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A class of Hessian quotient equations in de Sitter space

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Abstract. In this paper, we consider the closed spacelike solution to a class of Hessian quotient equations in de Sitter space. Under mild assumptions, we obtain an existence result using standard degree theory based on a priori estimates.

1 Introduction

Let M be an n-dimensional closed spacelike hypersurface embedded in some semi-Riemannian manifold $(N, \bar{g})$. With respect to the particular choice of unit normal, the Weingarten transformation $\mathcal{W}$ assigns to every point on M a symmetric matrix which measures the change rate of that unit normal there. The eigenvalues $(\kappa_1, \kappa_2, \ldots, \kappa_n)$ of Weingarten transformation $\mathcal{W}$ are usually referred to as principal curvatures. One classical topic in geometric analysis is the existence and uniqueness of closed spacelike hypersurface embedded in N such that some particular function G in principal curvatures coincides with the prescribed smooth positive function Ψ along M. Such problems are well known as prescribed curvature problems.

Assume further that the ambient space N is a product manifold $(a, b) \times \mathbb{S}^n$ with its fundamental structure

$$\bar{g} = \varepsilon\, dr^2 + \phi^2(r)\sigma, \tag{1.1}$$

where $\varepsilon = 1$ or -1, ϕ is a smooth positive function defined on $(a, b) \subset \mathbb{R}$ and $(\mathbb{S}^n, \sigma)$ is the standard spherical space. As is well known, the Euclidean space, the spherical space, and the hyperbolic space can be constructed in this way. In fact, the de Sitter space, or equivalently the Lorentzain space form of constant sectional curvature 1, can also be recovered in this manner. Let $\mathbb{L}^{n+2}$ be the Lorent–Minkowski space, the vector space $\mathbb{R}^{n+2}$ endowed with the structure

$$(x_1, x_2, \ldots, x_{n+2}) \cdot (y_1, y_2, \ldots, y_{n+2}) = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+2} y_{n+2}.$$

The de Sitter space of dimension $(n+1)$ can be parameterized by $\mathbb{X} : \mathbb{R} \times \mathbb{S}^n \to \mathbb{L}^{n+2}$,

$$\mathbb{X}(r, \zeta) = \sinh(r)\mathbb{E}_1 + \cosh(r)\Theta, \tag{1.2}$$

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where $\mathbb{E}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+2}$ and $\Theta = (0, \zeta)$ with $\zeta \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Let $\mathbb{S}^n_1$ be the upper branch of the de Sitter space, or equivalently be the product manifold

$$(0, +\infty) \times \mathbb{S}^n,$$

equipped with the induced Lorentzian structure

$$\bar{g} = -dr^2 + \phi^2(r)\sigma, \tag{1.3}$$

where $\phi(r) = \cosh r$. More generally, the standard sphere $(\mathbb{S}^n, \sigma)$ can be replaced by some other compact n-dimensional Riemannian manifold, and recover the GRW (generalized Robertson–Walker) spacetime by requiring $\varepsilon = -1$ in (1.1).

Assume also that the spacelike hypersurface M concerned can be written as the graph of some smooth function $\rho : \mathbb{S}^n \to (a, b)$, in other words,

$$M = \left\{ \big(\rho(\zeta), \zeta\big) \mid \zeta \in \mathbb{S}^n \right\} \subset N.$$

Under such circumstance, the support function is defined as

$$u = \varepsilon\, \bar{g}(\phi \partial_r, \nu) > 0, \tag{1.4}$$

where ν is the outward or the future-directed unit normal along M, which is same as that in introducing the Weingarten transformation $\mathcal{W}$. One major difference between spacelike hypersurface in Riemannian space $(\varepsilon = 1)$ and that in Lorentzian space $(\varepsilon = -1)$ is their unit normals belong, respectively, to the compact manifold $\mathbb{S}^n$ and the complete but non-compact hyperbolic space $\mathbb{H}^n$. For this reason, the prescribed function Ψ along spacelike hypersurface may depend more generally on the position vector and the normal vector in Riemannian setting but is more convenient to depend on the position vector and the support function when ambient space is Lorentzian.

It will be clear that the principal curvatures of spacelike hypersurface M can be completely determined by the covariant derivatives, up to second order, of the function ρ, while the position vector, the normal vector, and the support function depend only on the covariant derivatives, up to first order, of the function ρ. In this sense, the prescribed curvature problem can be reduced to, with a slight abuse of notations, the scalar partial differential equation of second order

$$G(D^2\rho, D\rho, \rho, \zeta) = \Psi(D\rho, \rho, \zeta), \tag{1.5}$$

where $D\rho$ and $D^2\rho = [\rho_{ij}]$ are, respectively, the gradient and the Hessian with respect to the spherical metric σ. More generally, we may consider the function G in principal curvatures as defined on $\mathcal{S}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^n$, where $\mathcal{S}^{n\times n}$ is the set of $n \times n$ symmetric matrices.

For each given function G, we are mainly concerned with the particular class of spacelike hypersurfaces, which are referred to as G-admissible ones, such that

$$\frac{\partial G}{\partial \rho_{ij}}(D^2\rho, D\rho, \rho, \zeta) > 0, \quad \forall \quad \zeta \in \mathbb{S}^n. \tag{1.6}$$

Now equation (1.5) is elliptic, to which we may apply the standard PDE theory to conclude its solvability. In particular, when $G = E_k/E_l$ for some $0 \le l < k \le n$, equation (1.5) will be elliptic for spacelike hypersurface whose principal curvatures belong to the cone

$$\Gamma_k = \{(\kappa_1, \kappa_2, \ldots, \kappa_n) \mid E_i(\kappa_1, \kappa_2, \ldots, \kappa_n) > 0,\ i = 1, \ldots, k\}$$

containing the positive cone

$$\Gamma_+ = \{(\kappa_1, \kappa_2, \ldots, \kappa_n) \mid \kappa_i > 0, \quad i = 1, \ldots, n\}.$$

Here, E_i is the normalized ith elementary symmetric polynomial and $E_0 = 1$.

In the case where $(N, \bar{g})$ is some Riemannian warped product with $\phi' > 0$, the particular function G in principal curvatures is E_k and the prescribed function Ψ depends either only on the position vector or on both the position vector and the unit normal of hypersurface M, the prescribed curvature problem has been settled, or partially settled by Andrade, Barbosa, and de Lira [1] and Chen, Li, and Wang [4] under mild assumptions on the prescribed function Ψ (see also [3, 11, 13–15, 18–21]).

