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Injectivity of the Connecting Homomorphisms in Inductive Limits of Elliott–Thomsen Algebras

Dedicated to Prof. Chunlan Jiang on the occasion of his 60th birthday

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Abstract. Let A be the inductive limit of a sequence

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots$$

with $A_n = \bigoplus_{i=1}^{n_i} A_{[n,i]}$, where all the $A_{[n,i]}$ are Elliott–Thomsen algebras and $\phi_{n,n+1}$ are homomorphisms. In this paper, we will prove that A can be written as another inductive limit

$$B_1 \xrightarrow{\psi_{1,2}} B_2 \xrightarrow{\psi_{2,3}} B_3 \longrightarrow \cdots$$

with $B_n = \bigoplus_{i=1}^{n_i'} B_{[n,i]'}$, where all the $B_{[n,i]'}$ are Elliott–Thomsen algebras and with the extra condition that all the $\psi_{n,n+1}$ are injective.

1 Introduction

In 1997, Li proved the result that if $A = \varinjlim(A_n, \phi_{m,n})$ is an inductive limit C*-algebra with $A_n = \bigoplus_{i=1}^{n_i} M_{[n,i]}(C(X_{[n,i]}))$, where all $X_{[n,i]}$ are graphs, n_i and $[n,i]$ are positive integers, then one can write $A = \varinjlim(B_n, \psi_{m,n})$, where

$$B_n = \bigoplus_{i=1}^{n_i'} M_{[n,i]'}\big(C(Y_{[n,i]'})\big)$$

are finite direct sums of matrix algebras over graphs $Y_{[n,i]'}$ with the extra property that the homomorphisms $\psi_{m,n}$ are injective [10]. This played an important role in the classification of simple AH algebras with one-dimensional local spectra (see [2, 3, 10–12]). This result was extended to the case of AH algebras [5], in which the space $X_{[n,i]}$ are replaced by connected finite simplicial complexes.

In this article, we consider the C*-algebra A that can be expressed as the inductive limit of a sequence

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots,$$

where all A_i are Elliott–Thomsen algebras and $\phi_{n,n+1}$ are homomorphisms. These algebras were introduced by Elliott in [4] and Thomsen in [6], and are also called one-dimensional non-commutative finite CW complexes. We will prove that A can

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be written as inductive limits of sequences of Elliott–Thomsen algebras with the property that all connecting homomorphisms are injective. The results in this paper will be used in to classify real rank zero inductive limits of one-dimensional non-commutative finite CW complexes.

2 Preliminaries

Definition 2.1 Let F_1 and F_2 be two finite dimensional C^*-algebras. Suppose that there are two homomorphisms $\varphi_0, \varphi_1: F_1 \to F_2$. Consider the C^*-algebra

$$A = A(F_1, F_2, \varphi_0, \varphi_1) = \left\{ (f, a) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_0(a), \quad f(1) = \varphi_1(a) \right\}.$$

These C^*-algebras have been introduced into the Elliott program by Elliott and Thomsen in [6]. Denote by $\mathcal{C}$ the class of all unital C^*-algebras of the form $A(F_1, F_2, \varphi_0, \varphi_1)$. (This class includes the finite dimensional C^*-algebras, the case $F_2 = 0$.) These C^*-algebras will be called *Elliott–Thomsen algebras.* Following [9], let us say that a unital C*-algebra $A \in \mathcal{C}$ is *minimal* if it is indecomposable, *i.e.,* not the direct sum of two or more C*-algebras in $\mathcal{C}$.

Proposition 2.2 ([9]) *Let* $A = A(F_1, F_2, \varphi_0, \varphi_1)$, *where* $F_1 = \bigoplus_{j=1}^{p} M_{k_j}(\mathbb{C})$, $F_2 = \bigoplus_{i=1}^{l} M_{l_i}(\mathbb{C})$ *and* $\varphi_0, \varphi_1: F_1 \to F_2$ *be two homomorphisms. Let* $\varphi_{0*}, \varphi_{1*}: K_0(F_1) = \mathbb{Z}^p \to K_0(F_1) = \mathbb{Z}^l$ *be represented by matrices* $\alpha = (\alpha_{ij})_{l\times p}$ *and* $\beta = (\beta_{ij})_{l\times p}$, *where* $\alpha_{ij}, \beta_{ij} \in \mathbb{Z}_+$ *for each pair* i, j. *Then*

$$K_0(A) = \operatorname{Ker}(\alpha - \beta), \quad K_1(A) = \mathbb{Z}^l / \operatorname{Im}(\alpha - \beta).$$

2.1 We use the notation $\#(\cdot)$ to denote the cardinal number of a set, the sets under consideration will be sets with multiplicity, and then we shall also count multiplicity when we use the notation #. We use • or •• to denote any possible positive integer. We shall use $\{a^{\sim k}\}$ to denote $\{\underbrace{a, \dots, a}_{k\ times}\}$. For example, $\{a^{\sim 3}, b^{\sim 2}\} = \{a, a, a, b, b\}$.

2.2 Let us use $\theta_1, \theta_2, \dots, \theta_p$ to denote the spectrum of F_1 and denote the spectrum of $C([0,1], F_2)$ by (t, i), where $0 \le t \le 1$ and $i \in \{1, 2, \dots, l\}$ indicates that it is in i-th block of F_2. So

$$\operatorname{Sp}\left(C([0,1], F_2)\right) = \coprod_{i=1}^{l} \left\{ (t, i),\ 0 \le t \le 1 \right\}.$$

Using identification of $f(0) = \varphi_0(a)$ and $f(1) = \varphi_1(a)$ for $(f, a) \in A$, $(0, i) \in \operatorname{Sp}(C[0,1])$ is identified with

$$(\theta_1^{\sim\alpha_{i1}}, \theta_2^{\sim\alpha_{i2}}, \dots, \theta_p^{\sim\alpha_{ip}}) \subset \operatorname{Sp}(F_1)$$

and $(1, i) \in \operatorname{Sp}(C([0,1], F_2))$ is identified with

$$(\theta_1^{\sim\beta_{i1}}, \theta_2^{\sim\beta_{i2}}, \dots, \theta_p^{\sim\beta_{ip}}) \subset \operatorname{Sp}(F_1)$$

as in $\operatorname{Sp}(A) = \operatorname{Sp}(F_1) \cup \coprod_{i=1}^{l} (0,1)_i$.

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2.3 With $A = A(F_1, F_2, \varphi_0, \varphi_1)$ as above, let $\varphi: A \to M_n(\mathbb{C})$ be a homomorphism; then there exists a unitary u such that

$$\varphi(f,a) = u^* \cdot \operatorname{diag}\big(\underbrace{a(\theta_1),\dots,a(\theta_1)}_{t_1},\dots,\underbrace{a(\theta_p),\dots,a(\theta_p)}_{t_p}, f(y_1),\dots,f(y_\bullet), 0_{\bullet\bullet}\big)\cdot u,$$

where $y_1, y_2, \dots, y_\bullet \in \coprod_{i=1}^{l}[0,1]_i$. For $y = (0,i)$ (also denoted by 0_i), one can replace $f(y)$ by

$$\big(\underbrace{a(\theta_1),\dots,a(\theta_1)}_{\alpha_{i1}},\dots,\underbrace{a(\theta_p),\dots,a(\theta_p)}_{\alpha_{ip}}\big)$$

in the above expression, and do the same with $y = (1,i)$. After this procedure, we can assume each y_k is strictly in the open interval $(0,1)_i$ for some i. We write the spectrum of φ by

$$\operatorname{Sp}\varphi = \{\theta_1^{\sim t_1}, \theta_2^{\sim t_2}, \dots, \theta_p^{\sim t_p}, y_1, y_2, \dots, y_\bullet\},$$

where $y_k \in \coprod_{i=1}^{l}(0,1)_i$.

If $f = f^* \in A$, we use $\operatorname{Eig}(\varphi(f))$ to denote the eigenvalue list of $\varphi(f)$, and then

$$\#(\operatorname{Eig}(\varphi(f))) = n \text{ (counting multiplicity).}$$

2.4 Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ be minimal. Write $a \in F_1$ as

$$a = (a(\theta_1), a(\theta_2), \dots, a(\theta_p)), \quad f(t) \in C\big([0,1], F_2\big)$$

as

$$f(t) = \big(f(t,1), f(t,2), \dots, f(t,l)\big),$$

where $a(\theta_j) \in M_{k_j}(\mathbb{C})$, $f(t,i) \in C([0,1], M_{l_i}(\mathbb{C}))$.

For any $(f,a) \in A$ and $i \in \{1,2,\dots,l\}$, define $\pi_t: A \to C([0,1], F_2)$ by $\pi_t(f,a) = f(t)$ and $\pi_t^i: A \to C([0,1], M_{l_i}(\mathbb{C}))$ by $\pi_t^i(f,a) = f(t,i)$, where $t \in (0,1)$ and $\pi_0^i(f,a) = f(0,i)$ (denoted by $\varphi_0^i(a)$), $\pi_1^i(f,a) = f(1,i)$ (denoted by $\varphi_1^i(a)$). There is a canonical map $\pi_e: A \to F_1$ defined by $\pi_e((f,a)) = a$, for all $j = \{1,2,\dots,p\}$.

