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ON THE STABILITY OF EQUIVARIANT BIFURCATION PROBLEMS AND THEIR UNFOLDINGS

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ABSTRACT. In their book *Singularities and Groups in Bifurcation Theory* M. Golubitsky, I. Stewart and D. Schaeffer have introduced an equivariant version of Martinet's notion of V (for variety)-equivalence with parameter. In this paper we give a unified proof that, in this context, infinitesimal stability is equivalent to stability at the local level of germs and that stability in the unfolding category is equivalent to versality.

0. **Introduction.** This paper investigates stability issues within the framework of singularity theory as developed by M. Golubitsky, I. Stewart, and D. Schaeffer in their book [GSS, 1988]. We have chosen our notation so that it conforms to the extent possible with [GSS, 1988].

Let Γ be a compact Lie group acting linearly on $\mathbb{R}^m$ and $\mathbb{R}^n$, and trivially on all parameter spaces $\mathbb{R}^k$ with k arbitrary. Given a smooth Γ-equivariant germ $g\colon (x,\lambda) \in (\mathbb{R}^m \times \mathbb{R}, 0) \longrightarrow \mathbb{R}^n$ such that $g(0,0) = 0$ and $d_x g(0,0) = 0$, the equation $g(x,\lambda) = 0$ is called a Γ-*equivariant bifurcation problem* with parameter $\lambda \in \mathbb{R}$. We review the groups $\mathcal{K}(\Gamma)$ of equivalence of germs and $\mathcal{K}_{\mathrm{un}}(\Gamma)$ of equivalence of unfoldings in § 1.

The equivalence of infinitesimal stability and stability for right-left equivalence in the equivariant context, at the local level of germs, was first considered by F. Ronga in 1974; however, this work was fundamentally incorrect, in both directions. E. Bierstone proposed a corrected and improved version of this result in [B, 1980], Chapter 5. The global version in both directions can be found in [B, 1977], and the implication from stability to infinitesimal stability was first presented in [P, 1976], pp. 93–174.

Another important issue is to investigate stability in the category of unfoldings. The desired result is: an unfolding is versal if and only if it is stable (in the category of unfoldings). For right equivalence of functions, in the equivariant context, S. Izumiya [I, 1980] was the first to establish this result. In a recent paper J. J. Gervais [G, 1988], using the same method as Izumiya, proved the equivalence of versal and stable unfoldings for $\mathcal{K}_{\mathrm{un}}(\Gamma)$.

Throughout this paper it is understood that stability, as we shall define it, is relative to the notions of equivalence induced by $\mathcal{K}(\Gamma)$ and $\mathcal{K}_{\mathrm{un}}(\Gamma)$. To relieve the burden of notation we omit specific references to these groups.

We propose a unified treatment of the stability of germs and that of their unfoldings by proving a single fundamental stability theorem (Theorem 4.1). This theorem on one

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hand, yields at once the equivalence of infinitesimal stability and stability for germs (Theorem 2.2); and on the other hand, with the addition of a standard corollary of the preparation theorem, establishes the equivalence between versality and stability for unfoldings (Theorem 3.2). Theorem 2.2 is a new result, while Theorem 3.2 gives a new proof of [G, 1988].

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1. Preliminaries.

