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STOCHASTIC PROCESSES WITH APPLICATIONS IN PHYSICS AND INSURANCE

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Background and motivation. Having been settled around the 1950s and matured afterwards, the theory of stochastic processes has been a common tool for describing time-evolving randomness in various systems, with applications in a number of disciplines, including physics, biology and finance.

Modelling the dynamics of numerous biophysical systems by stochastic processes has attracted significant interest in recent years, due to the improvement in technology enabling detailed and accurate observations. The most established process is Brownian motion, discovered by Robert Brown in 1828 by observing the movement of molecules whose mean square displacement is linear in time. Departing from the linear time dependence of mean square displacements, anomalous diffusion has been widely observed in many complex systems exhibiting temporal or spatial correlations. The typical identifier of anomalous diffusion is the power-law form $\simeq K_\alpha t^\alpha$ of the mean square displacement with the diffusion exponent α and diffusion coefficient K_α, further distinguished by subdiffusion ($\alpha \in (0, 1)$) and superdiffusion ($\alpha \in (1, +\infty)$).

Modelling anomalous diffusion via stochastic processes has received increasing popularity, as it has advantages compared to other mathematical frameworks, such as fractional Fokker–Planck equations, which only describe the single-point marginal law. One of the most widely observed anomalous behaviours is the trapping phenomenon, often modelled by an Ornstein–Uhlenbeck process subordinated by a random clock with the presence of an external harmonic force. The properties of such a process for modelling the position information vary depending on the dimension and the choice of the random clock process, through the lens of its probability density function. Different sequences of applying subordination and integration to the

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Ornstein–Uhlenbeck process for modelling the corresponding position yield disparate theoretical findings.

The other field of application of stochastic processes is the evaluation of ruin related quantities, which has been an active field of research in insurance mathematics and actuarial science. The compound Poisson risk process, also known as the Cramér–Lundberg risk process or classical risk process, was introduced in 1903 to model the risk surplus of an insurance company. Thereafter, many generalisations have been introduced and investigated, including the Sparre Andersen risk model, Lévy risk model and MAP risk model where the claim arrival is modelled by a Markovian arrival process.

The typical quantities of interest include the ultimate ruin probability, the Laplace transform of the time of ruin and the Gerber–Shiu function which provides a unified framework for the evaluation of a variety of risk quantities. Though the primary focus of the current literature is to find an explicit expression of the risk quantity of interest, a numerical method is required when the explicit solution is not available, which is common for complex risk models or problems considering additional features, such as a finite time horizon.

This thesis consists of five chapters, with Chapters 1, 2 and 3 investigating stochastic processes with applications in physics, and Chapters 4 and 5 in insurance mathematics. Summaries of the motivation and contributions of each chapter are outlined as follows.

Dimension-dependent properties of subdiffusions in harmonic force fields. Subdiffusion categorises the transport process that is slower than Brownian motion in the force-free limit, caused by sporadic trapping events. The stochastic representation of the fractional Fokker–Planck equation describes the macroscopic behaviour of test particles in a straightforward manner via the probability density function of the stochastic process, which is the solution of the fractional Fokker–Planck equation.

A cusp at the origin, which is interpreted as a subpopulation of test particles being immobilised near the initial position, has been observed and rigorously investigated in one dimension. However, since more observational results are collected in multi-dimensional systems, the corresponding higher-dimensional analysis is necessary and meaningful for model identification and inference.

The chapter investigates several properties of the subdiffusive anomalous diffusion process in harmonic force fields, modelled by a multivariate subordinated Ornstein–Uhlenbeck process. The results reveal theoretical properties through the lens of the probability density function of the corresponding stochastic process. The singularity of the probability density function at the origin in the multidimensional domain becomes more prominent as an explosion, rather than a cusp. Counter-intuitively, this explosion does not diminish regardless of the choice of the random clock, unlike the case in one dimension where the cusp becomes unobservable over time if the random clock is inverse tempered stable with gamma subordinators. Together with other properties including regularity everywhere except for the origin, modality and

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stationarity of the probability density function, the collected theoretical results have great potential to serve as inference for model identification and prediction in a multidimensional framework.

Time-squeezing and time-expanding transformations in harmonic force fields. Using the probability density function, we investigate whether an anomalous diffusion process is stochastic represented via subordination. Behaviour departing from the normal diffusion leads to the question: what results from varying the time-changing mechanisms in the subordinated process under the influence of an external harmonic force?

To further explore the topic, two representative time-changing mechanisms, time changing by a subordinator and an inverse subordinator, hereafter referred to as time-squeezing and time-expanding transformations, are thoroughly examined and contrasted [1]. We systematically derive a series of properties based upon the Ornstein–Uhlenbeck process after time-squeezing and time-expanding transformations, including the sample path properties, the marginal probability density function, degeneracy of increments, the stationary law, the second-order structure and mean square displacements.

