UNBOUNDED VECTOR MEASURES

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Introduction. The aim of this paper is to extend the idea of a measure which takes on values in Euclidean n-space so as to allow it to assume infinite values while preserving its countable additivity over a given σ-ring. It is shown that in order to do this it is necessary to restrict the range of the measure to one infinite value.

Liapounoff [1] and Halmos [2] have shown that the range of a non-atomic bounded vector measure is convex and that the range of any bounded vector measure is closed. It is shown here that while the former result remains true for unbounded vector measures, the latter does not.

Discussion. Let R be the space of real numbers and let E^n be Euclidean n-space regarded as a normed n-dimensional vector space. Since all norms on E^n are equivalent, we shall employ the Euclidean norm in all of the following without any loss of generality.

We construct a completion of E^n by adjoining to it infinite points $\alpha\infty$ corresponding to each of the points $\alpha \in E^n$ with $||\alpha|| = 1$ so that $\alpha\infty = \lim_{k\to\infty} \beta_k$ in a suitable sense, where $\beta_k \in E^n$.

The set $D^n = \{x \in E^n : ||x|| \leq 1\}$ is a compactification of E^n under the map

$$f(x) = x / (||x|| + 1)$$

of E^n into D^n. Let T be the completion of E^n given by

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$$T = E^n \cup \{\alpha\infty : \alpha \in E^n, \ ||\alpha|| = 1\}.$$

We shall describe the topology on T by giving a neighbourhood base for each of its points. The base for a point $x \in E^n$ is that of the usual topology on E^n, while the neighbourhood base for a point $\alpha\infty$ consists of the sets of the form

$$V(k, \epsilon) = \{x \in E^n : ||x|| > k, \ ||x/||x|| - \alpha|| < \epsilon\} \cup \{\beta\infty : ||\beta - \alpha|| < \epsilon\}$$

where k, ϵ are arbitrary positive numbers. This enables us to extend the notion of convergence in E^n to T. Furthermore if the function g is defined by

$$g(x) = f(x), \ x \in E^n \quad \text{and} \quad g(\alpha\infty) = \alpha,$$

then g is a homeomorphism of T onto D^n where D^n is provided with the usual topology.

Let X be a space and let S be a σ-ring of subsets of X. Let μ be a function defined on the sets of S which takes on values in T. We shall always assume that μ is countably additive on sets of finite measure; i.e., if $E \in S$, $||\mu(E)|| < \infty$ and $E = \bigcup_{n=1}^{\infty} E_n$ is a decomposition of E into disjoint measureable sets of finite measure, then

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

We shall also assume that $\mu(\phi) = 0$ where ϕ denotes the empty set and 0 is the null vector, and that the function μ is <u>strongly σ-finite</u>, i.e., if $E \in S$ then there exists a sequence $\{E_n\}$ of disjoint measurable sets of finite measure such that $E = \bigcup_{n=1}^{\infty} E_n$.

We shall make the following operational definitions for the use of the symbol $\alpha\infty$:

(1a) $\alpha\infty + \alpha\infty = \alpha\infty$; (1b) $\sum_{n=1}^{\infty} (\alpha\infty) = \alpha\infty$;

(2) $\lambda(\alpha\infty) = \alpha\infty$, where $\lambda \in R$, $\lambda > 0$;

(3) $\alpha\infty + x = \alpha\infty$, where $x \in E^n$.

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Definition 1. Let $E \in S$. We say that $\mu(E) = \alpha\infty$ if, for every countable disjoint decomposition of E into measurable sets, $E = \bigcup_{n=1}^{\infty} E_n$, satisfying $||\mu(E_n)|| < \infty$, $n = 1, 2, \ldots$ we have:

$$(1) \qquad \left|\left| \sum_{n=1}^{N} \mu(E_n) \right|\right| \to \infty \quad \text{as} \quad N \to \infty$$

and

$$(2) \qquad \sum_{n=1}^{N} \mu(E_n) \Big/ \left|\left| \sum_{n=1}^{N} \mu(E_n) \right|\right| \to \alpha \quad \text{as} \quad N \to \infty .$$

Since μ is strongly σ-finite over S, each set with measure $\alpha\infty$ has at least one such decomposition. If the limit in (2) does not exist for some decomposition, the function μ will not be a measure on S.