1.1. *Notation.* We write x (respectively y, μ, λ, α) for an element in the germ of $\mathbb{R}^m$ (respectively $\mathbb{R}^n$, $\mathbb{R}^k$, $\mathbb{R}$, $\mathbb{R}^q$) at the origin. The parameter space $\mathbb{R}^k$ equals $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}^q$; thus, either $\mu = \lambda$ or $\mu = (\lambda, \alpha)$. Denote by $\mathcal{E}_{x,\mu}(\Gamma)$ the ring of smooth Γ-invariant germs $f\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow \mathbb{R}$ (*i.e.* $f(\gamma . x, \mu) = f(x, \mu)$ for all $x \in \mathbb{R}^m$, $\mu \in \mathbb{R}^k$, $\gamma \in \Gamma$), and by $\mathcal{M}_{x,\mu}(\Gamma)$ its maximal ideal. For an integer s with $u \in \mathbb{R}^s$, $\mathcal{E}_u$ denotes the usual ring of smooth germs $h\colon (\mathbb{R}^s, 0) \longrightarrow \mathbb{R}$, and $\mathcal{M}_u$ its maximal ideal. Let $\overrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma)$ be the $\mathcal{E}_{x,\mu}(\Gamma)$-module of smooth Γ-equivariant germs $g\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow \mathbb{R}^n$ (*i.e.* $g(\gamma . x, \mu) = \gamma . g(x, \mu)$ for all $x \in \mathbb{R}^m$, $\mu \in \mathbb{R}^k$, $\gamma \in \Gamma$); denote by $\overrightarrow{\mathcal{M}}_{(x,\mu);y}(\Gamma)$ its submodule of germs vanishing at $(0,0)$. Let $\overleftrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma)$ be the $\mathcal{E}_{x,\mu}(\Gamma)$-module of smooth germs $S\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow \mathrm{End}(\mathbb{R}^n)$ satisfying $\gamma^{-1}.\ S(\gamma . x, \mu) . \gamma = S(x, \mu)$ for all $x \in \mathbb{R}^m$, $\mu \in \mathbb{R}^k$ and $\gamma \in \Gamma$. Finally in general $\overrightarrow{\mathcal{E}}_{x;y}$ represents the space of smooth germs $(\mathbb{R}^m, 0) \longrightarrow \mathbb{R}^n$ and $\overrightarrow{\mathcal{M}}_{x;y}$ its submodule of germs vanishing at zero. By the Malgrange-Poénaru finitude theorem ([GSS, 1988], Theorem 5.3, p. 51) $\overrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma)$ is generated, over $\mathcal{E}_{x,\mu}(\Gamma)$, by finitely many homogeneous Γ-equivariant polynomial mappings. An argument similar to [GSS, 1988], 1.3, p. 172, yields that $\overleftrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma)$ is also finitely generated over $\mathcal{E}_{x,\mu}(\Gamma)$. Recall that the usual space $J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n)$ of r-jets from $\mathbb{R}^m \times \mathbb{R}^k$ to $\mathbb{R}^n$ can be identified with $\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^n \times \{$polynomial mappings: $\mathbb{R}^m \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ with components of degree $\leq r$ and having no constant terms$\}$. Let $[(x_0, \mu_0), y_0, Z] \in J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n)$, the group Γ acts on $J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n)$ in the following natural fashion

$$\gamma . [(x_0, \mu_0,), y_0, Z] = [(\gamma . x_0, \mu_0), \gamma . y_0, \gamma . Z],$$

where $\gamma . Z$ is the polynomial mapping $(\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^n, 0)$ defined by $[\gamma . Z](x, \mu) = \gamma^{-1} . Z(\gamma . x, \mu)$. The fixed space of this action is the Γ-*equivariant r-jet space* from $\mathbb{R}^m \times \mathbb{R}^k$ to $\mathbb{R}^n$, and will be denoted by $J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma)$. Therefore there is a natural fibration $J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma) = (\mathbb{R}^m)^\Gamma \times \mathbb{R}^k \times (\mathbb{R}^n)^\Gamma \times \{\Gamma$-equivariant polynomial mappings: $\mathbb{R}^m \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ with components of degree $\leq r$ and no constant terms$\}$. Note that the above set of polynomials can be identified with

$$\overrightarrow{\mathcal{M}}_{(x,\mu);y}(\Gamma) / [\overrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma) \cap (\mathcal{M}_{x,\mu})^{r+1} . \overrightarrow{\mathcal{E}}_{(x,\mu);y}],$$

we call this space, following Arnold *et al.* [AGV, 1985], *the small* Γ-*equivariant r-jet space* and denote it by $s, J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma)$. We next define *the medium* Γ-*equivariant* *r-jet space* $m, J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma)$ to be $(\mathbb{R}^n)^\Gamma \times s, J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma)$. Thus, we have the natural projection

$$J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma) \xrightarrow{\pi^r} m, J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma).$$

Let $g \in \overrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma)$; it is easy to verify ([L, 1990], Lemma 1.4.1) that the r-jet of g at 0 belongs to $m, J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma)$. We shall denote it by $J_0^r g$. Thus there is a natural projection

$$J_0^r \colon \overrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma) \ni g \longmapsto J_0^r g \in m, J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma).$$

The Γ-equivariant Jet extensions are constructed as in the non-equivariant case. Let $g \colon \mathbb{R}^m \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ be a smooth Γ-equivariant mapping, $x_0 \in (\mathbb{R}^m)^\Gamma$, $\mu_0 \in \mathbb{R}^k$ and $y_0 = g(x_0, \mu_0) \in (\mathbb{R}^n)^\Gamma$. Then, $\hat{g}(x, \mu) = g(x + x_0, \mu + \mu_0) - y_0 \in \overrightarrow{\mathcal{E}}_{(x,\mu);y}(\Gamma)$; hence $J_0^r \hat{g}$ is defined.

