frac{ds_3}{dy}(y) - 2\frac{dp_3}{dy}(y).$$

Then we can obtain a fourth-order differential equations for $\phi^{(|l|)}(y)$ by computing

$$\left(\frac{d^2}{dy^2} + \tilde{s}_3(y)\frac{d}{dy} + \tilde{s}_4(y)\right) \cdot \text{(A-5)} + \left(\frac{d}{dy} + \tilde{p}_3(y)\right) \cdot \text{(A-8)}.$$

By using a symbolic computation system, we can confirm that $\psi^{(|l|)}(y)$ satisfies the differential equation (A-9). ■

6 Reduction of Differential Equations (The Odd Case)

Throughout this section, we assume that $\pi = \pi\big((-1)^{\lambda+1} \boxtimes D_\lambda, \nu_J\big)$ $(\lambda \geq 2)$ is an odd generalized principal series representation of G and that (η, V_η) is a non-unitary principal series representation $\big(\eta(l,\nu), V_{l,\nu}\big)$ of R. Then the corner K-type of π is given by $\tau = \tau_{(\lambda,\lambda-1)}$ (see Definition 2.6). From Proposition 3.3, we know that any spherical function $\phi \in C^\infty_{\eta,\tau}(A)$ of type (π, η, τ) can be written as

$$\phi(t) = \sum_{m \in M(l)} \left(\phi_0^{(m)}(t) g^{(m)}_{(m-1)/2} \otimes w_0 + \phi_1^{(m)}(t) g^{(m)}_{(m+1)/2} \otimes w_1\right),$$

where each $\phi_k^{(m)}(t)$ $(m \in M(l),\ k = 0, 1)$ is a C^∞-function on $\mathbb{R}$ and $\{w_k\}$ is the standard basis of $W_{(-\lambda+1,-\lambda)}$. We also set

$$\phi_+^{(m)}(t) = \frac{1}{2}\big(\phi_0^{(m)}(t) + \phi_1^{(m)}(t)\big); \quad \phi_-^{(m)}(t) = \frac{1}{2}\big(\phi_0^{(m)}(t) - \phi_1^{(m)}(t)\big).$$

Then, by Proposition 3.3, $\phi_+^{(m)}(t)$ (resp. $\phi_-^{(m)}(t)$) is an even or odd function according as $(m-1)/2$ (resp. $(m+1)/2$) $\in \mathbb{Z}$ is even or odd. As we do for even generalized principal series representations in the previous section, we derive single differential equations satisfied by $\phi_\pm^{(|l|)}(t)$ from the differential equations from (B-1) to (B-4) (Theorem 6.8).

6.1 Difference-Differential Equations

Proposition 6.1 *Suppose that $\phi \in C^\infty_{\eta,\tau}(A)$ satisfies the differential equation* (B-1) *and* (B-3) *in Section 4. Then the family $\{\phi_+^{(m)}(t), \phi_-^{(m)}(t) \mid m \in M(l)\}$ of C^∞-functions on $\mathbb{R}$ satisfies the following system of difference-differential equations* $\text{(B-5)}^{\pm}_{m \in M(l)}$ *and*

$(\text{B-6})^{\pm}_{m\in M(l)}$:

$$(\text{B-5})^{\pm}_{m}\quad \begin{aligned} & f_1(m-2)\left(\frac{1}{\cosh 2t}\mp 1\right)\phi^{(m-2)}_{\pm}(t)\\ &+\left(\frac{d}{dt}+(2\lambda-1)\tanh 2t+2\coth 2t\mp\frac{m+1}{\sinh 2t}\right)\phi^{(m)}_{\pm}(t)\\ &\pm\sqrt{-1}\nu B^{(m)}\left(\frac{\mp 1}{\cosh 2t}+m+1\right)\phi^{(m)}_{\mp}(t)\\ &+e_1(m+2)\left(\frac{1}{\cosh 2t}\pm 1\right)\phi^{(m+2)}_{\pm}(t)=0; \end{aligned}$$

$(\text{B-6})^{\pm}_{m}$

$$\begin{aligned} & f_1(m-4)f_1(m-2)\tanh^2 2t\phi^{(m-4)}_{\pm}(t)\\ &\quad-(4\lambda-2)f_1(m-2)\frac{\tanh 2t}{\cosh 2t}\phi^{(m-2)}_{\pm}(t)\\ &\quad-\frac{2\sqrt{-1}\nu l f_1(m-2)}{m^2-4}\tanh^2 2t\phi^{(m-2)}_{\mp}(t)\\ &\quad+\left\{\left(\frac{d}{dt}-(2\lambda-3)\tanh 2t+2\coth 2t\right)\left(\frac{d}{dt}+(2\lambda-1)\tanh 2t+2\coth 2t\right)\right.\\ &\qquad-\frac{(m+1)^2}{\sinh^2 2t}+\chi_0'-\left(1+\frac{1}{\cosh^2 2t}\right)\left(f_1(m)e_1(m+2)+f_1(m-2)e_1(m)\right)\\ &\qquad\left.+\left(\frac{1}{\cosh^2 2t}+(m+1)^2\right)\nu^2 {B^{(m)}}^2\pm 2(m+1)\frac{\coth 2t}{\sinh 2t}\right\}\phi^{(m)}_{\pm}(t)\\ &\quad+(4\lambda-2)\sqrt{-1}\nu B^{(m)}\frac{\tanh 2t}{\cosh 2t}\phi^{(m)}_{\mp}(t)-(4\lambda-2)e_1(m+2)\frac{\tanh 2t}{\cosh 2t}\phi^{(m+2)}_{\pm}(t)\\ &\quad-\frac{2\sqrt{-1}\nu l e_1(m+2)}{m(m+4)}\tanh^2 2t\phi^{(m+2)}_{\mp}(t)\\ &\quad+e_1(m+2)e_1(m+4)\tanh^2 2t\phi^{(m+4)}_{\pm}(t)=0. \end{aligned}$$