In the case where $(N, \bar{g})$ is the cosmological spacetime, some Lorentzian warped product meets the timelike convergence condition, and the particular function G in principal curvatures is E_1, Huisken and Ecker [9], applying parabolic method, gave an affirmative answer to such a prescribed curvature problem under monotonicity assumption on the prescribed function Ψ which is assumed to depend only on the position vector (see also [12] for similar results). Recently, Ballesteros-Chávez, Klingenberg, and Lambert [2] has obtained the existence result in the case where $N = \mathbb{S}^n_1$ and the particular function G in principal curvatures is $E_k^{1/k}$ $(k = 1, \ldots, n)$, under structural assumptions on the prescribed function Ψ which depends on the position vector and the tilt function defined there.

In this paper, we shall investigate some prescribed curvature type problem in $\mathbb{S}^n_1$. In detail, on spacelike hypersurface M embedded in $\mathbb{S}^n_1$, we consider the function, also denoted as G, in instead of its principal curvatures but the eigenvalues, denoted as $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, of its first Newton transformation, or equivalently of the matrix

$$\eta_{ij} = Hg_{ij} - h_{ij},$$

where $[g_{ij}]$ and $[h_{ij}]$ are matrices determined, respectively, by the first and the second fundamental form on spacelike hypersurface, and H is the sum of the principal curvatures. Such kind of problems can be converted into some elliptic PDE of type (1.5) as well when M is (η, k)-convex, i.e., the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ belongs to the cone Γ_k.

In the case where N is the Euclidean space or more generally a Riemannian warped product with $\phi' > 0$ and the function G in $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is some elementary symmetric polynomial or quotient of different elementary symmetric polynomials, such prescribed curvature type problems have been addressed in [6] and [8] (see also [5, 7]).

To conclude the solvability of such prescribed curvature type problems in $\mathbb{S}^n_1$, the prescribed function $\Psi : \mathbb{S}^n_1 \times (1, +\infty) \to \mathbb{R}$ is required to satisfy the following.

Assumption 1.1 (1) *There exist* $0 < r_1 < r_2 < +\infty$ *such that*

$$\Psi\big(r, \zeta, \phi(r)\big) < (n-1)\tanh(r), \quad \forall \quad \zeta \in \mathbb{S}^n,\ r < r_1,$$
$$\Psi\big(r, \zeta, \phi(r)\big) > (n-1)\tanh(r), \quad \forall \quad \zeta \in \mathbb{S}^n,\ r > r_2;$$

(2) *for every* $\mathbb{X} \in \mathbb{S}_1^n$, *there hold the asymptotics*

$$0 \le \lim_{u \to +\infty} \frac{\Psi(\mathbb{X}, u)}{u\Psi_u(\mathbb{X}, u)} < 1,$$
$$0 \le \lim_{u \to +\infty} \frac{\|\bar{\nabla}\Psi(\mathbb{X}, u)\|}{u\Psi_u(\mathbb{X}, u)} < +\infty,$$

where $\bar{\nabla}$ *is the Levi-Civita connection determined by (1.3) while the norm* $\|\cdot\|$ *is with respect to* $dr^2 + \phi^2(r)\sigma$.

The main result in this paper is the following.

Theorem 1.2 *Let* $n \ge 2$, $k \ge 2$, $0 \le l \le k-2$ *and* $\Psi : \mathbb{S}_1^n \times (1, +\infty) \to \mathbb{R}$ *be a smooth positive function satisfying Assumption 1.1. Then there exists a closed spacelike* (η, k)*-convex hypersurface* M *embedded in* $\mathbb{S}_1^n$ *such that along which there holds the equation*

$$\left[\frac{E_k(\lambda)}{E_l(\lambda)}\right]^{\frac{1}{k-l}} = \Psi\big(\mathbb{X}, u\big). \tag{1.7}$$

The rest of this paper is organized as follows: In Section 2, we recall necessary formulae concerning the geometry of semi-Riemannian manifold and its spacelike hypersurface. In Section 3, we obtain the a priori estimates up to first order for solution of equation (1.7). In Section 4, we derive the curvature estimate, which is crucial for the uniform ellipticity of equation (1.7). In Section 5, we apply the degree theory to conclude the existence and uniqueness of solution of equation (1.7).

2 Preliminary

2.1 Semi-Riemannian manifold and its spacelike hypersurface

For the semi-Riemannian manifold $(N, \bar{g})$, we define its curvature tensor of type $(1,3)$ and that of type $(0,4)$, respectively, as

$$\bar{R}(X, Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$$

and

$$\bar{R}(W, Z, X, Y) = \bar{g}\big(\bar{R}(X, Y)Z, W\big).$$

Lengthy computation leads to the fact that $\mathbb{S}_1^n$ is of constant sectional curvature 1.

Let M be a closed spacelike hypersurface embedded in the semi-Riemannian manifold $(N, \bar{g})$, with its induced metric g and its Levi-Civita connection ∇. The decomposition of the covariant derivative with respect to $\bar{\nabla}$ is abided by the Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \ \ \forall\ X, Y \in TM,$$

where B is the vector-valued second fundamental form. Here and in the sequel, we do not distinguish a tangential vector field to M and its push-forward. Let ν be the outward or future-directed timelike unit normal along M, then we obtain the corresponding Weingarten transformation $\mathcal{W}$ and scalar-valued second fundamental form h, respectively, through

$$\bar{\nabla}_X \nu = \mathcal{W}(X) \quad \text{and} \quad B(X,Y) = -\varepsilon h(X,Y)\nu$$

such that

$$h(X,Y) = g\big(\mathcal{W}(X), Y\big).$$

Furthermore, we have the Gauss equation

$$\bar{R}(W,Z,X,Y) = R(W,Z,X,Y) - \varepsilon\big[h(Y,Z)h(X,W) - h(X,Z)h(Y,W)\big]$$

and the Codazzi equation

$$\bar{R}(\nu,Z,X,Y) = \big(\nabla_Y h\big)(X,Z) - \big(\nabla_X h\big)(Y,Z),$$

where $R(\cdot,\cdot,\cdot,\cdot)$ is the curvature tensor determined by g and

$$\big(\nabla_Y h\big)(X,Z) = Y(h(X,Z)) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z).$$

Obviously, for spacelike hypersurface embedded in a semi-Riemannian space form, such as $\mathbb{S}^n_1$, the tensor ∇h of type $(0,3)$ possesses complete symmetry.