Some properties share similarities due to the presence of the external harmonic force, such as the stationary law and weak ergodicity breaking starting from a nonequilibrated state, with ergodicity in the equilibrated system. These two distinct time transformations result in varied domains of anomalous diffusion, squeezing for normal and superdiffusion and expanding for subdiffusion. Squeezed and expanded trajectories are visually distinguished by the vertical jump discontinuities of the former and horizontal flat periods of the latter. In addition, increments of the squeezed process never degenerate, which is completely opposite for that of the expanded process whose random clock is governed by an inverse stable subordinator, that is, increments of the expanded process eventually become degenerate.

Super- and subdiffusive positions in fractional Klein–Kramers equations. Continuing from the first and second chapters that model the position information by a subordinated Ornstein—Uhlenbeck process, we consider integration and subordination by an inverse stable subordinator simultaneously [3]. The different sequences of applying the integration and subordination result in varied realistic physical modelling and interpretations, one with the velocity occasionally paused yet within the realm of classical physics resulting in superdiffusive positions by integrating a subordinated Ornstein–Uhlenbeck process, and the other with the position trapped occasionally leading the position-velocity pair out of Newton's law of motion and causing subdiffusive positions by subordinating an integrated Ornstein–Uhlenbeck process. To describe various physical systems that exhibit a complex transient diffusion pattern, we contrast the physical relevance of those two very similar yet very distinct generalisations.

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Together with the Gaussian integrated Ornstein–Uhlenbeck process modelling the diffusive position of a Brownian particle subject to friction force, we derive a variety of key properties, including the second-order structure and ergodic behaviours with all relevant states of the initial velocity. In the long run, diffusion exponents of ensemble averaged mean square displacements distinguish the domains of anomalous transport that sub-, normal and superdiffusive positions characterise. In the short run, the agreement of the value of diffusion exponents and the corresponding position processes may not hold due to the influence of the initial velocity as well as the value of the stability index of the inverse stable subordinator. A similar distinction of time averaged mean square displacements of three positions is sustained. Furthermore, the three positions differ visually by their trajectories, particularly the linear growth of the superdiffusive position and flat periods of the subdiffusive position.

Moment and polynomial bounds for ruin-related quantities in risk theory. Various stochastic processes have been investigated for modelling the risk surplus of an insurance company, whose vulnerability to insolvency is of particular interest measured by several risk quantities. As a result of the growing complexity of the models and the problem settings, numerical methods become increasingly necessary due to the lack of analytical tractability.

Rather than approximations that most numerical methods aim to provide, the numerical method based upon semidefinite programming discussed in this chapter constructs deterministic upper and lower bounds for the risk quantities of interest [2]. There are several advantages of this novel numerical method: it provides a 100%-confidence interval for the range of the unknown value, allows general collective risk models with additional features such as dividend barriers and finite time horizon, and requires light computational effort.

The optimisation formulations consist of two sub-frameworks, the primal moment, providing a pointwise tighter upper or lower bound for the solution at a predetermined point, and the dual polynomial, providing an upper or lower bounding function uniformly over the entire problem domain. To examine the quality of the bounds provided by the proposed method, we perform numerical experiments for both infinite- and finite-time problems with several additional features, along with effective techniques that improve the quality of the bounds, including domain scaling, piecewise polynomial test functions and exponential tempering. The upper and lower bounds are excellently tight compared with closed form solutions for infinite-time problems. For finite-time problems where closed-form solutions are not available, our bounds can be employed to test the efficiency and complement the existing numerical approximation methods, for instance, assessing the validity of the 99%-confidence intervals of Monte Carlo simulation results with statistical errors.

The Gerber–Shiu discounted penalty function: from practical perspectives. The Gerber–Shiu function is a popular risk quantity that provides a unified framework for the evaluation of a variety of risk quantities, including the ruin probability and

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expected dividends until ruin discussed in the previous chapter. Although the primary focus of the existing literature is to find the explicit expression of the Gerber–Shiu function, numerical method is necessary, particularly when the risk model or the penalty function becomes complex.

To integrate and enhance the understanding of the Gerber–Shiu function with a wide collection of variant formulations for various surplus processes, we provide an exhaustive survey of the existing literature on analytical, semi-analytical and asymptotic methods on the Gerber–Shiu function, as well as numerical methods and statistical inference with a view towards potential application of the Gerber–Shiu function in practice. On the basis of an exhaustive collection of 198 references, we provide systematic categorisation, essential and representative formulae, and extra assumptions so as to achieve a full up-to-date coverage of the existing literature on the Gerber–Shiu function, which can also serve as a guidebook to model and method selection.

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