Here we set

$$\chi_0':=4\lambda^2-8\lambda-6-\chi_\pi(L)=2(\lambda-1)^2-2\nu_J^2;$$
$$f_1(m)=f_1(m,\nu):=\frac{-\sqrt{-1}}{4}(\nu+m+2)(m+1)(m+3)A^{(m)};$$
$$e_1(m)=e_1(m,\nu):=2\sqrt{-1}(\nu-m)C^{(m)}.$$

In these formulae, we understand that $\phi^{(m)}_{\pm}(t)=0$ *and* $f_1(m)=0$ *unless* $m\in M(l)$.

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Proof By Proposition 2.3, we have from (B-1)

$$\sum_{m\in M(l)}\Big[\Big(\frac{d}{dt}+(2\lambda-1)\tanh 2t+2\coth 2t\Big)\phi_0^{(m)}(t)$$
$$+\frac{1}{\cosh 2t}\big(f_1(m-2)\phi_0^{(m-2)}(t)-\sqrt{-1}\nu B^{(m)}\phi_0^{(m)}(t)+e_1(m+2)\phi_0^{(m+2)}(t)\big)$$
$$-f_1(m-2)\phi_1^{(m-2)}(t)-(m+1)\sqrt{-1}\nu B^{(m)}\phi_1^{(m)}(t)+e_1(m+2)\phi_1^{(m+2)}(t)$$
$$-\frac{m+1}{\sinh 2t}\phi_1^{(m)}(t)\Big]g^{(m)}_{(m-1)/2}=0.$$

Substitute $\phi_0^{(m)}(t) = \phi_+^{(m)}(t)+\phi_-^{(m)}(t)$ and $\phi_1^{(m)}(t) = \phi_+^{(m)}(t)-\phi_-^{(m)}(t)$ into this formula. Decomposing the left hand side into a sum of an even function and an odd function, we obtain $(\text{B-5})_m^+$ and $(\text{B-5})_m^-$. Similarly we can derive $(\text{B-6})_m^\pm$ from (B-3) by using Proposition 2.3 and

$$\mathcal{L}^\pm\cdot\mathcal{L}^\mp+\mathcal{M}^\mp\cdot\mathcal{M}^\pm=-2Je^\mp\cdot Je^\pm-\frac{2e^\mp\cdot e^\pm}{\sinh^2 2t}. \qquad\blacksquare$$

Remark 6.2 For $\phi\in C^\infty_{\eta,\tau}(A)$, the equation (B-2) (resp. (B-4)) is equivalent to (B-1) (resp. (B-3)).

6.2 Elimination of $\phi_\pm^{(|l|+4)}(t)$ and $\phi_\pm^{(|l|+2)}(t)$

Firstly we eliminate the term involving $\phi_\pm^{(|l|+4)}(t)$.

Proposition 6.3 *Suppose that a set $\{\phi_+^{(k)}(t),\phi_-^{(k)}(t)\mid k=|l|,|l|+2,|l|+4\}$ of C^∞-functions on $\mathbb{R}$ satisfies the differential equation $(\text{B-5})^\pm_{|l|+2}$ and $(\text{B-6})^\pm_{|l|}$ in Proposition 6.1. Then we have the following differential equations $(\text{B-7})^\pm$ for $\{\phi_+^{(k)}(t),\phi_-^{(k)}(t)\mid k=|l|,|l|+2\}$:*

$$(\text{B-7})^\pm\qquad Q_1^\pm\Big(t,\frac{d}{dt}\Big)\phi_\pm^{(|l|)}(t)+Q_2^\pm\Big(t,\frac{d}{dt}\Big)\phi_\mp^{(|l|)}(t)+Q_3^\pm\Big(t,\frac{d}{dt}\Big)\phi_\pm^{(|l|+2)}(t)$$
$$+Q_4^\pm\Big(t,\frac{d}{dt}\Big)\phi_\mp^{(|l|+2)}(t)=0,$$

where we set

$$Q_1^\pm\Big(t,\frac{d}{dt}\Big)$$
$$:=\Big(\frac{d}{dt}-(2\lambda-3)\tanh 2t+2\coth 2t\Big)\Big(\frac{d}{dt}+(2\lambda-1)\tanh 2t+2\coth 2t\Big)$$
$$-\frac{(|l|+1)^2}{\sinh^2 2t}+\chi_0'+\Big(\frac{1}{\cosh^2 2t}+(|l|+1)^2\Big)\frac{\nu^2}{(|l|+2)^2}$$
$$\pm 2(|l|+1)\frac{\coth 2t}{\sinh 2t}\mp\frac{2e_1(|l|+2)f_1(|l|)}{\cosh 2t};$$

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$$Q_2^+\left(t,\frac{d}{dt}\right)=Q_2^-\left(t,\frac{d}{dt}\right):=(4\lambda-2)\sqrt{-1}\nu B^{(|l|)}\frac{\tanh 2t}{\cosh 2t};$$

$$Q_3^\pm\left(t,\frac{d}{dt}\right):=e_1(|l|+2)\left(\frac{1}{\cosh 2t}\mp 1\right)\\ \times\left(\frac{d}{dt}+(2\lambda-1)\frac{(\cosh 2t\pm 1)^2}{\cosh 2t\sinh 2t}+2\coth 2t\mp\frac{|l|+3}{\sinh 2t}\right);$$

$$Q_4^\pm\left(t,\frac{d}{dt}\right):=-\sqrt{-1}\nu B^{(|l|)}e_1(|l|+2)\left(1\mp\frac{1}{\cosh 2t}\right)\left(\frac{\pm 1}{\cosh 2t}+|l|+1\right).$$

Here the constant χ_0' is as in Proposition 6.1.

