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RESEARCH ARTICLE

Yet another Freiheitssatz: Mating finite groups with locally indicable ones

Anton A. Klyachko[1,2,*] and Mikhail A. Mikheenko[1,2]

[1] Faculty of Mechanics and Mathematics, Moscow State University, Leninskie Gory, MSU, Moscow 119991, Russia
[2] Moscow Center for Fundamental and Applied Mathematics, Moscow, Russia
*Corresponding author. E-mail: klyachko@mech.math.msu.su

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Abstract
The main result includes as special cases on the one hand, the Gerstenhaber–Rothaus theorem (1962) and its generalisation due to Nitsche and Thom (2022) and, on the other hand, the Brodskii–Howie–Short theorem (1980–1984) generalising Magnus's Freiheitssatz (1930).

1. Introduction

An equation $w(x, y, \dots) = 1$ over a group G (where $w(x, y, \dots)$ is a word in the alphabet $G \sqcup \{x^{\pm1}, y^{\pm1}, \dots\}$) is called *solvable over* G if some overgroup $\widetilde{G} \supseteq G$ contains elements $\widetilde{x}, \widetilde{y}, \dots$ such that $w(\widetilde{x}, \widetilde{y}, \dots) = 1$.

The study of this notion has a long history, see, e.g., [1, 2, 4, 6–12, 15–18, 20–25, 27, 28, 32], and references therein. The following result is well known.

Gerstenhaber–Rothaus theorem (for a single equation) [12] (see also [29]). *If an equation* $g_1x^{\varepsilon_1} \dots g_nx^{\varepsilon_n} = 1$ *with unknown* x *over a finite group* $G \ni g_1, \dots, g_n$ *is nonsingular, i.e.,* $\sum \varepsilon_i \neq 0$, *then it is solvable over* G.

It is unknown whether the same holds for any (infinite) groups; this is the (strengthened) Kervaire–Laudenbach conjecture. Pestov [33] showed that this holds true for hyperlinear groups.* A further generalisation was obtained quite recently.

Nitsche–Thom theorem [32] (for a single equation). *An equation* $v(x, y, \dots) = 1$ *over any finite (and even any hyperlinear) group* G *is solvable over* G *if the content of* $v(x, y, \dots)$ *is nontrivial.*

Here, the *content* of a word v from the free product $G * F(x, y, \dots)$ of a group G and the free group $F(x, y, \dots)$ is the image of v under the natural homomorphism $\varepsilon : G * F(x, y, \dots) \to F(x, y, \dots)$ (whose kernel is the normal closure $\langle\langle G \rangle\rangle$ of G).

The following Freiheitssatz for *locally indicable* groups (i.e., groups whose nontrivial finitely generated subgroups admit epimorphisms onto $\mathbb{Z}$) is a result of quite another sort.

Brodskii–Howie–Short theorem [3, 4, 15, 35]. *The natural mappings* $C \to (C * D)/\langle\langle w \rangle\rangle \leftarrow D$ *are injective if the groups* C *and* D *are locally indicable, and the word* $w \in C * D$ *is not conjugate to an element of* $C \cup D$.

*Definitions, examples, and properties of hyperlinear (= Connes-embeddable) groups can be found, e.g., in [36]; we note only that the class of hyperlinear groups contains all finite group and their free products (possibly, even all group are hyperlinear – this is a well-known open question).

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This generalisation of Magnus's Freiheitssatz [30] can easily be reformulated in the language of equations:

any non-exotic equation over a locally indicable group is solvable over it

(where an equation $w(x, y, \dots) = 1$ over a group G is called *exotic* if the word $w(x, y, \dots) \in G * F(x, y, \dots)$ is conjugate to an element of G). And conversely: the Nitsche–Thom theorem can be reformulated as a Freiheitssatz:

> *if K is a free product of finite groups or, more generally, K is hyperlinear, F is a free group, and the image of a word $w \in K * F$ under the natural homomorphism $K * F \to F$ is non trivial, then the natural mapping $K \to (K * F)/\langle\langle w \rangle\rangle$ is injective.*

However, no relations between the Brodskii–Howie–Short and Gerstenhaber–Rothaus theorems were known so far. The purpose of this paper is to fill this lacuna, i.e., 'to mate' finite and locally indicable groups. The following fact includes all the results mentioned above as special cases.