Now we define, following Arnold *et al.* [AGV, 1985]:

The *medium r-jet extension* of g to be

$$\begin{aligned} m, J^r g \colon (\mathbb{R}^m)^\Gamma \times \mathbb{R}^k &\longrightarrow m, J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma) \\ (x_0, \mu_0) &\longmapsto \big(g(x_0, \mu_0), J_0^r \hat{g}\big) = m, J^r(x_0, \mu_0) g, \end{aligned}$$

and the *r-jet extension* of g to be

$$\begin{aligned} J^r g \colon (\mathbb{R}^m)^\Gamma \times \mathbb{R}^k &\longrightarrow J^r(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n; \Gamma) \\ (x_0, \mu_0) &\longmapsto \big((x_0, \mu_0), m, J^r(x_0, \mu_0) g\big) = J^r(x_0, \mu_0) g. \end{aligned}$$

1.2. *The equivalence groups.* Given $g \in \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$, a germ $G \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$ is called an *unfolding* of g if $G(x, \lambda, 0) = g(x, \lambda)$ for all $x \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$. The equivalence groups are defined as follows. For the study of germs let $\mathcal{K}(\Gamma) = \{ (S, X, \Lambda) \in \overleftrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma) \times \overrightarrow{\mathcal{M}}_{(x,\lambda);x}(\Gamma) \times \mathcal{M}_\lambda \mid \det S(0,0) > 0, \det(d_x X)_{(0,0)} > 0, d\Lambda(0) > 0 \}$. Under the operation

$$\begin{aligned} &(S_2, X_2, \Lambda_2).(S_1, X_1, \Lambda_1)(x, \lambda) \\ &\quad = \Big(S_1(x, \lambda) \circ S_2\big(X_1(x, \lambda), \Lambda_1(\lambda)\big), X_2\big(X_1(x, \lambda), \Lambda_1(\lambda)\big), \Lambda_2 \circ \Lambda_1(\lambda)\Big), \end{aligned}$$

$\mathcal{K}(\Gamma)$ is a group called the Γ-*equivalence group for germs*. For the study of unfoldings consider $\mathcal{K}_{\mathrm{un}}(\Gamma) = \{ (S, X, \Lambda, A) \in \overleftrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \times \overrightarrow{\mathcal{M}}_{(x,\lambda,\alpha);x}(\Gamma) \times \overrightarrow{\mathcal{M}}_{(\lambda,\alpha);\lambda} \times \overrightarrow{\mathcal{M}}_{\alpha;\alpha} \mid \det S(0,0,0) > 0, \det(d_x X)_{(0,0,0)} > 0, (d_\lambda \Lambda)_{(0,0)} > 0 \text{ and } \det(dA)_{(0)} > 0 \}$. Similarly $\mathcal{K}_{\mathrm{un}}(\Gamma)$ can be made a group; it is called the Γ-*equivalence group for unfoldings*. Observe that setting $\alpha = 0$ in $\mathcal{K}_{\mathrm{un}}(\Gamma)$ yields $\mathcal{K}(\Gamma)$. Also, $\mathcal{K}(\Gamma)$ and $\mathcal{K}_{\mathrm{un}}(\Gamma)$ are, in Damon's terminology, geometric subgroups of the group $\mathcal{K}$ of contact equivalence (*cf.* [D, 1984], example 3.1, p. 9). In spite of the similarity in the notation, it should be noted that our group $\mathcal{K}_{\mathrm{un}}(\Gamma)$ does not correspond to the Geometric group of unfoldings in Damon's book; it is rather his group of equivalence of unfoldings (*cf.* [D, 1984] p. 41).

There is a natural action of $\mathcal{K}_{\rm un}(\Gamma)$ on $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$

$$\mathcal{K}_{\rm un}(\Gamma) \times \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \longrightarrow \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$$
$$(S, X, \Lambda, A).\, G(x,\lambda,\alpha) = S(x,\lambda,\alpha) \circ G\big(X(x,\lambda,\alpha), \Lambda(\lambda,\alpha), A(\alpha)\big).$$

Setting $\alpha = 0$ gives the action of $\mathcal{K}(\Gamma)$ on $\overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$.