For any symmetric $(0,2)$ tensor, say T, on spacelike hypersurface (M,g,∇), the law for interchanging the order when taking covariant derivative twice is

$$\big(\nabla^2 T\big)(U,V;X,Y) - \big(\nabla^2 T\big)(U,V;Y,X) = T\big(U, R(X,Y)V\big) + T\big(R(X,Y)U, V\big),$$

this is the Ricci identity.

Since every compact spacelike hypersurface embedded in $\mathbb{S}^n_1$ can be written as graph over $\mathbb{S}^n$, i.e., we may assume that there exists some $\rho : \mathbb{S}^n \to \mathbb{R}^+$ such that

$$M = \big\{\big(\rho(\zeta),\zeta\big) \mid \zeta \in \mathbb{S}^n\big\}.$$

In general, we may assume the spacelike hypersurface embedded in Riemannian or Lorentzian warped product (1.1) can be written as graph over $\mathbb{S}^n$. Corresponding to the natural frame $\{\partial_i\}_{i=1}^n$ on $\mathbb{S}^n$, there is one special tangential frame

$$\{\partial_i + \rho_i \partial_r,\ i = 1,2,\ldots,n\},$$

under which the induced metric on M is

$$g_{ij} = \phi^2(\rho)\sigma_{ij} + \varepsilon\rho_i\rho_j, \tag{2.1}$$

where $D\rho = (\rho_1,\ldots,\rho_n)$ is the gradient of function ρ with respect to the Levi-Civita connection D on $\mathbb{S}^n$. Since M is spacelike, the induced metric is a Riemannian one, which is equivalent to

$$1 + \varepsilon\phi^{-2}|D\rho|^2 > 0,$$

where $|D\rho|^2 = \sigma^{ij}\rho_i\rho_j$. And the inverse of the matrix $[g_{ij}]$ is

$$g^{ij} = \phi^{-2}\left(\sigma^{ij} - \varepsilon\frac{\rho^i\rho^j}{\phi^2 v^2}\right).$$

Here and in the sequel, Einstein's summation convention is adopted.

It is trivial to check that

$$\nu = \frac{1}{v}\left(\partial_r - \varepsilon\phi^{-2}D\rho\right)$$

is the outward or the future-directed timelike unit normal to graphical hypersurface, where $v = \sqrt{1+\varepsilon\phi^{-2}|D\rho|^2}$. And the support function

$$u = \varepsilon\bar{g}(\phi\partial_r, \nu) = \frac{\phi}{v} \tag{2.2}$$

depends only on ρ and its derivatives of first order and is obviously larger (smaller) than ϕ in Lorentzian (Riemannian) setting.

Now the scalar-valued second fundamental form determines a symmetric matrix with components

$$h_{ij} = \frac{1}{v}\left(\phi\phi'\sigma_{ij} - \varepsilon\rho_{,ij} + 2\varepsilon\phi^{-1}\phi'\rho_i\rho_j\right), \tag{2.3}$$

where $\rho_{,ij}$ is the covariant derivatives with respect to σ. From (2.1), it yields that the Christoffel symbols of g_{ij} and of σ_{ij} are interrelated, we conclude further that

$$h_{ij} = v\left(-\varepsilon\rho_{;ij} + \phi\phi'\sigma_{ij}\right), \tag{2.4}$$

where $\rho_{;ij}$ is the Hessian with respect to g. Notice that the second fundamental form contains the derivatives of ρ up to second order.

Under any adapted frame $\{\xi_1, \ldots, \xi_n, \nu\}$ along spacelike hypersurface embedded in the semi-Riemannian manifold $(N, \bar{g})$, the Gauss and Codazzi equations obtain their local expressions as

$$R_{ijkl} = \bar{R}_{ijkl} + \varepsilon(h_{ik}h_{jl} - h_{il}h_{jk})$$

and

$$h_{ij;k} - h_{ik;j} = \bar{R}(\nu, \xi_i, \xi_j, \xi_k),$$

respectively. Furthermore, applying the Gauss equation, the Ricci identity and the Codazzi equation, we obtain the following Simons-type identity:

$$\begin{aligned} h_{ij;kl} = h_{kl;ij} &+ \varepsilon(h_i^m h_{mj}h_{kl} - h_i^m h_{ml}h_{kj} + h_k^m h_{mj}h_{il} - h_k^m h_{ml}h_{ij}) \\ &+ h_i^m \bar{R}_{mkjl} + h_j^m \bar{R}_{mkil} + h_k^m \bar{R}_{mijl} + h_l^m \bar{R}_{mijk} \\ &+ \varepsilon h_{kl}\bar{R}(\nu, \xi_i, \nu, \xi_j) - \varepsilon h_{ij}\bar{R}(\nu, \xi_k, \nu, \xi_l) \\ &+ (\bar{\nabla}_{\xi_l}\bar{R})(\nu, \xi_i, \xi_j, \xi_k) + (\bar{\nabla}_{\xi_j}\bar{R})(\nu, \xi_k, \xi_i, \xi_l). \end{aligned} \tag{2.5}$$

Let Φ be a primitive of function ϕ, then

$$\Phi_{;ij} = \varepsilon(\phi' g_{ij} - uh_{ij}). \tag{2.6}$$

Since the vector field $\phi\partial_r$ is conformal Killing with respect to $\bar{g}$, then

$$\nabla u = \mathcal{W}(\nabla\Phi),$$

which is locally equivalent to

$$u_{;i} = h_i^k \Phi_{;k}, \tag{2.7}$$

where $\nabla\Phi$ is the tangential part of $\bar{\nabla}\Phi = \varepsilon\phi\partial_r$, and furthermore

$$u_{;ij} = \varepsilon\phi' h_{ij} + g(\nabla\Phi, \nabla h_{ij}) - \varepsilon u h_{ik} h_j^k + \bar{R}(\nu, \xi_i, \nabla\Phi, \xi_j). \tag{2.8}$$