Freiheitssatz. *If groups C and D are locally indicable, K is a GR*-group, and the image of a word $w \in C * D * K$ under the natural homomorphism $C * D * K \to C * D$ is not conjugate to an element of $C \cup D$, then the natural mappings $C * K \to (C * D * K)/\langle\langle w \rangle\rangle \leftarrow D * K$ are injective.*

The definition of a GR*-group can be found in the next section; examples of GR*-groups are all hyperlinear groups, in particular, all free products of finite groups. Therefore, this theorem contains the above-mentioned results:

- if $K = \{1\}$, we obtain the Brodskii–Howie–Short theorem;
- if $C = F(x, y, \dots)$ and $D = F(x_1, y_1, \dots)$ are free groups, and $w = v(xx_1, yy_1, \dots)$, where $v(x', y', \dots) \in (K * F(x', y', \dots) \setminus \langle\langle K \rangle\rangle$, then we obtain the Nitsche–Thom theorem (for a single equation), generalising the Gerstenhaber–Rothaus theorem (for a single equation).

See Section 3 for the complete statement of our main result and Section 5 for the proof (while Section 4 contains the key lemma).

The full statements of the Gerstenhaber–Rothaus and Nitsche–Thom theorems dealing with systems of equations can be found in Sections 2 and 4. We do not try to generalise these full versions in this paper (because the Brodskii–Howie–Short theorem is essentially one-relator), but we use these results (moreover, our approach is heavily based on ideas from [32]). The last section contains a proof of the Nitsche–Thom theorem (which was essentially obtained in [32], but there are some nuances, see the last section).

The full version of the Brodskii–Howie–Short theorem is also stronger than the above statement (and we use it), namely, one can add the following words to the statement above: '*Moreover the group $(C * D)/\langle\langle w \rangle\rangle$ is locally indicable too, if the word w is not a proper power*' (and the following phrase to the 'equational' version of this theorem: '*, and a solution can be found in a locally indicable overgroup of the given group*').

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2. GR- and GR*-group

A system of equations over a group is called *nonsingular* if the integer rows composed of the exponent-sums of unknowns in equations are linearly independent. For example, the system (with unknowns x, y, z, t over a group $G \ni a, b, c, d$):

$$\begin{cases} axbycyz^5dz^{-2} = 1 \\ [xt, dz]dx^4cy^5bz^6 = 1 \\ ax^7y^8dz^k = 1 \end{cases} \text{has the exponent} - \text{sum matrix} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & k & 0 \end{pmatrix},$$

i.e., this system is singular if and only if $k = 9$.

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Gerstenhaber–Rothaus theorem [12]. *Any nonsingular system of equations over a finite group is solvable over this group*.

We suggest to call a group G a *GR-group*, if any nonsingular system of equations over G is solvable over G. Pestov [33] noticed that all hyperlinear group are GR-groups.

It is easy to see that

the class of GR-groups is a quasivariety,

i.e., this class consists of all groups satisfying some (possibly, infinite) system of *quasi-identities*, i.e., (finite) formulae of the form:

$$(\forall x, y, \dots)(u_1(x, y, \dots) = 1 \ \& \dots \ \& \ u_k(x, y, \dots) = 1 \Rightarrow v(x, y, \dots) = 1),$$

where $u_i(x, y, \dots)$ and $v(x, y, \dots)$ are words in the alphabet $\{x^{\pm 1}, y^{\pm 1}, \dots\}$.

A simple way to prove this is to apply the following characterisation of quasivarieties [31]:

a nonempty class of groups is a quasivariety if and only if it is closed under passage to subgroups and reduced (= filtered) products.

For the class of GR-groups, the both conditions hold obviously.

A disadvantage of the quasivariety of GR-groups is that it is unclear whether this class is closed under free products. We call a group G a *GR*-group* if it satisfies the following equivalent conditions:

(1) the free product $G * \mathbb{Z}$ is a GR-group;
(2) the free product $G * G$ is a GR-group;
(2') the free product of any family of groups isomorphic to G is a GR-group;
(4) G embeds into a GR-group $\widetilde{G} \supseteq G$ satisfying no nontrivial *mixed identity with constants from* G, i.e., for any word $v(x, y, \dots) \in G * F(x, y, \dots) \setminus \{1\}$ (where $F(x, y, \dots)$ is a free group), there exist $\widetilde{x}_v, \widetilde{y}_v, \dots \in \widetilde{G}$ such that $v(\widetilde{x}, \widetilde{y}, \dots) \neq 1$.