2.5 We use the convention that $A = A(F_1, F_2, \varphi_0, \varphi_1)$, $B = B(F_1', F_2', \varphi_0', \varphi_1')$, where

$$F_1 = \bigoplus_{j=1}^{p} M_{k_j}(\mathbb{C}), \quad F_2 = \bigoplus_{i=1}^{l} M_{l_i}(\mathbb{C}), \quad F_1' = \bigoplus_{j'=1}^{p'} M_{k'_{j'}}(\mathbb{C}), \quad F_2' = \bigoplus_{i'=1}^{l'} M_{l'_{i'}}(\mathbb{C}).$$

Set $L(A) = \sum_{i=1}^{l} l_i$, $L(B) = \sum_{i'=1}^{l'} l'_{i'}$. Denote by $\{e_{ss'}^i\}(1 \le i \le l, 1 \le s, s' \le l_i)$ the set of matrix units for $\oplus_{i=1}^{l} M_{l_i}(\mathbb{C})$ and by $\{f_{ss'}^j\}(1 \le j \le p, 1 \le s, s' \le k_j)$ the set of matrix units for $\oplus_{j=1}^{p} M_{k_j}(\mathbb{C})$.

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2.6 For each $\eta = \frac{1}{m}$ where $m \in \mathbb{N}_+$, let $0 = x_0 < x_1 < \cdots < x_m = 1$ be a partition of $[0,1]$ into m subintervals with equal length $\frac{1}{m}$. We will define a finite subset $H(\eta) \subset A_+$, consisting of two kinds of elements as described below.

(a) For each subset $X_j = \{\theta_j\} \subset \mathrm{Sp}(F_1) = \{\theta_1, \theta_2, \ldots, \theta_p\}$ and a list of integers $a_1, b_2, \ldots, a_l, b_l$ with $0 \leq a_i < a_i + 2 \leq b_i \leq m$, denote $W_j \triangleq \coprod_{\{i|\alpha_{ij} \neq 0\}} [0, a_i\eta]_i \cup \coprod_{\{i|\beta_{ij} \neq 0\}} [b_i\eta, 1]_i$. Then we call W_j the closed neighborhood of X_j; we define element $(f, a) \in A_+$ corresponding to $X_j \cup W_j$ as follows:

Let $a = (a(\theta_1), a(\theta_2), \ldots, a(\theta_p)) \in F_1$, where $a(\theta_j) = I_{k_j}$ and $a(\theta_s) = 0_{k_s}$ if $s \neq j$. For each $t \in [0,1]_i$, $i = \{1, 2, \ldots, l\}$, define

$$f(t,i) = \begin{cases} \varphi_0^i(a)\frac{\eta - \mathrm{dist}(t,[0,a_i\eta]_i)}{\eta} & \text{if } 0 \leq t \leq (a_i+1)\eta, \\ 0 & \text{if } (a_i+1)\eta \leq t \leq (b_i-1)\eta, \\ \varphi_1^i(a)\frac{\eta - \mathrm{dist}(t,[b_i\eta,1]_i)}{\eta} & \text{if } (b_i-1)\eta \leq t \leq 1. \end{cases}$$

All such elements $(f, a) = (f(t,1), f(t,2), \ldots, f(t,l)) \in A_+$ are included in the set $H(\eta)$ and are called *test functions of type* 1.

(b) For each closed subset $X = \bigcup_s [x_{r_s}, x_{r_{s+1}}]_i \subset [\eta, 1-\eta]_i$ (the finite union of closed intervals $[x_r, x_{r+1}]$ and points), so there are finite subsets for each i. Define (f, a) corresponding to X by $a = 0$ and for each $t \in (0,1)_r$, $r \neq i$, $f(t,r) = 0$ and for $t \in (0,1)_i$, define

$$f(t,i) = \begin{cases} 1 - \frac{\mathrm{dist}(t,X)}{\eta} & \text{if } \mathrm{dist}(t,X) < \eta, \\ 0 & \text{if } \mathrm{dist}(t,X) \geq \eta. \end{cases}$$

All such elements are called test functions of type 2.

Note that for any closed subset $Y \subset [\eta, 1-\eta]$, there is a closed subset X consisting of the union of the intervals and points such that $X \supset Y$ and for any $x \in X$, $\mathrm{dist}(x, Y) \leq \eta$.

2.7 Take η as above, define a finite set $\widetilde{H}(\eta)$ as follows:

In the construction of test functions of type 1, we can use $f_{ss'}^j \in F_1$ in place of $a \in F_1$, assume that all these elements are in $\widetilde{H}(\eta)$, and for all test functions $h \in H(\eta)$ of type 2, assume that all these elements $e_{ss'}^i \cdot h$ are in $\widetilde{H}(\eta)$.

Then there exists a natural surjective map $\kappa\colon \widetilde{H}(\eta) \to H(\eta)$. For any subset $G \subset H(\eta)$, define a finite subset $\widetilde{G} \subset \widetilde{H}(\eta)$ by

$$\widetilde{G} = \left\{ h \mid h \in \widetilde{H}(\eta),\ \kappa(h) \in G \right\}.$$

2.8 Suppose A is a C^*-algebra, $B \subset A$ is a subalgebra, $F \subset A$ is a finite subset, and let $\varepsilon > 0$. If for each $f \in F$, there exists an element $g \in B$ such that $\|f - g\| < \varepsilon$, then we say that F is *approximately contained* in B to within ε, and denote this by $F \subset_\varepsilon B$.

The following is clear by the standard techniques of spectral theory [1].

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Lemma 2.3 *Let* $A = \varinjlim(A_n, \phi_{m,n})$ *be an inductive limit of* C^**-algebras* A_n *with morphisms* $\phi_{m,n}: A_m \to A_n$. *Then* A *has* $RR(A) = 0$ *if and only if for any finite self-adjoint subset* $F \subset A_m$ *and* $\varepsilon > 0$, *there exists* $n \geq m$ *such that*

$$\phi_{m,n}(F) \subset_\varepsilon \{ f \in (A_n)_{sa} \mid f \text{ has finite spectrum} \}.$$

Lemma 2.4 ([13, Lemma 2.3]) *Let* $A \in \mathcal{C}$, *for any* $1 > \varepsilon > 0$ *and* $\eta = \frac{1}{m}$ *where* $m \in \mathbb{N}_+$. *If* $\phi, \psi: A \to M_n(\mathbb{C})$ *are unital homomorphisms with the condition that* $\operatorname{Eig}(\phi(h))$ *and* $\operatorname{Eig}(\psi(h))$ *can be paired to within* ε *one by one for all* $h \in H(\eta)$, *then for each* $i \in \{1, 2, \dots, l\}$, *then there exists* $X_i \subset \operatorname{Sp}\phi \cap (0,1)_i$, $X_i' \subset \operatorname{Sp}\psi \cap (0,1)_i$ *with* $X_i \supset \operatorname{Sp}\phi \cap [\eta, 1-\eta]_i$, $X_i' \supset \operatorname{Sp}\psi \cap [\eta, 1-\eta]_i$ *such that* X_i *and* X_i' *can be paired to within* 2η *one by one.*

3 Main Results

In this section, we will prove the following theorem.

Theorem 3.1 *Let* $A = \varinjlim(A_n, \phi_{m,n})$ *be an inductive limit of Elliott–Thomsen algebras. Then one can write* $A = \varinjlim(B_n, \psi_{m,n})$, *where all the* B_n *are Elliott–Thomsen algebras, and all the homomorphisms* $\psi_{m,n}$ *are injective.*

Lemma 3.2 ([10]) *Let* $Y \subset [0,1]$ *be a closed subset containing uncountably many points. Then there exists a surjective non-decreasing continuous map* $\rho: Y \to [0,1]$.

3.1 Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ be minimal. The topology base on

$$\operatorname{Sp}(A) = \{\theta_1, \theta_2, \dots, \theta_p\} \cup \coprod_{i=1}^{l} (0,1)_i$$

at each point θ_j is given by

$$\{\theta_j\} \cup \coprod_{\{i \mid \alpha_{ij} \neq 0\}} (0, \varepsilon)_i \cup \coprod_{\{i \mid \beta_{ij} \neq 0\}} (1-\varepsilon, 1)_i.$$

In general, this is a non-Hausdorff topology.

For closed subset $Y \subset \operatorname{Sp}(A)$ and $\delta > 0$, we will construct a space Z and a continuous surjective map $\rho: Y \to Z$ such that $Z \cap (0,1)_i$ is a union of finitely many intervals for each $i \in \{1, 2, \dots, l\}$, and $\operatorname{dist}(\rho(y), y) < \delta$ for all $y \in Y$. We can find a similar discussion in an old version of [8].