Proof We eliminate $\phi_\pm^{(|l|+4)}(t)$ from $(\text{B-5})^\pm_{|l|+2}$ by computing

$$-e_1(|l|+2)\left(\frac{1}{\cosh 2t}\pm 1\right)^{-1}\tanh^2 2t(\text{B-5})^\pm_{|l|+2}+(\text{B-6})^\pm_{|l|}. \qquad \blacksquare$$

Our next task is to eliminate the terms involving $\phi_\pm^{(|l|+2)}(t)$ from $(\text{B-7})^+$ and $(\text{B-7})^-$ by using $(\text{B-5})^+_{|l|}$ and $(\text{B-5})^-_{|l|}$.

Proposition 6.4 *Suppose that a set $\{\phi_+^{(k)}(t),\phi_-^{(k)}(t)\mid k=|l|,|l|+2\}$ of C^∞-functions on $\mathbb{R}$ satisfies the differential equation $(\text{B-5})^\pm_{|l|}$ and $(\text{B-7})^\pm$. Then we have*

$$(\text{B-8})^\pm \qquad V_1^\pm\left(t,\frac{d}{dt}\right)\phi_\pm^{(|l|)}(t)+V_2^\pm\left(t,\frac{d}{dt}\right)\phi_\mp^{(|l|)}(t)=0$$

with

$$V_1^\pm\left(t,\frac{d}{dt}\right):=\left(\frac{1}{\cosh 2t}\pm 1\right)^2 Q_1^\pm\left(t,\frac{d}{dt}\right)\\ +\tanh^2 2t\left\{\frac{d}{dt}+(2\lambda-1)\frac{(1\pm\cosh 2t)^2}{\cosh 2t\sinh 2t}\right.\\ \left.+2(\tanh 2t+\coth 2t)\mp\frac{|l|+3}{\sinh 2t}-\frac{2\sinh 2t}{\cosh 2t\pm 1}\right\}\\ \cdot\left(\frac{d}{dt}+(2\lambda-1)\tanh 2t+2\coth 2t\mp\frac{|l|+1}{\sinh 2t}\right)\\ -\frac{\nu^2}{(|l|+2)^2}\left(\frac{1}{\cosh 2t}\pm 1\right)^2\left(\frac{\pm 1}{\cosh 2t}+|l|+1\right)^2;$$

$$
\begin{aligned}
V_2^{\pm}\left(t, \frac{d}{dt}\right)
&:= \left(\frac{1}{\cosh 2t} \pm 1\right)^2 Q_2^{\pm}\left(t, \frac{d}{dt}\right) \\
&\quad + \tanh^2 2t \left\{ \frac{d}{dt} + (2\lambda - 1)\frac{(1 \pm \cosh 2t)^2}{\cosh 2t \sinh 2t} + 2(\tanh 2t + \coth 2t) \right. \\
&\qquad\qquad \left. \mp \frac{|l| + 3}{\sinh 2t} - \frac{2 \sinh 2t}{\cosh 2t \pm 1} \right\} (\pm\sqrt{-1}\nu) B^{(|l|)} \left(\frac{\mp 1}{\cosh 2t} + |l| + 1 \right) \\
&\quad \mp \sqrt{-1}\nu B^{(|l|)} \left(\frac{1}{\cosh 2t} \pm 1 \right)^2 \left(\frac{\pm 1}{\cosh 2t} + |l| + 1 \right) \\
&\quad \times \left(\frac{d}{dt} + (2\lambda - 1) \tanh 2t + 2 \coth 2t \pm \frac{|l| + 1}{\sinh 2t} \right).
\end{aligned}
$$

Here the differential operators $Q_1^{\pm}(t, \frac{d}{dt})$ and $Q_2^{\pm}(t, \frac{d}{dt})$ are as in Proposition 6.3.

Proof Straightforward computation. ■

As in the previous section, we concentrate our attention on the solutions of the system for $t > 0$. We make a change of variable from t to $x = x(t) = 1/\cosh 2t$. We can regard a C^∞-function on $\mathbb{R}_{>0}$ as a C^∞-functions in x on the interval $(0, 1)$. We denote it by $f(x)$ by a slight abuse of notation.

Lemma 6.5 *For a C^∞-function $f(t)$ on $\mathbb{R}_{>0}$, we have*

(i) $\frac{df}{dt}(t) = -2x(t)\sqrt{1 - x(t)^2}\,\frac{df}{dx}\big(x(t)\big)$;

(ii) $\frac{d^2 f}{dt^2}(t) = 4x(t)^2\big(1 - x(t)^2\big)\frac{d^2 f}{dx^2}\big(x(t)\big) + \big(4x(t) - 8x(t)^3\big)\frac{df}{dx}\big(x(t)\big)$.