Proposition. *These conditions are indeed equivalent.*

Proof. First, note that, if $|G| \leqslant 2$, then all the conditions are equivalent, because they are true in this case (the class of residually finite groups is closed with respect to free products [13] and is contained in the class of GR-groups by the Gerstenhaber–Rothaus theorem, therefore, if $|G| = 2$, then we can take, e.g., the free product of G and the free group of countable rank as $\widetilde{G}$ in (3)).

If $|G| > 2$, then (1) and (2) are equivalent, because $G * \mathbb{Z}$ and $G * G$ embed into each other[1] (and any subgroup of a GR-group is a GR-group).

Condition (2') is equivalent to (1) and (2) by similar reasons:

- the free product of any family of groups isomorphic to G is residually a finite free product $G * \dots * G$ (because any element of an arbitrary free product lies in a finite subproduct, and each subproduct is a retract of the whole product);
- GR is a residual property (because the class of GR-groups is a quasi-variety);
- a finite free product $G * \dots * G$ (with at least two factors) and $G * \mathbb{Z}$ embed into each other by the same 'converse of the Kurosh theorem'[1].

Moreover, Condition (1) implies Condition (3), because $G * \mathbb{Z}$ contains the free product $\widetilde{G} = G * F_\infty$ of G and the free group of infinite (countable) rank, which, obviously, has no mixed identities with constants from G.

It remains to prove the implication (3) ⇒ (1). The Cartesian product $H = \bigtimes_{v \in (\widetilde{G} * \langle t_\infty \rangle) \setminus \{1\}} \widetilde{G}_v$ of copies $\widetilde{G}_v$ of $\widetilde{G}$ is a GR-group (because the Cartesian product of any family of GR-groups is, obviously, a GR-group).

[1] Henceforth, we use the following well-known fact, which we leave to the reader as an easy exercise: *subgroups of a non-dihedral free product are described up to isomorphism as follows: these are all groups allowed by the cardinality restriction and the Kurosh subgroup theorem.*

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The group $G * \langle t\rangle_\infty$ embeds into H as follows: G embeds diagonally, and the element t is mapped to the element whose vth coordinate is $\widetilde{t}_v \in \widetilde{G}$ (i.e. $v(\widetilde{t}_v) \neq 1$). Clearly, this is an embedding. It remains to recall that a subgroup of a GR-group is a GR-group too. This completes the proof.

Question 1. Are the classes of GR- and GR*-groups closed with respect to direct and free products?

Simple considerations show that

- *the class of GR-groups is, obviously, closed with respect to direct products*;
- *if the class of GR-groups is closed under free products, then GR-groups and GR*-groups are the same* (this follows immediately from Condition 2) of the definition of GR*-groups; more generally, any GR-group decomposable nontrivially into a free product is a GR*-group);
- *if the class of GR*-groups is closed under direct products or, more generally, if there exists a GR*-group containing any two given GR*-groups as subgroups, then the class of GR*-groups is closed under free products*.

This means that the answers to Question 1 can be only the following:

- either $\mathrm{GR} = \mathrm{GR}^*$, and this class is closed with respect to both operations,
- or the class of GR-groups is closed under direct products, but not closed under free products, and the class of GR*-groups is
  - either closed under neither operations,
  - or closed with respect to both operations,
  - or closed under free products, but not closed under direct products.

Certainly, none of the presently known facts contradicts the equalities $\mathrm{GR} = \mathrm{GR}^* = \{\text{all groups}\}$ (Howie's conjecture). The class of hyperlinear groups is closed under free products [5]; hence, all hyperlinear groups are not only GR- but also GR*-groups.

3. Main result

Suppose that $A \triangleleft G$ and F is a free group. We call the natural epimorphism $\varepsilon_A \colon G * F \to (G/A) * F$ the *G/A-content*; the G/G-content is just the *content* ε (defined in [25, 32]).