LEMMA 2. Consider a sequence $\{x_n\}$ of vectors in E^n such that:

$$(3) \qquad x_n / ||x_n|| \to \alpha \quad \text{as} \quad n \to \infty$$

and

$$(4) \qquad ||x_n|| \to \infty \quad \text{as} \quad n \to \infty .$$

Let z be a fixed vector. Then

$$(5) \qquad \text{(a)} \quad ||x_n + z|| \to \infty,$$

$$(6) \qquad \text{(b)} \quad ||x_n|| / ||x_n + z|| \to 1,$$

$$(7) \qquad \text{(c)} \quad (x_n + z) / ||x_n + z|| \to \alpha .$$

THEOREM 3. Suppose E and F are two disjoint measurable sets such that $\mu(E) = \alpha\infty$, $\mu(F) = \beta\infty$ $(\alpha \neq \beta)$. Then μ is not a measure on S.

Proof. Since $\mu(E) = \alpha\infty$ and $\mu(F) = \beta\infty$, there exist decompositions of E and F into disjoint measurable sets of finite measure, $E = \bigcup_{n=1}^{\infty} E_n$ and $F = \bigcup_{n=1}^{\infty} F_n$, satisfying conditions (1) and (2).

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(a) We first show that $||\mu(E \cup F)||$ cannot be finite. This follows from (5) of Lemma 2, since there exists a strictly increasing sequence $\{t_k\}$ of positive integers with

$$|| \sum_{i=1}^{k} \mu(F_i) + \sum_{i=1}^{t_k} \mu(E_i) || > k \qquad k = 1, 2, \ldots$$

where the sequence of sets

$$F_1, E_1, \ldots, E_{t_1}, F_2, E_{t_1+1}, \ldots, E_{t_2}, \ldots ,$$

is a disjoint measurable decomposition of $(E \cup F)$ into sets of finite measure and the measures of the sequence of partial sums are unbounded.

(b) Suppose $\mu(E \cup F) = \gamma\infty$ for some γ.

Let N be any positive integer and let $z \in E^n$. From Lemma 2 it follows that

$$(8) \quad [z + \sum_{i=N}^{k} \mu(E_i)] / ||z + \sum_{i=N}^{k} \mu(E_i)|| \to \alpha \text{ as } k \to \infty$$

and

$$(9) \quad || z + \sum_{i=N}^{k} \mu(E_i) || \to \infty \text{ as } k \to \infty .$$

The same result holds for $F = \bigcup_n F_n$.

Suppose $||\alpha - \beta|| = \delta > 0$. Select ϵ such that $0 < \epsilon < \delta/2$. From condition (2) for the $\{E_n\}$, there exists N such that

$$|| \alpha - \sum_{i=1}^{N} \mu(E_i) / || \sum_{i=1}^{N} \mu(E_i) || \; || < \epsilon .$$

Let n_1 be the first such N. From the analogue of (8) for $F = \bigcup_n F_n$ there exists an integer M such that

$$||\beta - [\sum_{i=1}^{n_1} \mu(E_i) + \sum_{i=1}^{M} \mu(F_i)] / || \sum_{i=1}^{n_1} \mu(E_i) + \sum_{i=1}^{M} \mu(F_i) || \; || < \epsilon .$$

Let m_1 be the first such M. Having chosen $n_1, \ldots, n_{k-1}$;

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$m_1, \ldots, m_{k-1}$, (8) implies that there exists an $N > n_{k-1}$ such that

$$\left|\left|\alpha - \left[\sum_{i=1}^{m_{k-1}} \mu(F_i) + \sum_{i=1}^{N} \mu(E_i)\right] / \left|\left|\sum_{i=1}^{m_{k-1}} \mu(F_i) + \sum_{i=1}^{N} \mu(E_i)\right|\right|\,\right|\right| < \epsilon .$$

Let n_k be the first such N. Similarly we chose m_k.

Thus we have arranged the two sequences $\{E_n\}$ and $\{F_n\}$ into a single sequence:

$$E_1, \ldots, E_{n_1}, F_1, \ldots, F_{m_1}, E_{n_1+1}, \ldots, E_{n_2}, F_{m_1+1}, \ldots .$$

Rename these sets $H_1, H_2, H_3, \ldots$ while preserving the above order. Then $\{H_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets of finite measure whose union is $(E \cup F)$.