Using this lemma, we have

Proposition 6.6 *Suppose that $\phi_+^{(|l|)}(t)$ and $\phi_-^{(|l|+2)}(t)$ are C^∞-functions on the half line $\mathbb{R}_{>0}$. Put*

$$\psi_-^{(|l|)}(t) := (\sinh 2t)^{-1}\phi_-^{(|l|)}(t).$$

Then the differential equation (B-8)$^+$ *(resp.* (B-8)$^-$*) for $\phi_+^{(|l|)}(t)$ and $\phi_-^{(|l|)}(t)$ is equivalent to the following differential equation* (B-9) *(resp.* (B-10)*) for $\phi_+^{(|l|)}(x)$ and $\psi_-^{(|l|)}(x)$:*

(B-9)

$$2x\left(\frac{d^2}{dx^2} + v_1(x)\frac{d}{dx} + v_2(y)\right)\phi_+^{(|l|)}(x) + \sqrt{-1}\nu\,\mathrm{sgn}(l)\left(\frac{d}{dx} + v_3(x)\right)\psi_-^{(|l|)}(x) = 0;$$

(B-10)

$$
\begin{aligned}
&(-\sqrt{-1})\nu\,\mathrm{sgn}(l)\frac{x}{2(x^2 - 1)}\left(\frac{d}{dx} + v_4(x)\right)\phi_+^{(|l|)}(x) \\
&\qquad + \left(\frac{d^2}{dx^2} + v_5(x)\frac{d}{dx} + v_6(x)\right)\psi_-^{(|l|)}(x) = 0.
\end{aligned}
$$

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Here we denote the sign $l/|l| \in \{\pm 1\}$ *of* l *by* $\mathrm{sgn}(l)$ *and set*

$$\nu_1(x) := \big(2+2\lambda-|l|x+(4+|l|-2\lambda)x^2\big)\,/\big(2(x^3-x)\big);$$

$$\begin{aligned}\nu_2(x) := \{&4+4\lambda+\lambda^2-\nu_J^2+(2-2\lambda-2|l|\lambda+\lambda^2-\nu_J^2)x\\ &+(-5+|l|-4\lambda-\lambda^2+\nu_J^2)x^2+(-|l|-|l|^2+2\lambda+2|l|\lambda-\lambda^2+\nu_J^2)x^3\}\\ &\times\big(4(x^3-x)^2\big)^{-1};\end{aligned}$$

$$\nu_3(x) := (4+|l|+x)/\big(2(x^3-x)\big); \quad \nu_4(x) := (2+|l|-x)/\big(2(x^3-x)\big);$$

$$\nu_5(x) := \big(6+2\lambda+|l|x+(4+|l|-2\lambda)x^2\big)\,/\big(2(x^3-x)\big);$$

$$\begin{aligned}\nu_6(x) := \{&16+8\lambda+\lambda^2-\nu_J^2+(-2+2|l|+2\lambda+2|l|\lambda-\lambda^2+\nu_J^2)x\\ &+(-9+3|l|-8\lambda-\lambda^2+\nu_J^2)x^2+(|l|+|l|^2-2\lambda-2|l|\lambda+\lambda^2-\nu_J^2)x^3\}\\ &\times\big(4(x^3-x)^2\big)^{-1}.\end{aligned}$$

6.3 Single Differential Equations for $\phi_{\pm}^{(|l|)}(x)$

Before eliminating the terms involving $\phi_{+}^{(|l|)}(x)$ or $\psi_{-}^{(|l|)}(x)$, we change unknown functions:

Proposition 6.7 *Set*

$$\check{\phi}_{+}(x) := (1-x)^{-\rho_0}(1+x)^{-\sigma_0}\phi_{+}^{(|l|)}(x);$$
$$\check{\phi}_{-}(x) := (1-x)^{-\rho_0+1}(1+x)^{-\sigma_0}x^{-1}\psi_{-}^{(|l|)}(x),$$

with $\rho_0 := (-|l|-1)/4$ *and* $\sigma_0 := (-|l|-3)/4$. *Then the differential equation* (B-9) *(resp.* (B-10)*) is equivalent to* (B-11)$^+$ *(resp.* (B-11)$^-$*) below:*

$$(\text{B-11})^{\pm} \qquad \begin{aligned}&[x(\delta_x+\alpha_3+\alpha_4)(\delta_x+2\alpha_2)\mp(\delta_x-2\gamma_2)(\delta_x-2\gamma_4)]\check{\phi}_{\pm}(x)\\ &\pm\frac{\sqrt{-1}}{2}\nu\,\mathrm{sgn}(l)x(\delta_x+2\alpha_2)\check{\phi}_{\mp}(x)=0.\end{aligned}$$

Here δ_x *stands for the Euler operator* $x\frac{d}{dx}$ *and the constants* $\alpha_2, \alpha_3, \alpha_4, \gamma_2$ *and* γ_4 *are as in Theorem 5.6.*

Theorem 6.8 *Suppose that a set* $\{\check{\phi}_{+}(x), \check{\phi}_{-}(x)\}$ *of* C^∞*-functions on the interval* $(0,1)$ *satisfies the differential equations* (B-11)$^+$ *and* (B-11)$^-$. *Then we have the following differential equation* (B-12)$^+$ *(resp.* (B-12)$^-$*) for* $\check{\phi}_{+}(x)$ *(resp.* $\check{\phi}_{-}(x)$*):*
(B-12)$^{\pm}$

$$\Big[x^2\prod_{k=1}^{4}(\delta_x+2\alpha_k)\pm x(\delta_x+2\alpha_1)(\delta_x-2\gamma_2)(\delta_x-2\gamma_4)-\prod_{k=1}^{4}(\delta_x-2\gamma_k)\Big]\,\check{\phi}_{\pm}(x)=0;$$