Main theorem. *Suppose that a group G contains a normal subgroup A, which is GR-group, and the quotient group G/A is locally indicable. Then an equation $w(x, y, \dots) = 1$ is solvable over G if the G/A-content of $w(x, y, \dots)$ is not conjugate to an element of G/A in $(G/A) * F(x, y, \dots)$.*

This theorem implies immediately the Freiheitssatz stated in the introduction. Indeed, consider the group $G = C * D * K$, its normal subgroup $A = \langle\langle K\rangle\rangle$ (which is GR-group, because it is isomorphic to the free product of a family of groups isomorphic to the GR*-group K), and the embedding $\varphi : G \to G * \langle t\rangle_\infty$, where $C \ni c \mapsto c^t \in G * \langle t\rangle_\infty$ and the other free factors are mapped identically. The equation $\phi(w) = 1$ (with one variable t) is solvable over G by the main theorem, i.e., the composition $G \xrightarrow{\varphi} G * \langle t\rangle_\infty \to G * \langle t\rangle_\infty / \langle\langle \phi(w)\rangle\rangle$ is injective on $D * K$ and contains w in its kernel, as required. The injectivity of the natural mapping of $C * K$ can be proven similarly.

4. Key lemma

A proof of the following result (obtained essentially in [32]) can be found in the last section.

Nitsche–Thom theorem. *A system of equations $\{w_1(x, y, \dots) = 1,\ w_2(x, y, \dots) = 1, \dots\}$ over a GR-group G is solvable over G if the standard complex of the presentation $\langle x, y, \dots \mid \varepsilon(w_1) = 1,\ \varepsilon(w_2) = 1, \dots\rangle$, where ε stands for the content, admits a covering with trivial second homologies over $\mathbb{Z}$.*

To apply this theorem, we need the following lemma.

Lemma. *Suppose that a locally indicable group L acts freely on a set X, and* $L \times F(X) \xrightarrow{\circ} F(X)$ *is the natural extension of this action to the free group with basis X. Then, for each word* $v \in F(X) \setminus \{1\}$*, some covering of the standard complex K of the presentation* $H = \langle X \mid L \circ v \rangle$ *has trivial second homologies.*

Proof. Suppose that $v = u^k$ and the word u is not a proper power in $F(X)$. Let us prove that the required covering is the covering $p\colon \widetilde{K} \to K$ corresponding to the normal closure $\langle\langle L \circ u \rangle\rangle \triangleleft H = \pi_1(K)$ of the orbit of u. In an explicit form, the complex $\widetilde{K}$ is the Cayley graph of the group $A = \langle X \mid L \circ v \rangle$ with 'k times wrapped' two-discs glued to each cycle with label $l \circ u$, where $l \in L$ (so, the words $l \circ v = l \circ u^k$ are written on the boundaries of the discs).

The group $A = \langle X | L \circ v \rangle$ embeds into $\widehat{A} = Y * L / \langle\langle \widehat{u} \rangle\rangle$, where Y is a set of representatives of orbits of the action of L on X, and the word $\widehat{u}$ is obtained from u by replacing each letter x with lyl^{-1}, where $l \in L$ and $y \in Y$ are the (only) elements such that $x = l \circ y$. The embedding $A \to \widehat{A}$ is obvious: $x = l \circ y \mapsto lyl^{-1}$. The group $\widehat{A}$ is locally indicable [4] and $\widehat{A} = L \leftthreetimes A$. This group $\widehat{A}$ acts on the complex $\widetilde{K}$: the action of A is the standard action of a group on its Cayley graph, and L acts by conjugation on vertices: $l \circ a = lal^{-1}$ (then vertices joined by an edge labelled x are mapped to vertices joined by an edge labelled $l \circ x$). This action of $\widehat{A}$ is transitive on two-cells and on vertices and free on (oriented) edges.

The first is quite obvious; and the second can be explained easily: the stabiliser of an edge must stabilise its origin; the action is vertex-transitive; therefore, it suffices to show that the stabiliser of each edge e outgoing from the identity is trivial; so, the equality $(l, a) \circ e = e$ implies that $a = 1$ (otherwise, we would obtain an edge with another origin); now, the equality $l \circ e = e$ means that $l = 1$, because the action of L on X is free.