2.2 Elementary symmetric polynomials

For spacelike smooth hypersurface embedded in some semi-Riemannian manifold, its eigenvalues of the first Newton transformation and of the Weingarten transformation satisfy

$$\lambda_i = \sum_{j\neq i} \kappa_j, \quad \forall \quad i = 1, \ldots, n.$$

For $\lambda := (\lambda_1, \ldots, \lambda_n)$, its kth normalized elementary symmetric polynomial is defined as

$$E_k(\lambda) = \binom{n}{k}^{-1} \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1} \ldots \lambda_{i_k},$$

where $\binom{n}{k}$ is the combinatorial number. Let

$$G(\lambda) = \left[\frac{E_k(\lambda)}{E_l(\lambda)}\right]^{\frac{1}{k-l}},$$

then G is homogeneous of degree one such that $G(1, \ldots, 1) = 1$ and is also monotone increasing and concave in Γ_k. And the function G will be considered occasionally as in symmetric matrix $\eta = [\eta_{ij}]$ such that

$$G^{ij,pq}(\eta)\mathcal{S}_{ij}\mathcal{S}_{pq} = \sum_{i,j} \frac{\partial^2 G}{\partial\lambda_i \partial\lambda_j}(\lambda)\mathcal{S}_{ii}\mathcal{S}_{jj} + \sum_{i\neq j} \frac{G^i(\lambda) - G^j(\lambda)}{\lambda_i - \lambda_j}\mathcal{S}_{ij}^2, \tag{2.9}$$

where $(\mathcal{S}_{ij})$ is any symmetric matrix and

$$G^{ij,pq}(\eta) = \frac{\partial^2 G}{\partial\eta_{ij}\partial\eta_{pq}}(\eta) \quad \text{and} \quad G^i(\lambda) = \frac{\partial G}{\partial\lambda_i}(\lambda).$$

For convenience, we introduce the following notations:

$$G^{ij} := \frac{\partial G}{\partial\eta_{ij}}, \quad F^{ii} := \sum_{j\neq i} G^{jj}.$$

Indeed, $G(\lambda)$ is monotone increasing in Γ_k implies that $G(\eta)$ is elliptic, or the matrix $G^{ij}(\eta)$ is positive definite if the eigenvalues of matrix η belongs to Γ_k. Several formulae are stated in the following lemma in consideration of their great importance and interest.

Lemma 2.1

$$\sum_{i=1}^n F^{ii} = (n-1)\sum_{i=1}^n G^{ii},$$

$$\sum_{i=1}^n F^{ii} h_{ii} = \sum_{i=1}^n G^{ii}\eta_{ii},$$

and for every $p \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{n} F^{ii} h_{ii;p} = \sum_{i=1}^{n} G^{ii} \eta_{ii;p}.$$

From now on, we will assume that k is an integer larger than or equal to 2 and M is a compact spacelike (η, k)-convex hypersurface embedded in $\mathbb{S}_1^n$, such that along which (1.7) holds, then

$$H(\kappa) = \frac{n}{n-1} E_1(\lambda) > 0. \tag{2.10}$$

3 C^0 and gradient estimates

Suppose the function $\rho : \mathbb{S}^n \to \mathbb{R}^+$, corresponding to M, attains its maximum $\rho_{\max}$ at $\zeta_0 \in \mathbb{S}^n$, then

$$D\rho(\zeta_0) = 0, \quad D^2\rho(\zeta_0) \le 0.$$

From (2.1) and (2.4), it yields consequently that at the considered point, Weingarten transformation $\mathcal{W}$ satisfies

$$h_j^i \le \tanh(\rho_{\max}) \delta_j^i. \tag{3.1}$$

In other words, the principal curvatures there are less than or equal to $\tanh(\rho_{\max})$, which implies further that

$$\lambda_i \le (n-1) \tanh(\rho_{\max}), \quad \forall \quad i = 1, \ldots, n. \tag{3.2}$$

Since G is now monotone increasing and homogeneous of degree one, then

$$G(\lambda) \le (n-1) \tanh(\rho_{\max}).$$

Similarly, at the point where ρ attains its minimum $\rho_{\min}$,

$$G(\lambda) \ge (n-1) \tanh(\rho_{\min}).$$

From the first item in Assumption 1.1, we can conclude that

$$r_1 \le \rho \le r_2. \tag{3.3}$$

Now, due to (2.2) and (3.3), the gradient estimate for function ρ follows directly from the upper bound for support function u. Noticing also that the support function u is now naturally bounded from below by $\phi(r_1) > 1$. To obtain its upper bound, we apply the maximum principle to the auxiliary function

$$\varphi = \ln u - \gamma(\Phi),$$

where $\gamma(\Phi) = \beta \ln \Phi$ with $\beta > 1$ a constant to be determined. Without loss of generality, we may assume that $\Phi > 1$.

At the point where φ attains its maximum, there holds

$$\nabla \ln u = \gamma'(\Phi) \nabla \Phi, \tag{3.4}$$

while the Hessian

$$\varphi_{;ij} = -(\ln u)_{;i}(\ln u)_{;j} + \frac{1}{u}u_{;ij} - \gamma''(\Phi)\Phi_{;i}\Phi_{;j} - \gamma'(\Phi)\Phi_{;ij}$$

is nonpositive definite.

We may assume further that $\phi\partial_r = -\bar{\nabla}\Phi$ is not parallel to ν at the considered point, since otherwise the proof is done. Hence, we may choose the particular local orthonormal frame $\{\xi_i\}_{i=1}^n$ such that

$$g(\nabla\Phi, \xi_1) = |\nabla\Phi| \neq 0, \quad g(\nabla\Phi, \xi_j) = 0, \quad \forall \quad j = 2, \ldots, n,$$

where the norm $|\cdot|$ is with respect to g. From (3.4) and (2.7), it follows then

$$h_1^1 = u\gamma'(\Phi), \quad h_j^1 = 0, \quad \forall \quad j = 2, \ldots, n.$$