Here the constants α_k *and* γ_k *are as in Theorem 5.6.*

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Proof We eliminate the terms involving $\check{\phi}_-(x)$ from $(\text{B-11})^\pm$ by computing

$$[x(\delta_x + \alpha_3 + \alpha_4)(\delta_x + 2\alpha_2) + (\delta_x - 2\gamma_2 - 1)(\delta_x - 2\gamma_4 - 1)] \cdot (\text{B-11})^+ \\ - \frac{\sqrt{-1}}{2}\nu \operatorname{sgn}(l) x(\delta_x + 2\alpha_2) \cdot (\text{B-11})^-.$$

Taking the relations $\alpha_1 = \alpha_2 + 1/2$, $\gamma_1 = \gamma_2 + 1/2$ and $\gamma_3 = \gamma_4 + 1/2$ into account, we have $(\text{B-12})^+$. The equation $(\text{B-12})^-$ is obtained in the same manner. ∎

7 Main Results

The computation in the previous two sections leads to our main results (Theorems 7.2 and 7.5).

7.1 The Even Case

Let π be an even generalized principal series representation of G with the corner K-type $\tau = \tau_{(\lambda,\lambda)}$ (see Section 2.3) and (η, V_η) an irreducible admissible Hilbert representation of R with the minimal K'-type $F^{(m')}$. Then there exist $l \in \mathbb{Z}$ with $|l| = m'$ and $\nu \in \mathbb{C}$ such that (η, V_η) is infinitesimally equivalent to a subrepresentation of the non-unitary representation $\big(\eta(l,\nu), V_{l,\nu}\big)$ of R (see [Wa, 5.7.4]). Fixing such a pair (l,ν), we expand a spherical function $\phi(t)$ of type (π,η,τ) as

$$\phi(t) = \sum_{m \in M(l)} \phi^{(m)}(t) g^{(m)}_{m/2} \otimes w_0,$$

where each $\phi^{(m)}(t)$ is a C^∞-function on $\mathbb{R}$ (Proposition 3.2) and w_0 is a non-zero vector in $W_{(-\lambda,-\lambda)}$. Note that if (η, V_η) is finite dimensional, then $\phi^{(m)}(t) = 0$ for sufficiently large m. By Proposition 3.3, $\phi^{(m)}(t)$ is an even (resp. odd) function according as $m/2$ is an even (resp. odd) integer.

Definition 7.1 We call $\phi^{(|l|)}(t)$ *the lowest coefficient function* of a spherical function $\phi(t)$ of type (π,η,τ).

It is easily seen that this definition is, up to constant multiple, independent of the choice of (l,ν).

Theorem 7.2 *Let $\pi = \pi\big((-1)^\lambda \boxtimes D_\lambda, \nu_J\big)$ be an even irreducible generalized principal series representation of G with the corner K-type $\tau = \tau_{(\lambda,\lambda)}$ ($\lambda \geq 2$) (see Definition 2.6).*

(1) *For an arbitrary irreducible admissible Hilbert representation (η, V_η) of R, we have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C^\infty_\eta(R\setminus G)^0\big) \leq 1.$$

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(2) *For (η, V_η) in (1), let $\phi^{(|l|)}(t)$ be the lowest coefficient function of a spherical function $\phi(t)$ of type (π,η,τ) (see Definition 7.1). Then there exists a constant $C \in \mathbb{C}$ such that*

$$\phi^{(|l|)}(t) = C \times |\coth 2t| (\coth 2t)^{|l|/2} \times \int_{L(\sigma)} \prod_{k=1}^{4} \frac{\Gamma(\gamma_k - s)}{\Gamma(1-\alpha_k - s)} \left(\frac{1}{\cosh^2 2t}\right)^s ds,$$

where the path $L(\sigma)$ ($\sigma \in \mathbb{R}$) of integration is the vertical line from $\sigma - \sqrt{-1}\infty$ to $\sigma + \sqrt{-1}\infty$ with $\sigma < \mathrm{Re}(\gamma_k)$ ($1 \le k \le 4$). The parameters α_k and γ_k are as in Theorem 5.6.

Proof (1) Suppose that (η, V_η) is infinite dimensional. Then (η, V_η) is infinitesimally equivalent to $\big(\eta(l,\nu), V_{l,\nu}\big)$ with some $l \in \mathbb{Z}$ and $\nu \in \mathbb{C}$. Our discussion in Section 5 depends only on the $(\mathfrak{r}, K')$-module $V_{l,\nu}{}^{\circ}$ and is independent of the ambient Hilbert space $V_{l,\nu}$. Thus, we may suppose $V_\eta = V_{l,\nu}$. By Assumption 3.4, l is assumed to be an even integer. Since we assume that π is irreducible, we have a sequence of natural inclusions

$$\begin{aligned}
&\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R\setminus G)^0\big)\\
&\qquad\hookrightarrow \{F \in C_{\eta,\tau}^\infty(R\setminus G/K) \mid \text{(a-1)}, \text{(a-2)}\}\\
&\qquad\hookrightarrow \{\phi \in C_{\eta,\tau}^\infty(A) \mid \text{(A-3)}, \text{(A-4)}\}\\
&\qquad\hookrightarrow \Big\{ \big(\phi^{(m)}(t)\big)_{m\in M(l)} \in \bigoplus_{m\in M(l)} C^\infty(\mathbb{R}) \,\Big|\, \text{(A-3)}_m, \text{(A-4)}_m,\\
&\qquad\qquad\qquad\qquad \phi^{(m)}(-t) = (-1)^{m/2}\phi^{(m)}(t)\ (m \in M(l))\Big\}
\end{aligned}$$