The orbit of g (respectively G) under the action of $\mathcal{K}(\Gamma)$ (respectively $\mathcal{K}_{\rm un}(\Gamma)$) is denoted by $K(\Gamma).\, g$ (respectively $\mathcal{K}_{\rm un}(\Gamma).\, G$). Two germs g and h in $\overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$ (respectively G and H in $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$) are said to be Γ-*equivalent* if they belong to the same orbit.

1.3. *Tangent spaces.* The tangent space to $\mathcal{K}(\Gamma)$ is obtained by taking the derivative, at the origin, of one parameter unfoldings of elements of $\mathcal{K}(\Gamma)$. ([L, 1990], Lemma 3.5) or a computation similar to ([GSS, 1988], p. 210) yields

$$T\mathcal{K}(\Gamma) \cong \overleftrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma) \oplus \overrightarrow{\mathcal{E}}_{(x,\lambda);x}(\Gamma) \oplus \mathcal{E}_\lambda .$$

Similarly the tangent space to $\mathcal{K}_{\rm un}(\Gamma)$, at its identity, is given by

$$T_{\rm un}\mathcal{K}(\Gamma) \cong \overleftrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \oplus \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);x}(\Gamma) \oplus \overrightarrow{\mathcal{E}}_{(\lambda,\alpha);\lambda} \oplus \overrightarrow{\mathcal{E}}_{\alpha;\alpha} .$$

Note that in the literature, ([D, 1984], pp. 5 and 10), $T\mathcal{K}(\Gamma)$ is referred to as the extended tangent space.

Let $g \in \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$, the tangent space to $\overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$ at g can be identified with itself. Consider the orbit map:

$$a_g\colon \mathcal{K}(\Gamma) \longrightarrow \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$$
$$(S, X, \Lambda) \longmapsto (S, X, \Lambda).\, g,$$

and its derivative at the origin of $\mathcal{K}(\Gamma)$:

$$\begin{aligned} da_g\colon T\mathcal{K}(\Gamma) \cong \overleftrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma) \oplus \overrightarrow{\mathcal{E}}_{(x,\lambda);x}(\Gamma) \oplus \mathcal{E}_\lambda &\longrightarrow \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma) \\ (S, X, \Lambda) &\longmapsto S \circ g + d_x g(X) + d_\lambda g(\Lambda). \end{aligned} \tag{1.3.1}$$

One can similarly define the orbit map a_G for a germ $G \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$, and its derivative at the origin of $\mathcal{K}_{\rm un}(\Gamma)$:

$$\begin{aligned} da_G\colon T_{\rm un}\mathcal{K}(\Gamma) \cong \overleftrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \oplus \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);x}(\Gamma) \oplus \overrightarrow{\mathcal{E}}_{(\lambda,\alpha);\lambda} \oplus \overrightarrow{\mathcal{E}}_{\alpha;\alpha} &\longrightarrow \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \\ (S, X, \Lambda, A) &\longmapsto S \circ G + d_x G(X) + d_\lambda G(\Lambda) + d_\alpha G(A). \end{aligned}$$

In Damon's terminology ([D, 1984] (6.8.1), p. 29), $T\mathcal{K}(\Gamma)$ (respectively $T_{\rm un}\mathcal{K}(\Gamma)$) is a direct sum of finitely generated modules over the adequately ordered system of DA-algebras $\{\mathcal{E}_{x,\lambda}(\Gamma), \mathcal{E}_\lambda\}$ (respectively $\{\mathcal{E}_{x,\lambda,\alpha}(\Gamma), \mathcal{E}_{\lambda,\alpha}, \mathcal{E}_\lambda\}$). Furthermore, da_g and da_G are module homomorphisms over these systems ([D, 1984], example 12.1, p. 65).

The tangent space $T(g,\Gamma)$ (respectively $T_{\rm un}(G,\Gamma)$) to the orbit $\mathcal{K}(\Gamma).\, g$ (respectively $\mathcal{K}_{\rm un}(\Gamma).\, G$) is defined to be $da_g(T\mathcal{K}(\Gamma))$ (respectively $da_G(T\mathcal{K}_{\rm un}(\Gamma))$). Whence

$$T(g,\Gamma) = \tilde{T}(g,\Gamma) \oplus \mathcal{E}_\lambda\{d_\lambda g\},$$

where

$$\tilde{T}(g,\Gamma) = \{ S \circ g + d_x g(X) \mid S \in \overleftrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma), X \in \overrightarrow{\mathcal{E}}_{(x,\lambda);x}(\Gamma) \};$$

and

$$T_{\mathrm{un}}(G,\Gamma) = \tilde{T}_{\mathrm{un}}(G,\Gamma) \oplus \mathcal{E}_{\lambda,\alpha}\{ d_\lambda G\} \oplus \mathcal{E}_\alpha \{ d_\alpha G\},$$

where

$$\tilde{T}_{\mathrm{un}}(G,\Gamma) = \{ S \circ G + d_x G(X) \mid S \in \overleftrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma), X \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);x}(\Gamma) \}.$$