And after rotating $\xi_2, \ldots, \xi_n$ if necessary, we may assume further that $[h_{ij}]$ and then $[\eta_{ij}]$ are diagonal. Consequently at the considered point, there holds

$$\begin{aligned} 0 &\geq F^{ii}\varphi_{;ii} \\ &\geq F^{11}\big[\gamma''(\Phi) + \gamma'^2(\Phi)\big]\phi^2 - \phi'\frac{\Psi}{u} + \frac{1}{u}g(\nabla\Phi, \nabla\Psi) + \sum_{j=2}^n F^{jj}h_{jj}^2 \\ &\quad - \gamma'(\Phi)u\Psi + (n-1)\phi'\gamma'(\Phi)\sum_{i=1}^n G^{ii} - \gamma''(\Phi)F^{11}u^2, \end{aligned}$$

where we also used (2.6), (2.7), (2.8) and the basic fact

$$|\nabla\Phi|^2 = u^2 - \phi^2. \tag{3.5}$$

Since

$$\nabla\Psi = \nabla^{\mathbb{X}}\Psi + \Psi_u\nabla u,$$

where $\nabla^{\mathbb{X}}\Psi$ stands for the tangential part of the ambient vector field

$$\bar{\nabla}\Psi := \phi^{-2}D\Psi + \varepsilon\Psi_r\partial_r$$

and Ψ_u stands for the partial derivative of Ψ with respect to u, then

$$\frac{1}{u}g(\nabla\Phi, \nabla\Psi) = \frac{1}{u}g(\nabla\Phi, \nabla^{\mathbb{X}}\Psi) + \Psi_u\gamma'(\Phi)|\nabla\Phi|^2,$$

where we used the critical point condition (3.4) again. On the other hand, the Cauchy–Schwarz inequality implies

$$|\nabla^{\mathbb{X}}\Psi|_g^2 \leq \frac{2}{v^2}\big[\phi^{-2}|D\Psi|_\sigma^2 + \Psi_r^2\big] = \frac{2}{v^2}\|\bar{\nabla}\Psi\|^2.$$

Finally, we can conclude that

$$\begin{aligned} 0 &\geq \big[\gamma''(\Phi) + \gamma'^2(\Phi)\big]F^{11}\phi^2 - \gamma''(\Phi)F^{11}u^2 - \sqrt{2}\phi^{-1}\|\bar{\nabla}\Psi\|\sqrt{u^2-\phi^2} \\ &\quad + \gamma'(\Phi)u\left(1 - \frac{\phi^2}{u^2}\right)u\Psi_u - u\Psi\left(\gamma'(\Phi) + \frac{\phi'}{u^2}\right) + (n-1)\phi'\gamma'(\Phi)\sum_{i=1}^n G^{ii}. \end{aligned} \tag{3.6}$$

Now we may choose β large enough such that $\gamma''(\Phi) + \gamma'^2(\Phi) > 0$. Furthermore, thanks to the second item in Assumption 1.1, contradiction will emerge in (3.6) once the support function u is permitted to approach to $+\infty$.

4 Curvature estimate

In this section, we derive a priori estimate for the squared norm of the second fundamental form

$$\kappa_1^2 + \cdots + \kappa_n^2.$$

Due to (2.10), it suffices to prove that the largest principal curvature is bounded from above along M.

We may introduce the auxiliary function

$$\omega = \ln \kappa_{\max} - 2A\Phi,$$

where $\kappa_{\max}$ stands for the largest principal curvature and $A > 1$ is a constant to be determined. In general, $\kappa_{\max}$ is merely a continuous function along M, for which the maximum principle is not available.

Assume that ω achieves its maximum at $P_0 \in M$. We may choose the local orthonormal frame $\{e_1, \ldots, e_n\}$ such that $h_{ij} = h_{ii}\delta_{ij}$ and $h_{11} \geq h_{22} \geq \cdots \geq h_{nn}$ at P_0. Since $h_{11}(P_0) = \kappa_{\max}(P_0)$ and $h_{11} \leq \kappa_{\max}$ around P_0, then with slight abuse of notation, the new auxiliary function

$$\omega = \ln h_{11} - 2A\Phi$$

achieves its maximum at P_0 as well. Now, the new auxiliary function ω is of higher regularity around P_0 and without loss of generality, we may assume that $h_{11}(P_0)$ approaches to $+\infty$. It follows then, at P_0,

$$\nabla \ln h_{11} = 2A\nabla\Phi, \tag{4.1}$$

and

$$0 \geq F^{ii}\big(\ln h_{11}\big)_{;ii} - 2AF^{ii}(uh_{ii} - \phi' g_{ii}), \tag{4.2}$$

where we used equation (2.6).

Step 1: Applying (2.5) to the first term in (4.2), it yields that

$$\begin{aligned} F^{ii}\big(\ln h_{11}\big)_{;ii} = {} & \frac{1}{h_{11}} F^{ii} h_{ii;11} + \sum_{i=1}^{n} F^{ii} + F^{ii} h_{ii}^2 \\ & - \frac{\Psi}{h_{11}} - \Psi h_{11} - F^{ii}(\ln h_{11})_{;i}^2. \end{aligned}$$

Noticing also that

$$\begin{aligned} \frac{1}{h_{11}} F^{ii} h_{ii;11} &= \frac{1}{h_{11}} G^{ii} \eta_{ii;11} = \frac{1}{h_{11}}\big(\Psi_{;11} - G^{pq,rs}\eta_{pq;1}\eta_{rs;1}\big) \\ &= u\Psi_u h_{11} + \Phi_{;1}^2 \Psi_{uu} h_{11} + \frac{d_{\mathbb{X}}\Psi(\nabla_{e_1}\nabla_{e_1}\mathbb{X})}{h_{11}} \end{aligned}$$

$$+ 2\Phi_{;1} d^2_{\mathbb{X}u}\Psi(\nabla_{e_1}\mathbb{X}) - \phi'\Psi_u + \frac{d^2_{\mathbb{X}\mathbb{X}}\Psi(\nabla_{e_1}\mathbb{X}, \nabla_{e_1}\mathbb{X})}{h_{11}}$$
$$+ g(\nabla\Phi, \nabla \ln h_{11})\Psi_u - \frac{1}{h_{11}} G^{pq,rs}\eta_{pq;1}\eta_{rs;1},$$

where we used (2.7), (2.8) and $d_{\mathbb{X}}\Psi$, $d^2_{\mathbb{X}u}\Psi$, and $d^2_{\mathbb{X}\mathbb{X}}\Psi$ can be viewed as matrices of different sizes. Now, when restricted to M, the prescribed function Ψ is smoothly defined on some compact subset in $\mathbb{S}^n_1 \times [1, +\infty)$. Therefore,

$$F^{ii}\big(\ln h_{11}\big)_{;ii} \geq -C_0 h_{11} - \frac{C_0}{h_{11}} - C_0 + g\big(\nabla\Phi, \nabla \ln h_{11}\big)\Psi_u + F^{ii}h^2_{ii} + \sum_{i=1}^n F^{ii}$$
$$- \frac{2}{h_{11}} \sum_{j=2}^n G^{1j,j1}h^2_{11;j} - F^{ii}(\ln h_{11})^2_{;i},$$

where C_0 is a sufficiently large constant depending on $\inf \rho$, $|\rho|_{C^1}$ and $||\Psi||_{C^2}$, and we used (2.9) to deduce that

$$-G^{pq,rs}\eta_{pq;1}\eta_{rs;1} \geq -2\sum_{j=2}^n G^{1j,j1}\eta^2_{1j;1} = -2\sum_{j=2}^n G^{1j,j1}h^2_{11;j}.$$