by Propositions 4.2, 3.3, 5.1 and 5.2. Since $e(m,\nu) \neq 0$ for m ($> |l|$), the functions $\phi^{(m)}(t)$ ($m > |l|+4$) are determined recursively by $\phi^{(|l|)}(t)$, $\phi^{(|l|+2)}(t)$ and $\phi^{(|l|+4)}$ by virtue of $\text{(A-4)}_{m\in M(l)}$. As a result, we can regard the space $\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R\setminus G)^0\big)$ as a subspace of

$$\begin{aligned}
\Big\{ \big(\phi^{(|l|)}(t), \phi^{(|l|+2)}(t), \phi^{(|l|+4)}(t)\big) \in C^\infty(\mathbb{R})^{\oplus 3} \,\Big|\, \text{(A-3)}_{|l|}, \text{(A-3)}_{|l|+2}, \text{(A-4)}_{|l|},\\
\phi^{(m)}(-t) = (-1)^{m/2}\phi^{(m)}(t)\ (m = |l|, |l|+2, |l|+4)\Big\}.
\end{aligned}$$

Therefore it suffices to show that the dimension of the last space does not exceed one. Firstly we remark that $\phi^{(|l|+2)}(t)$ and $\phi^{(|l|)}(t)$ determine $\phi^{(|l|+4)}(t)$ by means of $\text{(A-4)}_{|l|}$. Further, since the kernel of differential operator $\big(\frac{d}{dt} - 2\tanh 2t + (|l|+4)\coth 2t\big)$ in the equation $\text{(A-3)}_{|l|}$ is spanned by $(\cosh 2t)(\sinh 2t)^{(-|l|-4)/2}$, the function $\phi^{(|l|+2)}(t)$ is uniquely determined by $\phi^{(|l|)}(t)$ under the condition of being a C^∞-function on $\mathbb{R}$. Thus, we may concentrate our attention on $\phi^{(|l|)}(t)$ (or on $\psi^{(|l|)}(y)$). We enumerate the characteristic indices at $y = y(0) = 1$ of the differential equation (A-9) in Theorem 5.6 as $(\rho_1,\rho_2,\rho_3,\rho_4) = (\frac{|l|+1}{2}, 0, 1, 2)$. By a well-known fact on generalized hypergeometric differential equations, there exists a set $\{\Phi_i(y) \mid 1 \le i \le$

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$4\}$ of linearly independent solutions of (A-9) around $y = 1$ such that $\Phi_i(y) = (1-y)^{\rho_i}\bigl(1+O(y-1)\bigr)$ as $y \in \mathbb{R}$ approaches 1 from the left. If $\phi^{(|l|)}(t)$ is the first component of an element of the space in question, then we have for $t > 0$

$$\phi^{(|l|)}(t) = \phi^{(|l|)}\bigl(y(t)\bigr) = (\tanh^2 2t)^{-(|l|+2)/4} \sum_{i=1}^{4} c_i \Phi_i\bigl(y(t)\bigr) \quad \text{with some } c_i \in \mathbb{C}.$$

Since

$$(\tanh^2 2t)^{-(|l|+2)/4}\Phi_i\bigl(y(t)\bigr) = t^{2\rho_i-(|l|+2)/2}\bigl(1+O(t^2)\bigr)$$

as t approaches zero from the right, we cannot extend $\phi^{(|l|)}(t)$ $(t \in \mathbb{R}_{>0})$ to a C^∞-function on the whole line $\mathbb{R}$ so as to satisfy the parity condition $\phi^{(|l|)}(-t) = (-1)^{|l|/2}\phi^{(|l|)}(t)$ unless $c_2 = c_3 = c_4 = 0$. Therefore, our assertion follows when (η, V_η) is infinite dimensional. Next we suppose that (η, V_η) is an irreducible finite dimensional representation of R with the minimal K'-type $F^{(m')}$. Then there exist $l \in \mathbb{Z}$ and $\nu \in \mathbb{C}$ such that V_η is a submodule of $V_{l,\nu}$ and that $|l| = m'$. Thus we may proceed in the same manner as in the case where η is infinite dimensional.

(2) By a general result on generalized hypergeometric differential equations (see [N, p. 310, (2.44)]), we know that an integral expression of $\Phi_1(y)$, $y \in (0,1)$ is given by

$$\Phi_1(y) = \frac{\Gamma(\frac{|l|+3}{2})}{2\pi\sqrt{-1}} \int_{L(\sigma)} \prod_{k=1}^{4} \frac{\Gamma(\gamma_k - s)}{\Gamma(1-\alpha_k - s)} y^s \, ds.$$

Here the path $L(\sigma)$ is taken as in the theorem. Extending the function $(\tanh^2 2t)^{-(|l|+2)/4}\Phi_1\bigl(y(t)\bigr)$ on $\mathbb{R}_{>0}$ to the whole line $\mathbb{R}$ so as to satisfy the parity condition, we have the formula in the theorem. ∎

Remark 7.3 The function $\Phi_1(y)$ in the above proof is known as a Meijer's G-function (see [Er, Ch. IV]).