For any closed subset $Y \subset \operatorname{Sp}(A)$, define index sets

$$\begin{aligned}
J_Y &= \{ j \mid \theta_j \in Y \}, \\
L_{0,Y} &= \{ i \mid (0,1)_i \cap Y = \varnothing \}, \\
L_{1,Y} &= \{ i \mid (0,1)_i \subset Y \}, \\
L_{l,Y} &= \{ i \mid i \notin L_{1,Y} \text{ and } \exists\, s > 0 \text{ such that } (0,s]_i \subset Y \}, \\
L_{ll,Y} &= \{ i \mid i \notin L_{1,Y} \cup L_{l,Y} \text{ and } \exists\, \{y_n\}_{n=1}^{\infty} \subset (0,1)_i \cap Y \text{ such that } \lim_{n\to\infty} y_n = 0_i \},
\end{aligned}$$

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$$L_{r,Y} = \{i \mid i \notin L_{1,Y} \text{ and } \exists t > 0 \text{ such that } [1-t,1)_i \subset Y\},$$
$$L_{rr,Y} = \{i \mid i \notin L_{1,Y} \cup L_{r,Y} \text{ and } \exists \{y_n\}_{n=1}^{\infty} \subset (0,1)_i \cap Y \text{ such that } \lim_{n\to\infty} y_n = 1_i\},$$
$$L_{a,Y} = \{i \mid i \notin L_{0,Y} \cup L_{1,Y}\}.$$

Then we have

$$L_{l,Y} \cup L_{ll,Y} \cup L_{r,Y} \cup L_{rr,Y} \subset L_{a,Y},$$
$$L_{0,Y} \cup L_{1,Y} \cup L_{a,Y} = \{1,2,\ldots,l\}.$$

Consider $Y \subset \mathrm{Sp}(A)$; if $i \in L_{1,Y} \cup L_{l,Y} \cup L_{ll,Y}$, assume that $(0,i) \in Y$ and if $i \in L_{1,Y} \cup L_{r,Y} \cup L_{rr,Y}$, assume that $(1,i) \in Y$. For $\delta > 0$, there exists $m \in \mathbb{N}_+$ such that $\frac{1}{m} < \frac{\delta}{2}$. Denote $Y_i = Y \cap [0,1]_i$, $i \in \{1,2,\ldots,l\}$, then we can construct a collection of finitely many points $\widehat{Y_i} = \{y_1, y_2, \ldots\} \subset Y_i$ as below.

(a) If $i \in L_{0,Y}$, let $\widehat{Y_i} = \varnothing$.
(b) If $i \in L_{1,Y}$, let $\widehat{Y_i} = \{(0,i), (\frac{1}{m}, i), \ldots, (1,i)\}$.
(c) For each $i \in L_{a,Y}$, consider the set $Y_i \cap [\frac{r-1}{m}, \frac{r}{m}]_i$. If $Y_i \cap [\frac{r-1}{m}, \frac{r}{m}]_i \neq \varnothing$, then set

$$x_i^r = \min\Big\{x \;\Big|\; x \in Y_i \cap \Big[\frac{r-1}{m}, \frac{r}{m}\Big]_i\Big\},$$
$$\widetilde{x}_i^r = \max\Big\{x \;\Big|\; x \in Y_i \cap \Big[\frac{r-1}{m}, \frac{r}{m}\Big]_i\Big\}.$$

Assume that $Y_i \cap [\frac{r-1}{m}, \frac{r}{m}]_i \neq \varnothing$ if and only if $r \in \{r_1, r_2, \ldots, r_\bullet\} \subset \{1,2,\ldots,m\}$. Then we have a finite set

$$\{x_i^{r_1}, \widetilde{x}_i^{r_1}, x_i^{r_2}, \ldots, x_i^{r_\bullet}, \widetilde{x}_i^{r_\bullet}\}.$$

Some of the points may be the same; we can delete the extra repeating points and denote the result by $\widehat{Y_i}$.

Denote $\widehat{Y} = \coprod_{i=1}^{l} \widehat{Y_i}$. Two points $(y_s,i),(y_t,i') \in \widehat{Y}$ are said to be *adjacent* if $(y_s,i),(y_t,i')$ are in the same interval (the case $i = i'$), and inside the open interval $(y_s,y_t)_i$, there is no other point in $\widehat{Y}$. Note that if $\{(y_s,i),(y_t,i)\}$ is an adjacent pair and $(y_s,y_t)_i \cap Y \neq \varnothing$, then $\mathrm{dist}((y_s,i),(y_t,i)) < \delta$, and for any $y \in Y \cap \coprod_{i=1}^{l}[0,1]_i$, there exists $y' \in \widehat{Y}$ such that $\mathrm{dist}(y,y') < \delta$.

It is obvious that Y_i can be written as the union of $[y_s,y_t]_i \cap Y_i$, where $\{(y_s,i),(y_t,i)\}$ runs over all adjacent pairs. We will define a space Z and a continuous surjective map $\rho\colon Y \to Z$ as follows (see also [10]).

First, $Y \cap \mathrm{Sp}(F_1) \subset Z$ and Z contains a collection of finitely many points $P(Z) = \{z_1, z_2, \ldots\}$, each $(z_s,i) \in P(Z)$ corresponding to one and only one $(y_s,i) \in \widehat{Y}$. To define the edges of Z, we consider an adjacent pair $\{(y_s,i),(y_t,i)\}$. We have the following two cases.

Case 1: If $[y_s,y_t]_i \cap Y$ has uncountably many points, then we let Z contain $[z_s,z_t]_i$, the line segment connecting $(z_s,i),(z_t,i)$. By Lemma 3.2, there exists a non-decreasing surjective map $\rho\colon [y_s,y_t]_i \cap Y \to [z_s,z_t]_i$ such that $\rho((y_s,i)) = (z_s,i)$, $\rho((y_t,i)) = (z_t,i)$. (Here both $[y_s,y_t]_i$ and $[z_s,z_t]_i$ are identified with interval $[0,1]$.)

Case 2: If $[y_s,y_t]_i \cap Y$ has at most countably many points, then it is defined that there is no edge connecting (z_s,i) and (z_t,i). Since $[y_s,y_t]_i \cap Y$ is a countable closed subset

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of $[y_s, y_t]_i$, there exists an open interval $(y'_s, y'_t)_i \subset (y_s, y_t)_i$ such that $(y'_s, y'_t)_i \cap Y = \varnothing$. Let $\rho\colon [y_s, y_t]_i \cap Y \to \{(z_s, i), (z_t, i)\}$ be defined by

$$\rho(y) = \begin{cases} (z_s, i) & \text{if } y \in [y_s, y'_s]_i \cap Y, \\ (z_t, i) & \text{if } y \in [y'_t, y_t]_i \cap Y. \end{cases}$$

By the above procedure for all adjacent pairs, we obtain a space Z such that $Z \cap (0,1)_i$ is a union of finitely many intervals for each $i \in \{1, 2, \ldots, l\}$.

Notice that ρ is defined on each $[y_s, y_t]_i \cap Y$ piece by piece, and $\rho((y_s, i)) = (z_s, i)$ for each s, i. The definitions of ρ on different pieces are consistent. Then we obtain a surjective map $\rho\colon Y \cap (0,1)_i \to Z \cap (0,1)_i$. Let $\rho\colon Y \cap \mathrm{Sp}(F_1) \to Z \cap \mathrm{Sp}(F_1)$ be defined by $\rho(\theta_j) = \theta_j$ for all $j \in J$.

Then we obtain a surjective map $\rho\colon Y \to Z$, and we have $\mathrm{dist}(\rho(y), y) < \delta$ for all $y \in Y$.

3.2 For any closed subset $X \subset \mathrm{Sp}(A)$, denote that $A|_X = \{f|_X \mid f \in A\}$. For the ideal $I \subset A$, there exists a closed subset $Y \subset \mathrm{Sp}(A)$ such that $I = \{f \in A \mid f|_Y = 0\}$. Then $A/I \cong A|_Y$.

Lemma 3.3 *Let $A \in \mathcal{C}$ be minimal, let $\varepsilon > 0$, $Y \subset \mathrm{Sp}(A)$ be a closed subset, and let $G \subset A|_Y$ be a finite subset. Suppose that $\delta > 0$ satisfies that $\mathrm{dist}(y, y') < \delta$ implies that $\|g(y) - g(y')\| < \varepsilon$ for all $g \in G$. Then there exists a closed subset $Z \subset \mathrm{Sp}(A)$ and a surjective map $\rho\colon Y \to Z$ such that $A|_Z \in \mathcal{C}$ and $G \subset_\varepsilon A|_Z$, where $A|_Z$ is considered as a subalgebra of $A|_Y$ by the inclusion $\rho^*\colon A|_Z \to A|_Y$.*

Proof For a closed subset $Y \subset \mathrm{Sp}(A)$ and $\delta > 0$, we can construct Z and ρ as in 3.1. The surjective map $\rho\colon Y \to Z$ induces a homomorphism

$$\rho^*\colon A|_Z \longrightarrow A|_Y,$$
$$(\rho^*(g))(y) = g(\rho(y)), \quad \forall y \in Y.$$

Then we have

$$\|\rho^*(g) - g\| = \max_{y \in Y} \|g(y) - g(\rho(y))\| < \varepsilon$$

for any $g \in G$, and $G \subset_\varepsilon A|_Z$.