Using the critical point condition (4.1), we obtain finally that

$$0 \geq -C_0 h_{11} + F^{ii}h^2_{ii} + (1 + 2A\phi')\sum_{i=1}^n F^{ii}$$
$$- \frac{C_0}{h_{11}} - C_0 - 2AC_0 - \frac{2}{h_{11}}\sum_{j=2}^n G^{1j,j1}h^2_{11;j} - F^{ii}(\ln h_{11})^2_{;i}. \tag{4.3}$$

Step 2 : Let

$$A = \max\left\{\frac{2nC_0}{(n-1)(n-k+1)}\frac{||\Psi||^{k-l-1}_{C^0}}{\inf \phi'}, 1\right\}.$$

We claim that there exists a constant

$$B = \max\{2n\delta^{-2}C_0, 1\}$$

such that if $h_{11} \geq B$, then

$$F^{ii}h^2_{ii} + 2A\phi'\sum_{i=1}^n F^{ii} \geq 2C_0 h_{11}, \tag{4.4}$$

where $0 < \delta < 1$ is some constant such that

$$\frac{\binom{n-1}{k-1}[1-(n-2)\delta]^{k-1} - (n-1)\binom{n-1}{k-2}\delta[1+(n-2)\delta]^{k-2}}{[1+(n-2)\delta]^l} \geq \frac{\binom{n-1}{k-1}}{2}. \tag{4.5}$$

We split its proof into two cases.

Case 1: $|h_{jj}| \leq \delta h_{11}$ for all $j \geq 2$.

In this case, we have

$$|\eta_{11}| \leq (n-1)\delta h_{11}, \quad [1-(n-2)\delta]h_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn} \leq [1+(n-2)\delta]h_{11}.$$

By the definition of G^{ii} and F^{ii}, we obtain

$$\begin{aligned}\sum_{i=1}^{n} F^{ii} &= (n-1)\sum_{i=1}^{n} G^{ii}\\ &= \frac{n-1}{k-l}\left[\frac{E_k(\eta)}{E_l(\eta)}\right]^{\frac{1}{k-l}-1}\frac{kE_l(\eta)E_{k-1}(\eta)-lE_k(\eta)E_{l-1}(\eta)}{E_l^2(\eta)}\\ &\geq \frac{n-1}{\binom{n}{k-1}}\left[\frac{E_k(\eta)}{E_l(\eta)}\right]^{\frac{1}{k-l}-1}\frac{\sigma_{k-1}(\eta|1)+\eta_{11}\sigma_{k-2}(\eta|1)}{E_l(\eta)}\\ &\geq \frac{(n-1)(n-k+1)}{2n}\Psi^{1-k+l}h_{11}^{k-l-1},\end{aligned}$$

where we used the Newton–MacLaurin inequalities and (4.5). Hence, in this case,

$$2A\phi'\sum_{i=1}^{n} F^{ii} \geq 2C_0h_{11},$$

under the assumption that $0 \leq l \leq k-2$.

Case 2: Either $h_{22} > \delta h_{11}$ or $h_{nn} < -\delta h_{11}$.

In this case, we have

$$F^{ii}h_{ii}^2 \geq F^{22}h_{22}^2 + F^{nn}h_{nn}^2 \geq \delta^2F^{22}h_{11}^2.$$

Since $\eta_{ii} = H - h_{ii}$, then at the considered point

$$\eta_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn},$$

from which it follows directly that

$$G^{11} \geq G^{22} \geq \cdots \geq G^{nn} \quad \text{and} \quad F^{11} \leq F^{22} \leq \cdots \leq F^{nn}.$$

And there holds further that

$$F^{22} \geq \frac{1}{n}\sum_{i=1}^{n} G^{ii} \geq \frac{1}{n},$$

since otherwise

$$(n-1)\sum_{i=1}^{n} G^{ii} = \sum_{i=1}^{n} F^{ii} < \frac{2}{n}\sum_{i=1}^{n} G^{ii} + (n-2)\sum_{i=1}^{n} G^{ii}.$$

In this case, we can conclude that

$$F^{ii}h_{ii}^2 \geq \frac{\delta^2}{n}h_{11}^2 \geq 2C_0h_{11}.$$

Step 3: We show that if $h_{11} \geq B$, then for every $2 \leq j \leq n$,

$$|h_{jj}| \leq AC_1,$$

where $C_1(> C_0)$ is a positive constant depending on $n, k, l, \delta, \inf\rho, |\rho|_{C^1}$ and $||\Psi||_{C^2}$.

Combining the results obtained in *Steps 1* and *2*, we conclude that

$$
\begin{aligned}
0 \geq & -\frac{2}{h_{11}} \sum_{j=2}^{n} G^{1j,j1} h_{11;j}^2 - F^{ii} (\ln h_{11})_{;i}^2 - \frac{C_0}{h_{11}} - C_0 - 2AC_0 \\
& + \frac{1}{2} F^{ii} h_{ii}^2 + (1 + A\phi') \sum_{i=1}^{n} F^{ii}.
\end{aligned} \tag{4.6}
$$

From (4.1) and the concavity of G, it follows that

$$
\begin{aligned}
0 &\geq -4A^2 F^{ii} \Phi_{;i}^2 - 4AC_0 + \frac{1}{2} F^{ii} h_{ii}^2 + (1 + A\phi') \sum_{i=1}^{n} F^{ii} \\
&\geq \frac{1}{2} F^{ii} h_{ii}^2 - 4AC_0 - \left[4A^2 C_0 - (1 + A\phi') \right] \sum_{i=1}^{n} F^{ii} \\
&\geq \left\{ \frac{1}{2n(n-1)} \sum_{j=2}^{n} h_{jj}^2 - 4AC_0 - \left[4A^2 C_0 - (1 + A\phi') \right] \right\} \sum_{i=1}^{n} F^{ii}
\end{aligned}
$$

where we used again the fact that for $2 \leq j \leq n$,

$$
F^{jj} \geq \frac{1}{n} \sum_{i=1}^{n} G^{ii} = \frac{1}{n(n-1)} \sum_{i=1}^{n} F^{ii}.
$$

The desired estimate follows directly.