7.2 The Odd Case

Next we suppose that π is an odd generalized principal series representation with the corner K-type $\tau = \tau_{(\lambda,\lambda-1)}$. For an irreducible admissible Hilbert representation (η, V_η) of R, we take a pair $(l,\nu) \in \mathbb{Z} \times \mathbb{C}$ as in the even case. Then a spherical function $\phi(t)$ of type (π, η, τ) can be written as

$$\begin{aligned}\phi(t) = \sum_{m \in M(l)} [\phi_+^{(m)}(t)(g^{(m)}_{(m-1)/2} \otimes w_0 + g^{(m)}_{(m+1)/2} \otimes w_1) \\ + \phi_-^{(m)}(t)(g^{(m)}_{(m-1)/2} \otimes w_0 - g^{(m)}_{(m+1)/2} \otimes w_1)],\end{aligned}$$

where each $\phi_\pm^{(m)}(t)$ is a C^∞-function on $\mathbb{R}$ and $\{w_k \mid k = 0,1\}$ is the standard basis of $W_{(1-\lambda,-\lambda)}$. Recall that $\phi_+^{(m)}(t)$ (resp. $\phi_-^{(m)}(t)$) is an even or odd function in t according as $(m-1)/2$ (resp. $(m+1)/2) \in \mathbb{Z}$ is an even or odd integer.

Definition 7.4 We call $\big(\phi_+^{(|l|)}(t), \phi_-^{(|l|)}(t)\big)$ *the pair of lowest coefficient functions* of a spherical function $\phi(t)$ of type (π, η, τ).

Again, this definition is, up to constant multiple, independent of the choice of (l, ν). In order to state our main results for the odd case, we introduce the symbol

$$\Gamma\begin{bmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{bmatrix} = \prod_{i=1}^{r} \Gamma(a_i) \Big/ \prod_{j=1}^{s} \Gamma(b_j).$$

Theorem 7.5 *Let* $\pi = \pi\big((-1)^{\lambda+1} \boxtimes D_\lambda, \nu_J\big)$ *be an odd irreducible generalized principal series representation of* G *with the corner* K*-type* $\tau = \tau_{(\lambda,\lambda-1)}$ $(\lambda \geq 2)$ *(see Definition 2.6).*

(1) *For an arbitrary irreducible admissible Hilbert representation* (η, V_η) *of* R*, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R \backslash G)^0\big) \leq 1.$$

(2) *For* (η, V_η) *in* (1)*, let* $\big(\phi_+^{(|l|)}(t), \phi_-^{(|l|)}(t)\big)$ *be the pair of lowest coefficient functions of a spherical function* $\phi(t)$ *of type* (π, η, τ) *(see Definition 7.4). Then there exists a constant* $C \in \mathbb{C}$ *satisfying*

$$\begin{aligned}\phi_+^{(|l|)}(t) = C \times |\coth 2t| (\coth 2t)^{(|l|-1)/2} \left(1 + \frac{1}{\cosh 2t}\right)^{-1/2} \\ \times \int_{L(\sigma)} \Bigg\{ \Gamma\begin{bmatrix} \gamma_1-s, & \gamma_2-s, & \gamma_3-s, & \gamma_4-s \\ 1-\alpha_1-s, & 1-\alpha_2-s, & 1-\alpha_3-s, & 1/2-\alpha_4-s \end{bmatrix} \\ + \Gamma\begin{bmatrix} \gamma_1-s, & \gamma_2-s, & \gamma_3-s, & \gamma_4-s \\ 1-\alpha_1-s, & 1-\alpha_2-s, & 1/2-\alpha_3-s, & 1-\alpha_4-s \end{bmatrix} \Bigg\} \left(\frac{1}{\cosh^2 2t}\right)^s ds\end{aligned}$$

and

$$\begin{aligned}\phi_-^{(|l|)}(t) = C \times (-\sqrt{-1})\,\mathrm{sgn}(l) |\coth 2t| (\coth 2t)^{(|l|+1)/2} \left(1 + \frac{1}{\cosh 2t}\right)^{1/2} \\ \times \int_{L(\sigma)} \Bigg\{ \Gamma\begin{bmatrix} \gamma_1-s, & \gamma_2-s, & \gamma_3-s, & \gamma_4-s \\ 1-\alpha_1-s, & 1-\alpha_2-s, & 1-\alpha_3-s, & 1/2-\alpha_4-s \end{bmatrix} \\ - \Gamma\begin{bmatrix} \gamma_1-s, & \gamma_2-s, & \gamma_3-s, & \gamma_4-s \\ 1-\alpha_1-s, & 1-\alpha_2-s, & 1/2-\alpha_3-s, & 1-\alpha_4-s \end{bmatrix} \Bigg\} \left(\frac{1}{\cosh^2 2t}\right)^s ds.\end{aligned}$$

Here the path $L(\sigma)$ *of integration and the parameters* α_k *and* γ_k *are as in Theorem 5.6.*

Proof As in the proof of Theorem 7.2, we have a natural inclusion

$$\begin{aligned}&\mathrm{Hom}_{(\mathfrak{g},K)}\big(\pi^0, C_\eta^\infty(R \backslash G)^0\big) \\ &\quad \hookrightarrow \{(\phi_+^{(|l|)}, \phi_-^{(|l|)}) \in C^\infty(\mathbb{R})^{\oplus 2} \mid \phi_\pm^{(|l|)}(-t) = (-1)^{(-|l|\pm 1)/2} \phi_\pm^{(|l|)}(t), \\ &\qquad (\text{B-8})^+, (\text{B-8})^-\}\end{aligned}$$

by Propositions 6.3 and 6.4. Hence, it suffices to show (2), which implies (1). We separate the argument according as

(i) $\nu \neq 0$ or
(ii) $\nu = 0$.