Now it is obvious that the sequence $\left\{\sum_{i=1}^{n} \mu(H_i) / \left|\left|\sum_{i=1}^{n} \mu(H_i)\right|\right|\right\}_{n=1}^{\infty}$ takes values in ϵ-neighbourhoods of α and β infinitely often, and since $||\alpha - \beta|| = \delta > 2\epsilon$ the sequence cannot converge. This contradicts the assumption that $\mu(E \cup F) = \gamma\infty$.

Thus we cannot define the measure of $(E \cup F)$ in such a way as to be consistent with our previous definitions and therefore μ is not a measure on the σ-ring S.

We have shown that in order for our set function μ to be defined on <u>all</u> sets of a σ-ring S, it is necessary that μ not take on distinct infinite values at disjoint measurable sets. Therefore we shall subsequently eliminate this situation from our considerations and assume that non-finite disjoint measurable sets have identical measures.

LEMMA 4. If E and F are measurable sets with $E \subset F$, then $||\mu(F)|| < \infty$ implies $||\mu(E)|| < \infty$.

<u>Proof.</u> The proof consists of assuming that $\mu(E) = \gamma\infty$ for some γ. Then any decomposition of E into disjoint sets of finite measure can be extended to a similar decomposition of F

such that the norms of the partial sums of the sequence of measures in unbounded. This would imply that $||\mu(F)|| = \infty$, a contradiction, and therefore $||\mu(E)|| < \infty$.

LEMMA 5. If E and F are two disjoint measurable sets with $\mu(E) = \mu(F) = \alpha\infty$, then $\mu(E \cup F) = \alpha\infty$.

Proof. By Lemma 4, $\mu(E \cup F)$ cannot be finite.

Let $E \cup F = \bigcup_{i=1}^{\infty} H_i$ be any decomposition of $(E \cup F)$ into disjoint, measurable sets of finite measure. Let $E_i = E \cap H_i$ and $F_i = F \cap H_i$, $i = 1, 2, \ldots$. Then $E = \bigcup_{i=1}^{\infty} E_i$ and $F = \bigcup_{i=1}^{\infty} F_i$ are decompositions of E and F respectively into disjoint sets of finite measure and thus satisfy conditions (1) and (2). It will suffice to prove that $E \cup F = \bigcup_{i=1}^{\infty} H_i$ satisfies conditions (1) and (2).

(a) If C is any ϵ-cone about the direction vector α (i.e., C is the cone based at the origin each of whose generators makes an angle of ϵ with the vector α) it is clear that $\mu(\bigcup_{i=1}^{n} E_i) \in C$ and $\mu(\bigcup_{i=1}^{n} F_i) \in C$ for $n \geq N_o$. Therefore $\mu(\bigcup_{i=1}^{n} E_i) + \mu(\bigcup_{i=1}^{n} F_i) = \mu(\bigcup_{i=1}^{n} H_i) \in C$ for $n \geq N_o$, that is,

$$\mu(\bigcup_{i=1}^{n} H_i) / ||\mu(\bigcup_{i=1}^{n} H_i)|| \to \alpha \quad \text{as} \quad n \to \infty .$$

(b) Also, since $\mu(\bigcup_{i=1}^{n} E_i)$, $\mu(\bigcup_{i=1}^{n} F_i) \in C$ for $n \geq N_o$, it follows that the angle between these vectors is less than 2ϵ (where $0 < \epsilon < \pi/6$) and so $||\sum_{i=1}^{n} \mu(E_i)||$, $||\sum_{i=1}^{n} \mu(F_i)|| \to \infty$ implies that

$$||\sum_{i=1}^{n} \mu(H_i)|| \to \infty .$$

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LEMMA 6. If E and F are two disjoint measurable sets with $\mu(E) = \alpha\infty$ and $||\mu(F)|| < \infty$, then $\mu(E \cup F) = \alpha\infty$.

Proof. The proof follows immediately from Definition 1.

Corollary to Theorem 3. If E and F are measurable sets with $\mu(E) = \alpha\infty$ and $\mu(F) = \beta\infty$, where $\alpha \neq \beta$, then μ is not a measure on the σ-ring S.