Step 4: We show that there exists a constant C_2 depending on $n, k, l, \delta, \inf \rho, |\rho|_{C^1}$ and $||\Psi||_{C^2}$ such that

$$
h_{11} \leq C_2.
$$

Without loss of generality, we may assume that

$$
h_{11} \geq \max \left\{ B, \frac{AC_1}{\alpha} \right\}, \tag{4.7}
$$

where $0 < \alpha < 1$ will be determined later.

Comparing with the result derived in *Step 3*, assumption (4.7) implies

$$
|h_{jj}| \leq \alpha h_{11}, \quad \text{for} \quad j \geq 2,
$$

and then

$$
\frac{1}{h_{11}} \leq \frac{1 + \alpha}{h_{11} - h_{jj}}, \quad \forall \quad j \geq 2.
$$

As a consequence,

$$
\begin{aligned}
\sum_{j=2}^{n} F^{jj} (\ln h_{11})_{;j}^2 &= \sum_{j=2}^{n} \frac{F^{jj} - F^{11}}{h_{11}^2} h_{11;j}^2 + \sum_{j=2}^{n} \frac{F^{11} h_{11;j}^2}{h_{11}^2} \\
&\leq \frac{1 + \alpha}{h_{11}} \sum_{j=2}^{n} \frac{F^{jj} - F^{11}}{h_{11} - h_{jj}} h_{11;j}^2 + \sum_{j=2}^{n} \frac{F^{11} h_{11;j}^2}{h_{11}^2}
\end{aligned}
$$

$$= \frac{1+\alpha}{h_{11}} \sum_{j=2}^{n} \frac{G^{11} - G^{jj}}{\eta_{jj} - \eta_{11}} h_{11;j}^2 + \sum_{j=2}^{n} \frac{F^{11} h_{11;j}^2}{h_{11}^2}$$
$$= -\frac{1+\alpha}{h_{11}} \sum_{j=2}^{n} G^{1j,j1} h_{11;j}^2 + \sum_{j=2}^{n} \frac{F^{11} h_{11;j}^2}{h_{11}^2}$$

where we used the basic fact that for every $2 \le j \le n$,

$$-G^{1j,j1} = \frac{G^{11} - G^{jj}}{\eta_{jj} - \eta_{11}}.$$

Hence, substituting this estimate back into (4.6), we can conclude that

$$0 \ge -F^{11} |\nabla \ln h_{11}|^2 - \frac{C_0}{h_{11}} - C_0 - 2AC_0 + C_0 h_{11}$$
$$= -4A^2 F^{11} |\nabla \Phi|^2 - \frac{C_0}{h_{11}} - C_0 - 2AC_0 + C_0 h_{11},$$

contradiction occurs once h_{11} is permitted to approach to $+\infty$, and simultaneously F^{11} approaches to 0.

5 Proof of Theorem 1.2

In this section, we use the degree theory for nonlinear elliptic equation developed by Y. Y. Li in [17] to prove Theorem 1.2.

We define

$$C_{\text{ad}}^{4,\alpha}(\mathbb{S}^n) = \left\{\rho \in C^{4,\alpha}(\mathbb{S}^n) : \text{the graph of } \rho \text{ is spacelike and } (\eta, k) - \text{convex}\right\},$$

and consider the family of functionals

$$\mathcal{F} : C_{\text{ad}}^{4,\alpha}(\mathbb{S}^n) \times [0,1] \to C^{2,\alpha}(\mathbb{S}^n)$$

defined by

$$\mathcal{F}(\rho, t) = G\big(\lambda(\rho)\big) - \Xi(\rho, \zeta, u, t),$$

where

$$\Xi(\rho, \zeta, u, t) = t\Psi(\rho, \zeta, u) + (1-t)\Theta(\rho, \zeta, u)$$

and

$$\Theta(\rho, \zeta, u) = (n-1) u^p \rho \tanh(\rho), \quad p > 1.$$

Claim 1 There exists an unique $\hat{\rho} \in C_{\text{ad}}^{4,\alpha}(\mathbb{S}^n)$ such that

$$G(\lambda(\hat{\rho})) = \Theta(\hat{\rho}, \zeta, u(\hat{\rho})).$$

In fact, when we choose $\hat{\rho} : \mathbb{S}^n \to \mathbb{R}$ to be of constant value, then the graphical hypersurface determined by $\hat{\rho}$ is spacelike and umbilical, with principal curvatures $\tanh(\hat{\rho})$ and support function $\cosh(\hat{\rho})$. Hence, $G(\lambda(\hat{\rho})) = (n-1)\tanh(\hat{\rho})$, and once there holds further $\hat{\rho}\cosh^p(\hat{\rho}) = 1$, then $\hat{\rho}$ is a solution. Clearly, such a $\hat{\rho}$ does exist, since the continuous function $f(x) = x\cosh^p(x)$ satisfies $f(0) = 0$ and $f(1) > 1$.

Suppose that there exists another $\rho \in C^{4,\alpha}_{\mathrm{ad}}(\mathbb{S}^n)$ such that $\mathcal{F}(\rho, 0) = 0$ and furthermore that $\max\rho = \rho(\zeta_0) > \hat{\rho}$. As in obtaining the C^0 estimates, we have

$$\begin{aligned}\Theta(\rho(\zeta_0), \zeta_0, u(\zeta_0)) &= (n-1)\cosh^p(\rho(\zeta_0))\rho(\zeta_0)\tanh(\rho(\zeta_0))\\ &= G(\lambda(\rho(\zeta_0))) \le (n-1)\tanh(\rho(\zeta_0)),\end{aligned}$$

which is a contradiction, as $f(x) = x\cosh^p(x)$ is a monotone increasing function. Therefore, $\max\rho \le \hat{\rho}$. The same argument implies $\min\rho \ge \hat{\rho}$, and finally $\rho \equiv \hat{\rho}$.

Claim 2 At $\hat{\rho}$ the linearization of $\mathcal{F}$ is invertible.