Now, the situation is simple. Since the action on two-cells is transitive, the nontriviality of the second homologies implies an equality $r\bar{u} = 0$, where

- $\bar{u}$ is the sum of edges of the boundary of a two-cell (which does not vanish, because the universal covering of a group with one relator $u = 1$, which is not a proper power, is contractible, i.e., one-relator torsion-free groups are aspherical, see [29]),
- and r is a nonzero element of the group ring $\mathbb{Z}[\widehat{A}]$ of $\widehat{A}$.

The freeness of the action on edges leads immediately to zero divisors in the group ring $\mathbb{Z}[\widehat{A}]$ of the locally indicable group $\widehat{A}$, which is a contradiction [14].

5. Proof of the main theorem

Since the word $\varepsilon_A(w) \in (G/A) * F(x, y, \dots)$ is not conjugate to an element of G/A, the equation $\varepsilon_A(w) = 1$ has a solution in a locally indicable group B containing G/A (by the Brodskii–Howie–Short theorem). Therefore, we have the embeddings of the G into the unrestricted wreath products: $G \subseteq A \bar{\wr} (G/A) \subseteq A \bar{\wr} B = A^B \leftthreetimes B$ (where the first embedding is the Kaloujnine–Krasner theorem [26], see also [19]). Therefore, replacing G with $A \bar{\wr} B$, and A with its Cartesian power $A^B \triangleleft A \bar{\wr} B$ (which is also a GR-group), we can assume that the equation $\varepsilon_A(w) = 1$ has a solution $\widehat{x}, \widehat{y}, \dots \in G/A$; hence, making an obvious change of variables $(x \mapsto x\widehat{x},\ y \mapsto y\widehat{y}, \dots)$, we obtain that $\varepsilon_A(w)$ is contained in the normal closure $\langle\langle F \rangle\rangle$ of $F = F(x, y, \dots)$ in $(G/A) * F$. Thus, we assume that $w = w(x, y, \dots)$ can be rewritten as a word $\widehat{w}$ in the alphabet $\{x^{\pm b}, y^{\pm b}, \dots \mid b \in B = G/A\} \sqcup A$ (because we assume already that $G = A \leftthreetimes B$).

We arrive to the situation of the Lemma. For $X = \{bxb^{-1}, byb^{-1}, \dots \mid b \in B\}$ and $v = \varepsilon(\widehat{w}) \in F(X)$, where $\varepsilon\colon F(X) * A \to F(X)$ is the natural retraction, and $L = B$, the Lemma says that a covering of the standard complex of the presentation $H = \langle X \mid B \circ \varepsilon(\widehat{w}) \rangle$ has trivial second homologies. By the Nitsche–Thom theorem, this means the solvability of the system of equations $\{b \circ w = 1 \mid b \in B\}$ (with unknowns $X = B \circ \{x, y, \dots\}$) over the group A (where the action of B on X is extended to $A * F(X)$ naturally: $b \circ a := bab^{-1}$ for $a \in A$). In other words, the natural mapping $A \to \widetilde{A} = (A * F(X)) / \langle\langle \{b \circ w = 1 \mid b \in B\} \rangle\rangle$ is injective. The group B acts on $\widetilde{A}$, and this action extends the action of B on A by conjugations.

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Therefore, the natural mapping $G = A \leftthreetimes B \to \widetilde{G} = \widetilde{A} \leftthreetimes B$ is injective. Moreover, $w(x, y, \dots) = 1$ in $\widetilde{G}$, which means the solvability of the equation.

6. Subgroup presentations and the Nitsche–Thom theorem

In this section, we prove the Nitsche–Thom theorem as stated in Section 4. This result was essentially proven in [32], but unfortunately, a weaker version was explicitly stated there (see [32], Theorem 1.3 and Remark 2.2). Our approach somewhat differs from the proof that can be extracted from [32].

The following fact is well known, see, e.g., [37], Theorem 2.2.1, or [29], Proposition II.4.1.

Schreier's theorem on subgroup presentations [34]. *Suppose that U is a subgroup of a group $G = \langle X|R \rangle$, and $T \subseteq F(X)$ is a Schreier system of the right coset representatives of $\pi^{-1}(U)$ in the free group $F(X)$, where $\pi : F(X) \to G = F/\langle\langle R \rangle\rangle$ is the natural epimorphism. Then the subgroup U is generated by the images of the words $y_{t,x} = tx(\overline{tx})^{-1} \in F(X)$ and has a presentation, where generators are all nontrivial words $\{y_{t,x} \mid t \in T, x \in X\}$, and the defining relators are all words $\{trt^{-1} \mid t \in T, x \in X\}$ rewritten as words in nontrivial generators $y_{t,x}$ considered as letters.*

Here, the bar means taking a representative: $\bar{v} \in T$ (where $v \in F(X)$) is the unique word from T such that $\pi^{-1}(U)\bar{v} = \pi^{-1}(U)v$; and T is *Schreier* in the usual sense: any prefix of any word from T lies in T.