Proof. We shall consider all possibilities for the measures of the disjoint sets $E \cap F$, $E - F$, and $F - E$, where $\mu(E) = \alpha\infty$ and $\mu(F) = \beta\infty$, $\alpha \neq \beta$. We only consider the case where $||\mu(E \cap F)|| < \infty$, since all other cases follow in the same general manner.

If (i) $||\mu(E - F)|| < \infty$, then

$$||\mu(E)|| = ||\mu(E \cap F) + \mu(E - F)|| \leq ||\mu(E \cap F)|| + ||\mu(E - F)|| < \infty,$$

which gives a contradiction. Therefore $||\mu(E - F)|| = \infty$ and similarly $||\mu(F - E)|| = \infty$.

(ii) $\mu(E - F) = \gamma\infty$, for some γ, then Lemma 5 implies that $\mu(E) = \gamma\infty$. Therefore $\gamma = \alpha$. Similarly $\mu(F - E) = \beta\infty$. Thus $(E - F)$ and $(F - E)$ are disjoint, measurable sets with different infinite measures and Theorem 3 then implies that μ is not a measure on S.

In the above manner the existence of two measurable sets with different infinite measures always leads to the existence of two disjoint sets with different infinite measures which according to Theorem 3 implies that μ is not a measure on S. Thus the range of our measure cannot contain two different infinite values if it is to be consistently defined on all sets of the given σ-ring. This situation motivates the following definition.

Definition 7. A set function μ, defined on a σ-ring S with values in the space T, countably additive on sets of finite measure, strongly σ-finite, and assuming one and only one infinite value (in the sense of Definition 1) will be called an unbounded vector measure. We shall usually denote its unique infinite value by $\alpha\infty$.

THEOREM 8. Let μ be an unbounded vector measure on the σ-ring S. Since μ is countably additive on sets of finite measure, it is countably additive on all sets of S.

Proof. The proof follows immediately from the definition of an unbounded vector measure and from Lemma 4, noting that we must make use of the operational definitions for the use of the symbol $\alpha\infty$ which were adopted at the beginning of the paper.

We now consider the possibility of extending the results of Liapounoff [1] and Halmos [2] to the case of unbounded vector measures.

Definition 9. If ν is a signed (scalar) measure and E is a measurable set, $\nu(E) \neq 0$, then E is called an *atom* of ν if $F \subset E$, F measurable, implies that $\nu(F) = \nu(E)$ or $\nu(F) = 0$.

A bounded vector measure can be expressed in the form $\mu = (\mu_1, \ldots, \mu_n)$ where μ_i is a signed measure $i = 1, \ldots, n$. It is called *non-atomic* if none of its coordinates have any atoms.

An unbounded vector measure is said to be *non-atomic* if

(i) it is non-atomic on measurable sets of finite measure;

(ii) $\mu(E) = \alpha\infty$ implies there exists $F \subset E$, $F \in S$, such that $0 < ||\mu(F)|| < \infty$.

Definition 10. A half-cylinder on a set S contained in E^n is defined to be the set of all vectors of the form $(x + \beta t)$, where $x \in S$, $t \geq 0$, and β is a fixed direction. An open half-cylinder on a set S is the interior of the associated half-cylinder.

LEMMA 11. Let A be an unbounded convex set in E^n. Then there exists a translate of some m-dimensional vector subspace of E^n $(1 \leq m \leq n)$ containing A such that A contains a non-trivial open half-cylinder in the m-dimensional space.

Proof. Consider the set of all translates of vector subspaces of E^n which contain A. Select the one of minimal dimension, m, and take this to be our space. Since A is unbounded and the unit sphere is compact we can always find a sequence $\{x_n\}_{n=1}^{\infty}$ of points belonging to A such that:

$$(10) \qquad ||x_n|| \to \infty \quad \text{as} \quad n \to \infty$$

and

(11) $x_n / ||x_n|| \to \alpha$ as $n \to \infty$, for some unit vector α.

There certainly exist m independent points of A, $y_1, \ldots, y_m$, such that the direction α does not lie in the hyperplane spanned by these points. Let H be the open half-cylinder with base the simplex determined by the vertices $y_1, \ldots, y_m$ and extending to infinity in the direction α.