(i) The case of $\nu \neq 0$. Define two functions $\Phi_+(x)$ and $\Phi_-(x)$ on the interval $(0,1)$ by the following integrals

$$\Phi_\pm(x) := \int_{L(\sigma)} \left\{ \Gamma \left[\begin{smallmatrix} \gamma_1 - s, & \gamma_2 - s, & \gamma_3 - s, & \gamma_4 - s \\ 1-\alpha_1 - s, & 1-\alpha_2 - s, & 1-\alpha_3 - s, & 1/2-\alpha_4 - s \end{smallmatrix} \right] \right.$$
$$\left. \pm \Gamma \left[\begin{smallmatrix} \gamma_1 - s, & \gamma_2 - s, & \gamma_3 - s, & \gamma_4 - s \\ 1-\alpha_1 - s, & 1-\alpha_2 - s, & 1/2-\alpha_3 - s, & 1-\alpha_4 - s \end{smallmatrix} \right] \right\} x^{2s}\, ds.$$

Here the path $L(\sigma)$ of integration is taken as in the theorem. Then it is easy to see that $\big(\check{\phi}_+(x), \check{\phi}_-(x)\big) = \big(\Phi_+(x), (-\sqrt{-1})\operatorname{sgn}(l)\Phi_-(x)\big)$ satisfies the differential equations (B-11)$^+$ and (B-11)$^-$ in Proposition 6.7. Hence, $\Phi_+(x)$ (resp. $\Phi_-(x)$) satisfies (B-12)$^+$ (resp. (B-12)$^-$) in Theorem 6.8. On the other hand, the characteristic indices at $x = 1$ of the differential equation (B-12)$^+$ (resp. (B-12)$^-$) are $0, 1, 2$ and $|l|/2$ (resp. $0, 1, 2$ and $(|l| + 2)/2$). We claim that $\Phi_+(x)$ (resp. $\Phi_-(x)$) is the solution of (B-12)$^+$ (resp. (B-12)$^-$) corresponding to the characteristic index $|l|/2$ (resp. $(|l| + 2)/2$). In fact, the result of Nörlund quoted in the proof of Theorem 7.2 tells us that, for $(i, j) = (3, 4)$ and $(4, 3)$,

$$\frac{\Gamma(\frac{|l|+2}{2})}{2\pi\sqrt{-1}} \int_{L(\sigma)} \Gamma \left[\begin{smallmatrix} \gamma_1 - s, & \gamma_2 - s, & \gamma_3 - s, & \gamma_4 - s \\ 1-\alpha_1 - s, & 1-\alpha_2 - s, & 1-\alpha_i - s, & 1/2-\alpha_j - s \end{smallmatrix} \right] x^{2s}\, ds$$
$$= (1 - x^2)^{|l|/2}\big(1 + O(x - 1)\big)$$

as $x \in \mathbb{R}$ approaches 1 from the left. From these formulae and $|l|/2 \notin \mathbb{Z}$, our claim follows. By the parity condition $\phi_\pm^{(|l|)}(-t) = (-1)^{(-|l|\pm 1)/2}\phi_\pm^{(|l|)}(t)$, we conclude that $\big(\check{\phi}_+(x), \check{\phi}_-(x)\big) = \big(c_+\Phi_+(x), c_-\Phi_-(x)\big)$ with some $c_\pm \in \mathbb{C}$. Substituting this into (B-11)$^+$, we have $c_- = (-\sqrt{-1})\operatorname{sgn}(l)c_+$. This shows the theorem for the case of $\nu \neq 0$.

(ii) Next we suppose that $\nu = 0$. Then the system of differential equations (B-11)$^\pm$ in Proposition 6.7 becomes

$$(\sharp)_\pm \qquad [x(\delta_x + \alpha_3 + \alpha_4)(\delta_x + 2\alpha_2) \mp (\delta_x - 2\gamma_2)(\delta_x - 2\gamma_4)]\check{\phi}_\pm(x) = 0.$$

Since $x = 1$ is not a singularity of differential equation $(\sharp)_-$, any non-zero solution of $(\sharp)_-$ around $x = 1$ is of the form $\sum_{n\geq 0} c_n(1-x)^n$ with $(c_0, c_1) \neq (0, 0)$. The parity condition for $\phi_-^{(|l|)}(t)$ prevents us from extending

$$\big(1 - x(t)\big)^{(-|l|-3)/4}\big(1 + x(t)\big)^{(-|l|-1)/4}\sum_{n\geq 0} c_n\big(1 - x(t)\big)^n, \quad (t > 0)$$

to a C^∞-function on $\mathbb{R}$. Hence $\phi_-^{(|l|)}(t)$ must be identically zero. On the other hand, the solution of $(\sharp)_+$ compatible with the parity condition for $\phi_+^{(|l|)}(t)$ is given by

$$\check{\phi}_+(x) = \int_{L(\sigma)} \Gamma\begin{bmatrix} 2\gamma_2 - 2s, & 2\gamma_4 - 2s \\ 1 - 2\alpha_2 - 2s, & 1 - \alpha_3 - \alpha_4 - 2s \end{bmatrix} x^{2s}\, ds,$$

up to constant multiple. Now we note that

$$\alpha_1 = \alpha_2 + 1/2, \quad \alpha_3 = \alpha_4 = -\lambda/2, \quad \gamma_1 = \gamma_2 + 1/2 \quad \text{and} \quad \gamma_3 = \gamma_4 + 1/2.$$

From these and the duplication formula $\Gamma(s)\Gamma(s+1/2) = \pi^{1/2}2^{1-2s}\Gamma(2s)$ of Gamma function, the last integral equals to $2^{-(|l|+4)/2}\Phi_+(x)$. Therefore, the formulae in the statement (2) are valid for the case of $\nu = 0$, too. ∎

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