The geometric interpretation of this presentation, which we call *Schreier*, is also well known:

- the group G is the fundamental group of the standard complex K corresponding to the presentation $\langle X|R \rangle$ (i.e., K has a single vertex, edges correspond to generators from X, and two-dimensional cells correspond to relators from R);
- the subgroup U is the fundamental group of the complex $\widetilde{K}$ (with a base vertex) covering the complex K; vertices of $\widetilde{K}$ correspond to right cosets of U in G (and of $\pi^{-1}(U)$ in $F(X)$);
- a Schreier system of representatives corresponds to a maximal subtree in the 1-skeleton of $\widetilde{K}$;
- generators $y_{t,x}$ correspond to edges of $\widetilde{K}$ (or, more precisely, each edge $e \in \widetilde{K}$ corresponds to a path starting at the base vertex going through the maximal subtree to the start point of e, passing e, and returning to the base vertex via the maximal subtree); so, nontrivial generators $y_{t,x}$ correspond to edges not belonging to the maximal subtree;
- finally, relators of U correspond to two-cells of $\widetilde{K}$.

This geometric interpretation shows that

> *the standard complex of the presentation $G = \langle X|R \rangle$ admits a covering with trivial second homologies if and only if the relators of the Schreier presentation of some subgroup $U \subseteq G$ form a nonsingular system in the sense of Section 2.*

Thus, we can restate the Nitsche–Thom theorem in group-theoretical terms (with no topology or homologies).

Nitsche–Thom theorem (a pure group-theoretical form). *A system of equations $W = 1$ (possibly, infinite and, possibly, with an infinite set of unknowns X) over a GR-group G is solvable over G if there exists a subgroup of $\langle X|\varepsilon(W) \rangle$, the relators of whose Schreier presentation form a nonsingular system.* (Here, ε stands for the content, see Introduction.)

Proof. Suppose that $A \subseteq \langle X|\varepsilon(W) \rangle$ is a subgroup with a nonsingular Schreier presentation. Consider a presentation $G = \langle Z|V \rangle$ of the group G, the corresponding presentation of the group $(G * F(X))/\langle\langle W \rangle\rangle = \langle Z \sqcup X | V \sqcup W \rangle = H$, the natural epimorphism $\theta : H \to H/\langle\langle G \rangle\rangle = \langle X|\varepsilon(W) \rangle$, and the subgroup $\theta^{-1}(A) \subseteq H$. It is easy to see that the Schreier presentation for the subgroup $\theta^{-1}(A) \subseteq H$ transforms into the Schreier presentation for the subgroup $A \subseteq \langle X|\varepsilon(W) \rangle$ by deleting all generators corresponding to conjugate of generators from Z (in geometrical terms, each vertex of the covering complex corresponding to the subgroup $\theta^{-1}(A) \subseteq H$ is contained in a subcomplex isomorphic to the standard

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complex of the presentation $G = \langle Z|V\rangle$; and, contracting each such subcomplex to a single point, we obtain the covering complex corresponding to the subgroup $A \subseteq \langle X|\varepsilon(W)\rangle$).

Thus, the relators of the group $\theta^{-1}(A)$ form a nonsingular system of equations over the free product of $|\langle X|\varepsilon(W)\rangle : A|$ copies of G. Since G is a GR-group, the natural mapping $G \to \theta^{-1}(A) \subseteq H$ is injective[2] (to show this, we can take the quotient group of $\theta^{-1}(A)$ by the normal closure of the product of all copies of G, except one). This implies immediately the injectivity of the natural homomorphism $G \to H$, as required.

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[2] But we cannot assert that the entire free product of $|\langle X|\varepsilon(W)\rangle : A|$ copies of G embeds into $\theta^{-1}(A)$; it embeds if G is not only GR- but also a GR*-group.

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