Let z be any point of H and let C_i, $i = 1, \ldots, m$, be the cone with base point y_i all of whose generators make the same angle with the vector α as does the line $y_i z$. Then (10) implies that

$$x_k \in \text{interior } C_1 \cap \cdots \cap \text{interior } C_m$$

for all but finitely many of the members of the sequence $\{x_k\}$. This fact and the convexity of A imply that z is contained in A and therefore that H is a subset of A.

Note. If A is an unbounded convex subset of E^n then the above proof can easily be altered to show that for any vector β in the interior of A or in the interior of A considered as a subset of some translate of a vector subspace of E^n, we have $(\beta + t\alpha) \in A$, for all $t \geq 0$. The counter example given below shows that this property need not extend to every point in A.

THEOREM 12. Let μ be a non-atomic, unbounded vector measure. Then

(a) if E and F are any two sets of finite measure, then for each λ, $0 \leq \lambda \leq 1$, there exists a measurable set $G(\lambda)$ such that

$$(G(\lambda)) = \lambda\mu(E) + (1 - \lambda)\mu(F);$$

(b) if $\alpha\infty$ is contained in the range of μ, then there exists $t_o > 0$ and a vector $\beta \in E^n$ such that $(\beta + t\alpha)$ is contained in the range of μ for all $t > t_o$.

(These two properties may together be regarded as an extension of the idea of convexity to the case of T, the completion of E^n defined above. Thus Theorem 12 states that the range of a non-atomic, unbounded vector measure is a convex

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subset of T.)

Proof. (a) Since E and F have finite measure so does $E \cup F$. Let μ be the restriction of μ to $(E \cup F)$. Then Lemma 4 implies that $\bar{\mu}$ is a bounded vector measure[1]. Thus the result proved by Halmos in [2] shows that the range of $\bar{\mu}$ is convex which implies (a).

(b) Decompose the range of μ into the disjoint union $E_1 \cup E_2$, where E_1 consists of all the finite points and E_2 of all the infinite points in the range of μ. Since $E_2 = \{\alpha\infty\}$ and each set of infinite measure can be decomposed into a sequence of sets of finite measure satisfying

$$\left\|\mu\left(\bigcup_{i=1}^{n} E_i\right)\right\| = \left\|\sum_{i=1}^{n} \mu(E_i)\right\| \to \infty ,$$

E is an unbounded convex set. Thus Lemma 11 certainly implies that E contains a half-line of the form

$$\beta + \gamma t \qquad t > t_o , \qquad \beta, \gamma \in E^n .$$

However the range of an unbounded vector measure can only tend to infinity in one direction, namely α. Therefore we must have $\gamma = \alpha$, which concludes the proof of (b).

Counter Example 13. Here we give an example of an unbounded vector measure whose range is not closed. For convenience we give the example in the complex plane.

Let μ be Lebesgue measure on the real line. Consider the unbounded complex measure, ν, given by

$$\nu(E) = \int_{E \cap (1,\infty)} d\mu + i \int_{E \cap (1,\infty)} [1/(1+t^2)]\, d\mu .$$

Obviously ν only takes on values in the first quadrant and tends to infinity in the direction of the positive real axis. It may easily be verified that ν takes on values as close to any point $(x, 0)$ on the positive real axis as we please. However ν takes on no values on the real axis other than the origin. Thus the range of ν is not closed.

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[1] See Gould [5], page 195.

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REFERENCES

1. A. Liapounoff, Sur les fonctions-vecteurs completement additives, Bull. Acad. Sci. URSS Ser. Math. Vol. 4 (1940) 465-478.

2. P. R. Halmos, The range of a vector measure, Bull. A. M. S. vol. 54(1948), 416-421.

3. P. R. Halmos, On the set of values of a finite measure, Bull. A. M. S. vol. 53(1947), 138-144, (lemmas 1 and 2). (Note that the statement and proof of lemma 5 are wrong.)

4. P. R. Halmos, Measure theory, Princeton (1950).

5. G. G. Gould, Integration over vector-valued measures, Proc. Lon. Math. Soc., vol. 15, part 2 (1965), 193-225.

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