We need to verify $A|_Z \in \mathcal{C}$. Defining index sets for Z, we will have

$$J_Z = J_Y, \qquad L_{0,Z} = L_{0,Y},$$
$$L_{1,Z} \supset L_{1,Y}, \qquad L_{ll,Z} = L_{rr,Z} = \varnothing.$$

We will define positive numbers s_i for all $i \in L_{l,Z}$, positive numbers t_i for all $i \in L_{r,Z}$, and positive numbers $a_i < b_i$ for all $i \in L_{a,Z}$ to satisfy that $s_i < a_i < b_i$ (if $i \in L_{l,Z}$) and $a_i < b_i < t_i$ (if $i \in L_{r,Z}$) as below.

For $i \in L_{l,Z}$, let $s_i = \max\{s \mid (0,s]_i \subset Z\}$. For $i \in L_{r,Z}$, let $t_i = \min\{t \mid [t,1)_i \subset Z\}$. Note that if $i \in L_{l,Z} \cap L_{r,Z}$, then $s_i < t_i$.

For $i \in L_{l,Z}$, choose a_i with $s_i < a_i < 1$ such that $(s_i, a_i)_i \cap Y = \varnothing$. For $i \in L_{a,Z} \setminus L_{l,Z}$, choose a_i with $0 < a_i < \delta$ such that $(0, a_i)_i \cap Y = \varnothing$ (we do not need to define s_i in this case). Evidently the numbers a_i satisfies that $a_i < t_i$ provided $i \in L_{r,Z}$.

For $i \in L_{r,Z}$, choose b_i with $a_i < b_i < t_i$ such that $(b_i, t_i)_i \cap Y = \varnothing$. For $i \in L_{a,Z} \backslash L_{r,Z}$, choose b_i with $b_i > 1 - \delta$ such that $(b_i, 1)_i \cap Y = \varnothing$ (we do not need to define t_i in this case).

Define closed subsets of $\mathrm{Sp}(A)$ as below:

$$Z_1 = \coprod_{i \in L_{a,Z}} [a_i, b_i]_i,$$
$$Z_2 = \{\theta_j,\ j \in J\} \cup \coprod_{i \in L_{1,Z}} (0,1)_i \cup \coprod_{i \in L_{l,Z}} (0, s_i]_i \cup \coprod_{i \in L_{r,Z}} [t_i, 1)_i.$$

Then $Z_1 \cap Z_2 = \varnothing$ and $Z \subset Z_1 \cup Z_2$, we have $A|_Z \cong A|_{Z_2} \oplus A|_{Z_1}$, where $A|_{Z_1}$ is a direct sum of matrices over interval algebras or matrix algebras.

Now we consider $A|_{Z_2}$, for each $i \in L_{l,Z}$, we denote $F_2^i = M_{l_i}(\mathbb{C})$ by $F_{2,l}^i$, and for each $i \in L_{r,Z}$, we denote $F_2^i = M_{l_i}(\mathbb{C})$ by $F_{2,r}^i$. Let

$$E_1 = \bigoplus_{j \in J_Z} F_1^j \oplus \bigoplus_{i \in L_{l,Z}} F_{2,l}^i \oplus \bigoplus_{i \in L_{r,Z}} F_{2,r}^i$$
$$E_2 = \bigoplus_{i \in L_{1,Z}} F_2^i \oplus \bigoplus_{i \in L_{l,Z}} F_{2,l}^i \oplus \bigoplus_{i \in L_{r,Z}} F_{2,r}^i.$$

Write $a \in F_1$ by $a = (a(\theta_1), a(\theta_2), \dots, a(\theta_p))$. Define $\pi\colon F_1 \to F_1$ by

$$\pi(a) = a' = \big(a'(\theta_1), a'(\theta_2), \dots, a'(\theta_p)\big),$$

where

$$a'(\theta_j) = \begin{cases} a(\theta_j) & \text{if } j \in J_Z, \\ 0_{k_j} & \text{if } j \notin J_Z. \end{cases}$$

Then there exist a natural inclusion ι and a projection ι^* such that

$$\iota \circ \iota^* = \pi\colon F_1 \to F_1,$$
$$\iota^* \circ \iota = \mathrm{id}\colon \bigoplus_{j \in J_Z} F_1^j \longrightarrow \bigoplus_{j \in J_Z} F_1^j.$$

Then we have if $i \in L_{1,Z} \cup L_{l,Z}$, then $\varphi_0^i(a) = \varphi_0^i(\pi(a))$ for any $a \in F_1$, and if $i \in L_{1,Z} \cup L_{r,Z}$, then $\varphi_1^i(a) = \varphi_1^i(\pi(a))$ for any $a \in F_1$.

Let $\psi_0\colon E_1 \to E_2$ be defined as follows:

(1) For the part $\bigoplus_{j \in J_Z} F_1^j$ in E_1, the partial map of ψ_0 is defined to be

$$\bigoplus_{i \in L_{1,Z}} \varphi_0^i \circ \iota \oplus \bigoplus_{i \in L_{l,Z}} \varphi_0^i \circ \iota \oplus \bigoplus_{i \in L_{r,Z}} 0.$$

(2) For the part $\bigoplus_{i \in L_{l,Z}} F_{2,l}^i$ in E_1, the partial map of ψ_0 is zero.

(3) For the part $\bigoplus_{i \in L_{r,Z}} F_{2,r}^i$ in E_1, the partial map of ψ_0 is defined to be

$$\bigoplus_{i \in L_{1,Z}} 0 \oplus \bigoplus_{i \in L_{l,Z}} 0 \oplus \bigoplus_{i \in L_{r,Z}} \mathrm{id}_i,$$

where id_i ($i \in L_{r,Z}$) is the identity map from $M_{l_i}(\mathbb{C})$ to $M_{l_i}(\mathbb{C})$.

Similarly, let $\psi_1\colon E_1 \to E_2$ be defined as follows:

(1) For the part $\bigoplus_{j \in J_Z} F_1^j$ in E_1, the partial map of ψ_1 is defined to be

$$\bigoplus_{i \in L_{1,Z}} \varphi_1^i \circ \iota \oplus \bigoplus_{i \in L_{l,Z}} 0 \oplus \bigoplus_{i \in L_{r,Z}} \varphi_1^i \circ \iota.$$

(2) For the part $\bigoplus_{i\in L_{l,Z}} F_{2,l}^i$ in E_1, the partial map of ψ_0 is defined to be

$$\bigoplus_{i\in L_{1,Z}} 0 \oplus \bigoplus_{i\in L_{l,Z}} id_i \oplus \bigoplus_{i\in L_{r,Z}} 0,$$

where id_i ($i \in L_{l,Z}$) is the identity map from $M_{l_i}(\mathbb{C})$ to $M_{l_i}(\mathbb{C})$.

(3) For the part $\bigoplus_{i\in L_{r,Z}} F_{2,r}^i$ in E_1, the partial map of ψ_0 is zero.

Evidently, $A|_{Z_2} \cong B(E_1, E_2, \psi_0, \psi_1) \in \mathcal{C}$; then we have $A|_Z \in \mathcal{C}$. ∎

We will apply some techniques from [14] and obtain some perturbation results.

Lemma 3.4 *Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ be minimal, $B = M_n(\mathbb{C})$, and let $F \subset A$ be a finite subset. Given $1 > \varepsilon > 0$, there exist $\eta, \varepsilon' > 0$ such that, if unital homomorphisms $\phi, \psi\colon A \to B$ satisfy the conditions*

(i) $\mathrm{Sp}\,\phi = \mathrm{Sp}\,\psi$,

(ii) $\|\phi(h) - \psi(h)\| < \varepsilon'$ *for all* $h \in H(\eta) \cup \widetilde{H}(\eta)$,

then there is a continuous path of homomorphisms $\phi_t\colon A \to B$ such that $\phi_0 = \phi$, $\phi_1 = \psi$, and $\|\phi_t(f) - \phi(f)\| < \varepsilon$ for all $f \in F$, $t \in [0,1]$.

Proof Without loss of generality, we can suppose that for each $f \in F$, $\|f\| \le 1$. Since $F \subset A$ is a finite set, there exists an integer $m > 0$ such that for any $\mathrm{dist}(x, x') < \frac{2}{m}$, $\|f(x) - f(x')\| < \frac{\varepsilon}{2}$ holds for all $f \in F$, and ε' will be specified later. Set $\eta = \frac{1}{2mn}$; then we have finite subsets $H(\eta)$ and $\widetilde{H}(\eta)$.