Since for the spacelike hypersurface determined by function ρ,

$$\begin{aligned}\eta^i_j(\rho) = \frac{n\phi'}{\phi v^2}\delta^i_j - \frac{\phi'}{\phi^3 v^4}\left(\phi^2\delta^i_j - \rho^i\rho_j\right)\\ + \frac{1}{\phi^2 v}\left[\delta^i_j\left(\sigma^{mq} + \frac{\rho^m\rho^q}{\phi^2 v^2}\right)\rho_{,mq} - \left(\sigma^{im} + \frac{\rho^i\rho^m}{\phi^2 v^2}\right)\rho_{,mj}\right],\end{aligned}$$

then

$$\delta_\rho\mathcal{F}(\vartheta, t) := \frac{d}{ds}\left[G(\rho + s\vartheta) - \Xi(\rho + s\vartheta, t)\right]\Big|_{s=0} = a^{ij}\vartheta_{,ij} + b^i\vartheta_i + c\vartheta, \tag{5.1}$$

where the matrix

$$a^{ij} = G^q_m\frac{\partial\eta^m_q}{\partial\rho_{,ij}} = \frac{1}{v}\left[\left(\sum_{m=1}^n G^m_m\right)g^{ij} - G^i_m g^{mj}\right]$$

is positive definite and

$$c = G^j_i\frac{\partial\eta^i_j(\rho)}{\partial\rho} - \frac{\partial\,\Xi(\rho, \zeta, t)}{\partial\rho}.$$

We are interested in (5.1) when $t = 0$, that is, when $\rho \equiv \hat{\rho}$. In this case, we have

$$u \equiv \cosh(\hat{\rho}), \quad \eta^i_j \equiv (n-1)\tanh(\hat{\rho})\delta^i_j, \quad G \equiv (n-1)\tanh(\hat{\rho}), \quad G^j_i \equiv \frac{1}{n}\delta^i_j,$$

and consequently when $t = 0$

$$c = -p(n-1)\hat{\rho}\cosh^{p-2}(\hat{\rho})\sinh^2(\hat{\rho}) - (n-1)\cosh^{p-1}(\hat{\rho})\sinh(\hat{\rho}) < 0,$$

where we used the condition that $\hat{\rho}\cosh^p(\hat{\rho}) = 1$.

From the strong maximum principle, it follows that the unique solution to $\delta_\rho\mathcal{F}(\cdot, 0)(\vartheta) = 0$ is $\vartheta \equiv 0$, or equivalently that $\ker(\delta_\rho\mathcal{F}) = \{0\}$ at $t = 0$. And the standard theory of second-order elliptic equations implies further that $\delta_\rho\mathcal{F}$ is invertible at $t = 0$, as required.

Claim 3 For all $t \in [0,1]$, the admissible solution ρ^t of $\mathcal{F}(\rho, t) = 0$, if any, satisfies

$$\underline{\rho} \le \rho^t \le \overline{\rho},$$

where $0 < \underline{\rho} < \overline{\rho}$ are constants independent of t.

In fact, there exist positive constants R_1 and R_2 such that

$$\Theta(r, \zeta, \cosh(r)) < (n-1)\tanh(r), \quad r < R_1,$$
$$\Theta(r, \zeta, \cosh(r)) > (n-1)\tanh(r), \quad r > R_2.$$

Setting $\underline{\rho} = \min\{r_1, R_1\}$ and $\overline{\rho} = \max\{r_2, R_2\}$, then for all $t \in [0,1]$,

$$\Xi(r, \zeta, \cosh(r), t) < (n-1)\tanh(r), \quad r < \underline{\rho},$$
$$\Xi(r, \zeta, \cosh(r), t) > (n-1)\tanh(r), \quad r > \overline{\rho}.$$

The claim follows then from the comparison principle. Furthermore, it is trivial to check that $\Theta(\rho, \zeta, u)$ satisfies as well the second item in Assumption 1.1. Following the same arguments as in Sections 3 and 4, we conclude that for $t \in [0,1]$, ρ^t is spacelike and equation $\mathcal{F}(\rho, t) = 0$ is uniformly elliptic. According to the regularity theory developed in [10] and [16], there is constant C depending only on $k, l, n, \underline{\rho}, \overline{\rho}$ and $\|\Psi\|_{C^{2,\alpha}(\mathbb{S}^n)}$ such that for all $t \in [0,1]$

$$u(\rho^t) < C \quad \text{and} \quad |\rho^t|_{C^{4,\alpha}(\mathbb{S}^n)} < C.$$

Moreover, there are positive constants μ and $C_{\mathcal{W}}$ such that for all $t \in [0,1]$,

$$\Xi(\rho^t(\zeta), \zeta, u(\rho^t), t) > \mu, \quad \zeta \in \mathbb{S}^n,$$

and

$$|\kappa_i(\rho^t)| < C_{\mathcal{W}}, \quad i \in \{1, 2, \ldots, n\}.$$

Claim 4 $\mathcal{F}(\rho, 1) = 0$ is solvable.

Let

$$\Gamma = \left\{ \lambda \in \Gamma_k \,\middle|\, G(\lambda) > \frac{1}{2}\mu,\ |\lambda| < 2\sqrt{n}(n-1)C_{\mathcal{W}} \right\}$$

and

$$\mathcal{B}_R = \left\{ \rho \in C^{4,\alpha}(\mathbb{S}^n) \,\middle|\, \frac{1}{2}\underline{\rho} < \rho < 2\overline{\rho},\ \frac{1}{2} < u(\rho) < R,\ |\rho|_{C^{4,\alpha}(\mathbb{S}^n)} < R,\ \lambda(\rho) \in \Gamma \right\}.$$

From *Claim 3*, it yields that there is some sufficiently large constant R such that for all $t \in [0,1]$, the admissible solution ρ^t of $\mathcal{F}(\cdot, t) = 0$, if any, belongs to $\mathcal{B}_R$, while $\mathcal{F}(\cdot, t) = 0$ has no solution on $\partial\mathcal{B}_R$. Therefore, the degree $\deg(\mathcal{F}(\cdot, t), \mathcal{B}_R, 0)$ is well defined for $0 \le t \le 1$. Using *Claims 2* and *3* and the homotopic invariance of the degree, we have

$$\deg\left(\mathcal{F}(\cdot, 1), \mathcal{B}_R, 0\right) = \deg\left(\mathcal{F}(\cdot, 0), \mathcal{B}_R, 0\right) = \pm 1.$$

So we obtain a solution at $t = 1$. This completes the proof of Theorem 1.2.

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