There exist unitaries U, V such that

$$\phi(f,a) = U^*\phi'(f,a)U, \quad \psi(f,a) = V^*\phi'(f,a)V.$$

Here we denote $\phi'\colon A \to B$ by

$$\phi'(f,a) = \mathrm{diag}\left(a(\theta_1)^{\sim t_1}, \dots, a(\theta_p)^{\sim t_p}, f(x_1), f(x_2), \dots, f(x_\bullet)\right),$$

where $x_1, x_2, \dots \in \coprod_{i=1}^{l} (0,1)_i$.

Divide $(0,1)_i$ into $2mn$ intervals of equal length $\frac{1}{2mn}$. For each sub-interval $(\frac{k-1}{m}, \frac{k}{m})_i$, $k = 1, 2, \dots, m$, there exist an integer a_k^i such that

$$(a_k^i\eta, a_k^i\eta + 2\eta)_i \subset \left(\frac{k-1}{m}, \frac{k}{m}\right)_i \quad \text{and} \quad (a_k^i\eta, a_k^i\eta + 2\eta)_i \cap \mathrm{Sp}\,\phi = \varnothing.$$

Then we have

$$\mathrm{Sp}\,\phi' = \mathrm{Sp}\,\phi' \cap \coprod_{i=1}^{l} \left([0, a_1^i\eta]_i \cup [a_m^i\eta + 2\eta, 1]_i \cup \bigcup_{k=1}^{m-1} [a_k^i\eta + 2\eta, a_{k+1}^i\eta]_i\right).$$

For each $X_j = \{\theta_j\}$ and $W_j \triangleq \coprod_{\{i|\alpha_{ij}\neq 0\}} [0, a_1^i\eta]_i \cup \coprod_{\{i|\beta_{ij}\neq 0\}} [a_m^i\eta + 2\eta, 1]_i$, we can define h_j corresponding to $X_j \cup W_j$ for all $j \in \{1, 2, \dots, p\}$, and we can define h_k^i corresponding to $[a_k^i\eta + 2\eta, a_{k+1}^i\eta]_i$ for each $i \in \{1, 2, \dots, l\}$, $k \in \{1, 2, \dots, m-1\}$.

Denote

$$G = \left\{h_1, h_2, \dots, h_p, h_1^1, \dots, h_{m-1}^1, \dots, h_1^l, \dots, h_{m-1}^l\right\},$$

We will construct $\widetilde{G}$ as in 2.7:

$$\widetilde{G} = \left\{h \mid h \in \widetilde{H}(\eta),\ \kappa(h) \in G\right\}.$$

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To define $\phi'': A \to B$, change all the elements

$$x \in \operatorname{Sp}\phi' \cap (0, a_1^i \eta]_i \quad \text{to} \quad 0_i \sim \{\theta_1^{\sim\alpha_{i1}}, \dots, \theta_p^{\sim\alpha_{ip}}\},$$
$$x \in \operatorname{Sp}\phi' \cap (a_m^i \eta + 2\eta, 1)_i \quad \text{to} \quad 1_i \sim \{\theta_1^{\sim\beta_{i1}}, \dots, \theta_p^{\sim\beta_{ip}}\},$$

change all the elements $x \in \operatorname{Sp}\phi' \cap [a_{k-1}^i \eta + 2\eta, a_k^i \eta]_i$ to $(\frac{k-1}{m}, i) \in [a_{k-1}^i \eta + 2\eta, a_k^i \eta]_i$ for each $i \in \{1, 2, \dots, l\}$, $k \in \{2, \dots, m\}$. Set $\omega_k^i = \#(\operatorname{Sp}\phi' \cap [a_{k-1}^i \eta + 2\eta, a_k^i \eta]_i)$.

There exists a unitary W such that

$$W\phi''(f)W^* = \begin{pmatrix} a(\theta_1) \otimes I_{t_1'(x)} & & & & & \\ & \ddots & & & & \\ & & a(\theta_p) \otimes I_{t_p'(x)} & & & \\ & & & f((\frac{1}{m}, 1)) \otimes I_{\omega_1^1} & & \\ & & & & \ddots & \\ & & & & & f((\frac{m-1}{m}, l)) \otimes I_{\omega_m^l} \end{pmatrix}.$$

From the construction of ϕ'', we have

$$\phi'(h) = \phi''(h), \quad \forall h \in G \cup \widetilde{G}.$$

Let $P_j = W\phi'(h_j)W^*$, $P_k^i = W\phi'(h_k^i)W^*$; then $P_1, \dots, P_p, P_1^1, \dots, P_1^l, \dots, P_{m-1}^l$ are projections; some of them may be zero. We rewrite the nonzero ones as $P_1, \dots, P_{n'}$. Note that $n' \le n$, and we can write

$$P_1 = \begin{pmatrix} I_{r_1} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \dots, P_{n'} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & I_{r_{n'}} \end{pmatrix}.$$

Since

$$\|\phi(h) - \psi(h)\| < \varepsilon', \quad \forall\, h \in H(\eta) \cup \widetilde{H}(\eta),$$

we have the inequality

$$\|U^* W^* P_r W U - V^* W^* P_r W V\| < \varepsilon', \quad r = 1, 2, \dots, n'.$$

Set $\widetilde{W} = WVU^*W^*$. Let us write the unitary $\widetilde{W} = \left(\begin{smallmatrix} w_{11} & w_{1*} \\ w_{*1} & w_{**} \end{smallmatrix}\right)$, where the size of w_{11} is the same as the rank of P_1. Then we have $\|w_{1*}\| < \varepsilon'$ and $\|w_{*1}\| < \varepsilon'$. Applying this computation to $P_2, \dots, P_{n'}$, we then have

$$\left\| \widetilde{W} - \begin{pmatrix} w_{11} & & \\ & \ddots & \\ & & w_{n'n'} \end{pmatrix} \right\| < n'^2 \varepsilon' \le n^2 \varepsilon'.$$

Writing $T = \left(\begin{smallmatrix} w_{11} & & \\ & \ddots & \\ & & w_{n'n'} \end{smallmatrix}\right)$, T is invertible if $n^2\varepsilon' < 1$. There is a unitary S such that $T = |T^*|S$, so

$$\|\widetilde{W}S^* - |T^*|\| < n^2\varepsilon'.$$

Since $\widetilde{W}S^*$ is a unitary and $|T^*|$ is close to I to within $n^2\varepsilon'$, we have

$$\|\widetilde{W}S^* - I\| \le \|\widetilde{W}S^* - |T^*|\| + \||T^*| - I\| < 2n^2\varepsilon'.$$

Let R_t ($t \in [\frac{2}{3}, 1]$) be a unitary path in a $2n^2\varepsilon'$ neighbourhood of I such that $R_{\frac{2}{3}} = \widetilde{W}S^*$ and $R_1 = I$.

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Since

$$\left\| U^* W^* \big(W\phi'(h) W^* \big) WU - V^* W^* \big(W\phi'(h) W^* \big) WV \right\| < \varepsilon', \ \forall h \in H(\eta) \cup \widetilde{H}(\eta).$$

Then we have

$$\left\| U^* W^* \big(W\phi'(h) W^* \big) WU - V^* W^* R_t \big(W\phi'(h) W^* \big) R_t^* WV \right\| < 4n^2\varepsilon' + \varepsilon' < 5n^2\varepsilon',$$

for all $h \in H(\eta) \cup \widetilde{H}(\eta)$, $t \in [\frac{2}{3}, 1]$. When $t = \frac{2}{3}$, we have

$$\left\| S\big(W\phi'(h) W^* \big) - \big(W\phi'(h) W^* \big) S \right\| < 5n^2\varepsilon', \quad \forall h \in H(\eta) \cup \widetilde{H}(\eta).$$

For any $h \in G \cup \widetilde{G}$, we have $\phi'(h) = \phi''(h)$. Then

$$\| S(W\phi''(h)W^*) - (W\phi''(h)W^*)S \| < 5n^2\varepsilon', \quad \forall\, h \in G \cup \widetilde{G}.$$

Recall that S has diagonal form $S = \operatorname{diag}(S_1, \dots, S_{n'})$; write $S = (w^r_{st})$ as

$$S = \begin{pmatrix} \begin{pmatrix} w^1_{11} & \cdots & w^1_{1r_1} \\ \vdots & \ddots & \vdots \\ w^1_{r_1 1} & \cdots & w^1_{r_1 r_1} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} w^{n'}_{11} & \cdots & w^{n'}_{1r_{n'}} \\ \vdots & \ddots & \vdots \\ w^{n'}_{r'_n 1} & \cdots & w^{n'}_{r_{n'} r_{n'}} \end{pmatrix} \end{pmatrix}.$$

Then the matrix w^r_{st} commutes with the matrix units to within $5n^2\varepsilon'$, so there exist $d^r_{st} \in \mathbb{C}$ such that

$$\| w^r_{st} - d^r_{st} I^r_{st} \| < 5n^4\varepsilon',$$

where I^r_{st} is the identity matrix with suitable size. Write $D = (d^r_{st} I^r_{st})$ as

$$\begin{pmatrix} \begin{pmatrix} d^1_{11} I^1_{11} & \cdots & d^1_{1r_1} I^1_{1r_1} \\ \vdots & \ddots & \vdots \\ d^1_{r_1 1} I^1_{r_1 1} & \cdots & d^1_{r_1 r_1} I^1_{r_1 r_1} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} d^{n'}_{11} I^{n'}_{11} & \cdots & d^{n'}_{1r'_n} I^{n'}_{1r'_n} \\ \vdots & \ddots & \vdots \\ d^{n'}_{r'_n 1} I^{n'}_{r'_n 1} & \cdots & d^{n'}_{r_{n'} r_{n'}} I^{n'}_{r_{n'} r_{n'}} \end{pmatrix} \end{pmatrix}.$$

Then we have

$$\| S - D \| < 5n^6\varepsilon',$$
$$D(W\phi''(f)W^*) = (W\phi''(f)W^*)D, \quad \forall\ f \in A.$$

Hence,

$$\left\| D\big(W\phi'(f) W^* \big) - \big(W\phi'(f) W^* \big) D \right\| < 2\|D\|\varepsilon' < 2(1+5n^6\varepsilon')\varepsilon' < 12n^6\varepsilon', \quad \forall f \in F.$$

Decompose $D = |D^*|O$ in the commutant of $W\phi''(f)W^*$. Let R'_t ($t \in [\frac{1}{3}, \frac{2}{3}]$) be an exponential unitary path in that commutant such that $R'_{\frac{1}{3}} = O^*$ and $R'_{\frac{2}{3}} = I$.

Notice that

$$\| S^* O^* - |D^*| \| < 5n^6\varepsilon'.$$

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Using the same technique as above, we have

$$\|S^*O^* - I\| < 10n^6\varepsilon',$$

Hence there is a unitary path R_t'' $(t \in [0, \frac{1}{3}])$ in a $10n^6\varepsilon'$ neighbourhood of I such that $R_0'' = I$ and $R_{\frac{1}{3}}'' = S^*O^*$.

Finally, choose ε' such that $4n^2\varepsilon' + 12n^6\varepsilon' + 20n^6\varepsilon' < \varepsilon$. We can take ε' to be $\frac{\varepsilon}{40n^6}$, and define a unitary path u_t on $[0,1]$ as follows:

$$u_t^* = \begin{cases} U^*W^*R_t''W & \text{if } t \in [0, \frac{1}{3}], \\ U^*W^*S^*R_t'W & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ V^*W^*R_tW & \text{if } t \in [\frac{2}{3}, 1]. \end{cases}$$

Denote

$$\phi_t(f) = u_t^* \cdot \operatorname{diag}\big(a(\theta_1)^{\sim t_1}, \dots, a(\theta_p)^{\sim t_p}, f(x_1), f(x_2), \dots, f(x_\bullet)\big) \cdot u_t.$$

Then $\phi_0 = \phi$, $\phi_1 = \psi$, $u_0 = U$, $u_1 = V$, and we will have

$$\|\phi_t(f) - \phi(f)\| < \varepsilon$$

for all $f \in F$, $t \in [0,1]$. ∎

Lemma 3.5 *Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ be minimal, let $B = M_n(\mathbb{C})$, and let $F \subset A$ be a finite subset. Given $1 > \varepsilon > 0$, there exist $\eta, \eta_1, \varepsilon' > 0$, such that if $\phi, \psi\colon A \to B$ are unital homomorphisms that satisfy the following conditions:*

(i) $\|\phi(h) - \psi(h)\| < 1$, $\forall\ h \in H(\eta_1)$;

(ii) $\|\phi(h) - \psi(h)\| < \frac{\varepsilon'}{8}$, $\forall\ h \in H(\eta) \cup \widetilde{H}(\eta)$,

then there is a continuous path of homomorphisms $\phi_t\colon A \to B$ such that $\phi_0 = \phi$, $\phi_1 = \psi$ and

$$\|\phi_t(f) - \phi(f)\| < \varepsilon$$

for all $f \in F$, $t \in [0,1]$. Moreover, for each $y \in (\operatorname{Sp}\phi \cup \operatorname{Sp}\psi) \cap \coprod_{i=1}^{l}(0,1)_i$, we have

$$\overline{B_{4\eta_1}(y)} \subset \bigcup_{t\in[0,1]} \operatorname{Sp}\phi_t,$$

where $\overline{B_{4\eta_1}(y)} = \{x \in \coprod_{i=1}^{l}[0,1]_i : \operatorname{dist}(x,y) \le 4\eta_1\}$.

Proof Take ε', η, m as in Lemma 3.4. Let $\eta_1 = \frac{1}{m_1} < \frac{\eta}{2}$ satisfy $\|h(x) - h(x')\| < \frac{\varepsilon'}{8}$ for any $\operatorname{dist}(x,x') \le 4\eta_1$ and for all $h \in H(\eta) \cup \widetilde{H}(\eta)$.

There exist unitaries U, V such that

$$\phi(f,a) = U^* \cdot \operatorname{diag}\big(a(\theta_1)^{\sim s_1}, \dots, a(\theta_p)^{\sim s_p}, f(x_1), f(x_2), \dots, f(x_\bullet)\big) \cdot U,$$
$$\psi(f,a) = V^* \cdot \operatorname{diag}\big(a(\theta_1)^{\sim t_1}, \dots, a(\theta_p)^{\sim t_p}, f(y_1), f(y_2), \dots, f(y_{\bullet\bullet})\big) \cdot V.$$

where $f \in A$, $x_1, x_2, \dots, y_1, y_2, \dots \in \coprod_{i=1}^{l}(0,1)_i$.

From condition (i) and Lemma 2.4, for each $i \in \{1,2,\dots,l\}$, there exists $X_i \subset \operatorname{Sp}\phi \cap (0,1)_i$, $X_i' \subset \operatorname{Sp}\psi \cap (0,1)_i$ with $X_i \supset \operatorname{Sp}\phi \cap [\eta_1, 1-\eta_1]_i$, $X_i' \supset \operatorname{Sp}\psi \cap [\eta_1, 1-\eta_1]_i$ such that X_i and X_i' can be paired to within $2\eta_1$ one by one. Denote the one to one correspondence by $\pi\colon X_i \to X_i'$.

To define ϕ', change all the elements $x_k \in (0,\eta_1)_i \backslash X_i$ to $0_i \sim \{\theta_1^{\sim\alpha_{i1}},\dots,\theta_p^{\sim\alpha_{ip}}\}$ and $x_k \in (1-\eta_1,1)_i \backslash X_i$ to $1_i \sim \{\theta_1^{\sim\beta_{i1}},\dots,\theta_p^{\sim\beta_{ip}}\}$, and finally, change all the $x_k \in X_i$ to $\pi(x_k) \in X'_i$. To define ψ', change all the elements $y_k \in (0,\eta_1)_i \backslash X'_i$ to $0_i \sim \{\theta_1^{\sim\alpha_{i1}},\dots,\theta_p^{\sim\alpha_{ip}}\}$ and $y_k \in (1-\eta_1,1)_i \backslash X'_i$ to $1_i \sim \{\theta_1^{\sim\beta_{i1}},\dots,\theta_p^{\sim\beta_{ip}}\}$. Then we have

$$\operatorname{Sp}\phi' \cap (0,1)_i = \operatorname{Sp}\psi' \cap (0,1)_i$$

for all $i = 1,2,\dots,l$.

Since $2\eta_1 < \eta = \frac{1}{2mn}$, then for each $[0,1]_i$, there exist integers a_i, b_i with $1 < a_i < a_i + 2 \le b_i < m_1$ such that

$$\operatorname{Sp}\phi \cap (a_i\eta_1, b_i\eta_1)_i = \operatorname{Sp}\psi \cap (a_i\eta_1, b_i\eta_1)_i = \varnothing.$$

Then for $X_j = \{\theta_j\}$ and $W_j \triangleq \coprod_{\{i|\alpha_{ij}\neq 0\}}[0,a_i\eta_1]_i \cup \coprod_{\{i|\beta_{ij}\neq 0\}}[b_i\eta_1,1]_i$, we can define h_j corresponding to X_j and W_j in $H(\eta_1)$, then $\phi(h_j), \psi(h_j)$ are projections and

$$\phi(h_j) = \phi'(h_j), \quad \psi(h_j) = \psi'(h_j), \quad \|\phi(h_j) - \psi(h_j)\| < 1,$$

for each $j = 1,2,\dots,p$, this fact means that

$$\operatorname{Sp}\phi' \cap \operatorname{Sp}(F_1) = \operatorname{Sp}\psi' \cap \operatorname{Sp}(F_1).$$

Now we have $\operatorname{Sp}\phi' = \operatorname{Sp}\psi'$.

For each $x_k \in \operatorname{Sp}\phi \cap (0,1)_i$, define a continuous map

$$\gamma_k \colon \left[0,\tfrac{1}{3}\right] \longrightarrow \coprod_{i=1}^{l}[0,1]_i$$

with the following properties:

(i) $\gamma_k(0) = x_k;$

(ii) $$\gamma_k\left(\tfrac{1}{3}\right) = \begin{cases} 0_i & \text{if } x_k \in (0,\eta_1)_i \backslash X_i, \\ \pi(x_k) & \text{if } x_k \in X_i, \\ 1_i & \text{if } x_k \in (1-\eta_1,1)_i \backslash X_i; \end{cases}$$

(iii) $$\Im\gamma_k = \overline{B_{4\eta_1}(x_k)} = \Big\{x \in \coprod_{i=1}^{l}[0,1]_i; \operatorname{dist}(x,x_k) \le 4\eta_1\Big\}.$$

Define ϕ_t on $[0,\frac{1}{3}]$ by

$$\phi_t(f) = U^* \cdot \operatorname{diag}\big(a(\theta_1)^{\sim s_1},\dots,a(\theta_p)^{\sim s_p}, f(\gamma_1(x)), f(\gamma_2(x)),\dots, f(\gamma_\bullet(x))\big) \cdot U.$$

Then $\phi_{\frac{1}{3}} = \phi'$ and

$$\|\phi(h) - \phi'(h)\| < \frac{\varepsilon'}{8}, \quad \forall\, h \in H(\eta) \cup \widetilde{H}(\eta).$$

Similarly, for each $y_k \in \operatorname{Sp}\psi \cap (0,1)_i$, define a continuous map

$$\gamma'_k \colon \left[\frac{2}{3},1\right] \longrightarrow \coprod_{i=1}^{l}[0,1]_i$$

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with the following properties:

(i) $$\gamma_k'\Big(\frac{2}{3}\Big) = \begin{cases} 0_i & \text{if } y_k \in (0,\eta_1)_i \backslash X_i', \\ y_k & \text{if } y_k \in X_i', \\ 1_i & \text{if } y_k \in (1-\eta_1,1)_i \backslash X_i'; \end{cases}$$

(ii) $$\gamma_k'(1) = y_k;$$

(iii) $$\Im \gamma_k' = \overline{B_{4\eta_1}(y_k)} = \Big\{ y \in \coprod_{i=1}^{l} [0,1]_i; \operatorname{dist}(y, y_k) \le 4\eta_1 \Big\}.$$

Define ϕ_t on $[\frac{2}{3},1]$ by

$$\phi_t(f) = V^* \cdot \operatorname{diag}\big(a(\theta_1)^{\sim t_1}, \dots, a(\theta_p)^{\sim t_p}, f(\gamma_1'(y)), f(\gamma_2'(y)), \dots, f(\gamma_{\bullet\bullet}'(y)) \big) \cdot V.$$

Then $\phi_{\frac{2}{3}} = \psi'$, and

$$\|\psi(h) - \psi'(h)\| < \frac{\varepsilon'}{8}, \qquad \forall h \in H(\eta) \cup \widetilde{H}(\eta).$$
$$\|\phi'(h) - \psi'(h)\| < \frac{\varepsilon'}{8} + \frac{\varepsilon'}{8} + \frac{\varepsilon'}{8} < \frac{\varepsilon'}{2}, \qquad \forall h \in H(\eta) \cup \widetilde{H}(\eta).$$

Apply Lemma 3.4; then there is a continuous path of homomorphisms $\phi_t: A \to B$, $t \in [\frac{1}{3}, \frac{2}{3}]$, such that $\phi_{\frac{1}{3}} = \phi'$, $\phi_{\frac{2}{3}} = \psi'$ and

$$\|\phi_t(f) - \phi'(f)\| < \frac{\varepsilon}{2}, \quad \forall f \in F.$$

Now we have a continuous path of homomorphisms $\phi_t: A \to B$ such that $\phi_0 = \phi$, $\phi_1 = \psi$ and $\|\phi_t(f) - \phi(f)\| < \varepsilon$ for all $f \in F$, $t \in [0,1]$.

From property (iii) of γ_k and γ_k', for any $y \in (\operatorname{Sp}\phi \cup \operatorname{Sp}\psi) \cap \coprod_{i=1}^{l} (0,1)_i$, we have

$$\overline{B_{4\eta_1}(y)} \subset \bigcup_{t \in [0,1]} \operatorname{Sp}\phi_t.$$

where $\overline{B_{4\eta_1}(y)} = \{x \in \coprod_{i=1}^{l} [0,1]_i : \operatorname{dist}(x,y) \le 4\eta_1\}$. ∎

Theorem 3.6 *Let $A, B \in \mathcal{C}$, let $F \subset A$ be a finite subset, let $Y \subset \operatorname{Sp}(B)$ be a closed subset, and let $G \subset B|_Y$ be a finite subset. Let $\phi: A \to B|_Y$ be a unital injective homomorphism; then for any $\varepsilon > 0$, there exist a closed subset $Z \subset Y$ and a unital injective homomorphism $\psi: A \to B|_Z$ such that*

(i) $\|\phi(f) - \psi(f)\| < \varepsilon$, $\forall f \in F$;
(ii) $G \subset_\varepsilon B|_Z \in \mathcal{C}$.

Proof Set $n = L(B)$, choose $\varepsilon', \eta, \eta_1$ as in Lemma 3.5; then there exists $\delta > 0$ such that for any $\operatorname{dist}(y, y') < \delta$, we have the following:

$$\|\phi_y(h) - \phi_{y'}(h)\| < 1 \qquad \forall h \in H(\eta_1),$$
$$\|\phi_y(h) - \phi_{y'}(h)\| < \frac{\varepsilon'}{8} \qquad \forall h \in H(\eta) \cup \widetilde{H}(\eta),$$
$$\|g(y) - g(y')\| < \varepsilon \qquad \forall g \in G.$$

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Applying Lemma 3.3, we can obtain a closed subset Z and a surjective map $\rho\colon Y \to Z$ such that $G \subset_{\varepsilon} B|_Z \in \mathcal{C}$.

We will define an injective homomorphism $\psi\colon A \to B|_Z$ as follows.

Recall the construction of $\widehat{Y}$ and $P(Z)$ in 3.1. Let $P(Z) = \{z_1, z_2, \dots\}$ be the points corresponding to the finite points $\{y_1, y_2, \dots\} = \widehat{Y}$. Define

$$\psi_{z_k}(f) = \psi_{\rho(y_k)}(f) = \phi_{y_k}(f), \quad \forall f \in A, \quad z_k \in \{z_1, z_2, \dots\}.$$

For each adjacent pair $\{(y_s, i), (y_t, i)\}$, if $(y_s, y_t)_i \cap Y$ has at most countably many points, then $(z_s, z_t)_i \cap Z = \varnothing$, and we do not need to define ψ on $(z_s, z_t)_i$, if $(y_s, y_t)_i \cap Y$ has uncountable many points, then we have $\operatorname{dist}((y_s, i), (y_t, i)) < \delta$ and $[z_s, z_t]_i \subset Z$. Then by Lemma 3.5, we can define ψ on $[z_s, z_t]_i$ and

$$\|\psi_z(f) - \phi_{(y_s, i)}(f)\| < \varepsilon, \quad \forall f \in F, \quad \forall z \in [z_s, z_t]_i.$$

Applying the above procedure to all adjacent pairs in $\widehat{Y}$, we can define ψ on each $[z_s, z_t]_i \subset Z$ piece by piece, then we obtain ψ on $Z \cap \coprod_{i=1}^{l} [0,1]_i$. For each $\theta_j \in Z \cap \operatorname{Sp}(F_1)$, define $\psi_{\theta_j}(f) = \phi_{\theta_j}(f)$ for all $\theta_j \in Y \cap \operatorname{Sp}(F_1)$. Then we have defined ψ on Z and ψ satisfies property (i).

To prove ψ is injective, we only need to verify that $\operatorname{Sp}\psi = \bigcup_{z \in Z} \operatorname{Sp}\psi_z = \operatorname{Sp}(A)$. The proof is similar to the corresponding part of [10].

Write $A = \bigoplus_{k=1}^{m} A_k$ with all A_k minimal. Then $\operatorname{Sp}(A) = \coprod_{k=1}^{m} \operatorname{Sp}(A_k)$. Define an index set $\Lambda \subset \{1, 2, \dots, m\}$ such that A_k is a finite dimensional C*-algebra if and only if $k \in \Lambda$. For $k \in \Lambda$, $\phi|_{A_k} \neq 0$ means that $\operatorname{Sp}(A_k) \subset \operatorname{Sp}\phi$, by the definition of ψ, we have $\psi|_{A_k} \neq 0$, then $\operatorname{Sp}(A_k) \subset \operatorname{Sp}\psi$.

Consider $\widetilde{A} = \widetilde{A}(\widetilde{F}_1, \widetilde{F}_2, \widetilde{\varphi}_0, \widetilde{\varphi}_1) = \bigoplus_{k \notin \Lambda} A_k$. We define two sets $Y', Y'' \subset Y$, for each adjacent pair $\{(y_s, i), (y_t, i)\}$. If $(y_s, y_t)_i \cap Y$ has at most countably many points, let $(y_s, y_t)_i \cap Y \subset Y'$. If $(y_s, y_t)_i \cap Y$ has uncountably many points, let $[y_s, y_t]_i \cap Y \subset Y''$. Then we have $Y' \cap Y'' = \varnothing$ and $Y' \cup Y'' = Y \cap \coprod_{i=1}^{l} [0,1]_i$. Note that Y' has at most countably many points.

For any point $x_0 \in \coprod_{i=1}^{l} (0,1)_i$ and $\overline{B_{\eta_1}(x_0)} = \{x \in \operatorname{Sp}(\widetilde{A}) : \operatorname{dist}(x, x_0) \leq \eta_1\}$, $\overline{B_{\eta_1}(x_0)} \cap (\bigcup_{y \in Y'} \operatorname{Sp}\phi_y)$ have at most countably many points. Following the injectivity of ϕ, we have

$$\overline{B_{\eta_1}(x_0)} \subset \operatorname{Sp}\phi = \bigcup_{y \in Y''} \operatorname{Sp}\phi_y \cup \bigcup_{y \in Y'} \operatorname{Sp}\phi_y \cup \bigcup_{y \in Y \cap \operatorname{Sp}(\widetilde{F}_1)} \operatorname{Sp}\phi_y.$$

Then the set $\bigcup_{y \in Y''} \operatorname{Sp}\phi_y \cap \overline{B_{\eta_1}(x_0)}$ has uncountably many points. Recall the definition of Y''; there is at least one adjacent pair $\{(y_s, i), (y_t, i)\}$ such that $[y_s, y_t]_i \cap Y$ has uncountably many points. Then we have ψ defined on $[z_s, z_t]_i \subset Z$.

Choose

$$x_1 \in \bigcup_{y \in [y_s, y_t]_i \cap Y''} \operatorname{Sp}\phi_y \cap \overline{B_{\eta_1}(x_0)};$$

then there exists $x_2 \in \operatorname{Sp}\phi_{(y_s, i)}$ such that $\operatorname{dist}(x_1, x_2) \leq 2\eta_1$. We have

$$\operatorname{dist}(x_0, x_2) \leq \operatorname{dist}(x_0, x_1) + \operatorname{dist}(x_1, x_2) \leq 3\eta_1 < 4\eta_1.$$

By Lemma 3.5, we will have

$$x_0 \in \overline{B_{4\eta_1}(x_2)} \subset \bigcup_{z \in [z_s, z_t]_i} \operatorname{Sp}\psi_z.$$

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This means that $\coprod_{i=1}^{l}(0,1)_i \subset \operatorname{Sp}\psi$.

Note that, if we choose x_0 such that $x_0 \in \coprod_{i=1}^{l}(0,\eta_1)_i \cup (\eta_1,1)_i$, then we will have $0_i, 1_i \in \operatorname{Sp}\psi$ for all $i \in \{1,2,\dots,l\}$, this means that $\operatorname{Sp}(\widetilde{F}_1) \subset \operatorname{Sp}\psi$.

Now we have

$$\operatorname{Sp}\psi = \bigcup_{z\in Z} \operatorname{Sp}\psi_z = \operatorname{Sp}(\widetilde{A}) \cup \coprod_{k\in\Lambda} \operatorname{Sp}(A_k) = \operatorname{Sp}(A).$$

This ends the proof of the injectivity of ψ. ∎

Remark 3.7 Theorem 3.6 still holds if we let ϕ be non-unital, then the homomorphism ψ will also be non-unital.

Proof of [10, Theorem 3.1] Let $\widetilde{A}_n = \phi_{n,\infty}(A_n)$, $n = 1,2,\dots$. Then we can write $A = \lim_{n\to\infty}(\widetilde{A}_n, \tilde{\phi}_{n,m})$, where the homomorphism $\widetilde{\phi}_{n,m}$ are induced by $\phi_{n,m}$, and they are injective.

Let $\varepsilon_n = \frac{1}{2^n}$, $\{x_i\}_{i=1}^{\infty}$ be a dense subset of A. We will construct an injective inductive limit $B_1 \to B_2 \to \cdots$ as follows.

Consider $G_1 = x_1 \subset A$. There is an $\widetilde{A}_{i_1}$, and a finite subset $\tilde{G}_1 \subset \widetilde{A}_{i_1}$ such that $G_1 \subset_{\frac{\varepsilon_1}{2}} \widetilde{G}_{i_1}$.

For $\widetilde{G}_1 \subset \widetilde{A}_{i_1}$, apply Lemma 3.3; there exists a sub-algebra $B_1 \subset \widetilde{A}_{i_1}$ such that $B_1 \in \mathcal{C}$ and $\widetilde{G}_1 \subset_{\frac{\varepsilon_1}{2}} \widetilde{B}_1$. This give us an injective homomorphism $B_1 \hookrightarrow \widetilde{A}_{i_1}$. Let $\{b_{1j}\}_{j=1}^{\infty}$ be a dense subset of B_1. Set $\widetilde{F}_1 = \{b_{11}\} \subset B_1$ and $G_2 = \{x_1, x_2\} \subset A$. There exist $\tilde{A}_{i_2}$, $i_2 > i_1$ and a finite subset $\widetilde{G}_2 \subset \widetilde{A}_{i_2}$ such that $G_2 \subset_{\frac{\varepsilon_2}{2}} \widetilde{G}_2$. Apply Theorem 3.6 and Remark 3.7 to $\widetilde{F}_1 \subset B_1$, $\widetilde{G}_2 \subset \widetilde{A}_{i_2}$, and the injective map $B_1 \hookrightarrow \widetilde{A}_{i_1} \to \widetilde{A}_{i_2}$; there exist a sub-algebra $B_2 \subset \widetilde{A}_{i_2}$ and an injective homomorphism $\psi_{1,2}\colon B_1 \to B_2$ such that $\widetilde{G}_2 \subset_{\frac{\varepsilon_2}{2}} \widetilde{B}_2$ and such that the diagram

$$\begin{array}{ccc} \widetilde{A}_{i_1} & \xrightarrow{\widetilde{\phi}_{i_1,i_2}} & \widetilde{A}_{i_2} \\ \uparrow & & \uparrow \\ B-1 & \xrightarrow{\widetilde{\phi}_{i_1,i_2}} & B_2 \end{array}$$

almost commutes on $\widetilde{F}_1$ to within ε_1. Let $\{b_{2j}\}_{j=1}^{\infty}$ be a dense subset of B_2. Choose

$$\widetilde{F}_2 = \{b_{21}, b_{22}\} \cup \{\psi_{1,2}(b_{11}), \psi_{1,2}(b_{12})\}, \quad G_3 = \{x_2, x_2, x_3\}$$

in the place of $\widetilde{F}_1$ and G_2 respectively, and repeat the above construction to obtain $\widetilde{A}_{i_3}$, $B_3 \subset \widetilde{A}_{i_3}$ and an injective map $\psi_{2,3}\colon B_2 \to B_3$ (using ε_2 and ε_3 in place of ε_1 and ε_2, respectively).

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In general, we can construct the diagram

$$\begin{array}{ccccccccc}
\widetilde{A}_{i_1} & \xrightarrow{\widetilde{\phi}_{i_1,i_2}} & \widetilde{A}_{i_2} & \xrightarrow{\widetilde{\phi}_{i_2,i_3}} & \widetilde{A}_{i_3} & \longrightarrow \cdots & \widetilde{A}_{i_k} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
B_1 & \xrightarrow{\psi_{1,2}} & B_2 & \xrightarrow{\psi_{2,3}} & B_3 & \longrightarrow \cdots & B_k & \longrightarrow & \cdots
\end{array}$$

with the following properties:

(i) The homomorphism $\psi_{k,k+1}$ are injective;

(ii) For each k, $G_k = \{x_1, x_2, \dots, x_k\} \subset_{\varepsilon_k} \widetilde{\phi}_{i_k,\infty}(B_k)$, where B_k is considered to be a sub-algebra of $\widetilde{A}_{i_k}$;

(iii) The diagram

$$\begin{array}{ccc}
\widetilde{A}_{i_k} & \xrightarrow{\widetilde{\phi}_{i_k,i_{k+1}}} & \widetilde{A}_{i_{k+1}} \\
\uparrow & & \uparrow \\
B_k & \xrightarrow{\psi_{k,k+1}} & B_{k+1}
\end{array}$$

almost commutes on $\widetilde{F}_k = \{b_{ij}; 1 \le i \le k, 1 \le j \le k\}$ to within ε_k, where $\{b_{ij}\}_{j=1}^{\infty}$ is a dense subset of B_i.

Then by [3, 2.3 and 2.4], the above diagram defines a homomorphism from $B = \varinjlim(B_n, \psi_{n,m})$ to $A = \varinjlim(\widetilde{A}_n, \widetilde{\phi}_{n,m})$. It is routine to check that the homomorphism is in fact an isomorphism. This concludes the proof. ∎

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