The *codimension* $\operatorname{cod} g$ of g is by definition: $\dim_{\mathbb{R}} \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)/\, T(g,\Gamma)$.

1.4. *The orbit of an r-jet.* The action of $\mathcal{K}_{\mathrm{un}}(\Gamma)$ on $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$ induces a quotient action of the corresponding group of r-jets: let $\mathcal{K}^r_{\mathrm{un}}(\Gamma)$ denote the group of r-jets of elements of $\mathcal{K}_{\mathrm{un}}(\Gamma)$ at 0. Then $\mathcal{K}^r_{\mathrm{un}}(\Gamma)$ acts naturally on $m, J^r(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, \mathbb{R}^n; \Gamma)$ by $(J^r_0 S, J^r_0 X, J^r_0 \Lambda, J^r_0 A). J^r_0 G = J^r_0\big((S,X,\Lambda,A).G\big)$. Given $G \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$, the orbit of $J^r_0 G$ under the action of $\mathcal{K}^r_{\mathrm{un}}(\Gamma)$ is called the *medium orbit* of the r-jet of G, and is denoted by $m, \mathcal{K}^r_{\mathrm{un}}(\Gamma).G$. The *orbit* $\mathcal{K}^r_{\mathrm{un}}(\Gamma).G$ of $J^r_0 G$ is by definition $(\mathbb{R}^m)^\Gamma \times \mathbb{R} \times \mathbb{R}^q \times m, \mathcal{K}^r_{\mathrm{un}}(\Gamma).G$. Identify $m, J^r(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, \mathbb{R}^n; \Gamma)$ with its tangent space. Denote by $m, T^r_{\mathrm{un}}(G,\Gamma)$ the tangent space to the medium orbit $m, \mathcal{K}^r_{\mathrm{un}}(\Gamma).G$. Then

$$m, T^r_{\mathrm{un}}(G,\Gamma) = J^r_0\big(\overrightarrow{\mathcal{M}}_{(x,\lambda,\alpha);y} \cap T_{\mathrm{un}}(G,\Gamma)\big).$$

Equivalently, $Z \in m, T^r_{\mathrm{un}}(G,\Gamma)$ if and only if there is a decomposition

$$Z = S \circ G + d_x G(X) + d_\lambda G(\Lambda) + d_\alpha G(A) \tag{1.4.1}$$

modulo $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \cap (\mathcal{M}_{x,\lambda,\alpha})^{r+1}.\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}$, where $X \in \overrightarrow{\mathcal{M}}_{(x,\lambda,\alpha);x}(\Gamma)$, $\Lambda \in \overrightarrow{\mathcal{M}}_{(\lambda,\alpha);\lambda}$ and $A \in \overrightarrow{\mathcal{M}}_{\alpha;\alpha}$.

Indeed, let $t \longmapsto (\tilde{S}_t, \tilde{X}_t, \tilde{\Lambda}_t, \tilde{A}_t)$ be a curve in $\mathcal{K}_{\mathrm{un}}(\Gamma)$ passing through its identity at $t = 0$. A vector tangent to $m, \mathcal{K}^r_{\mathrm{un}}(\Gamma).G$ at $J^r_0 G$ is $Z = J^r_0\big(\frac{d}{dt}\,|_{t=0}\, (\tilde{S}_t, \tilde{X}_t, \tilde{\Lambda}_t, \tilde{A}_t).G\big)$.

To obtain similar notions about germs in $\overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$ it suffices to set $\alpha = 0$ in the above.

2. **Stability of germs.**

2.1. *Definitions.* Let $g \in \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$. We call g *infinitesimally stable* if the derivative (1.3.1) is surjective; that is to say, $T(g,\Gamma) = \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$ or $\operatorname{cod} g = 0$.

Assume g is the germ at 0 of the Γ-equivariant mapping $\tilde{g}\colon \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}^n$. Denote the space of such mappings by $C^\infty(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^n; \Gamma)$ and equip it with the C^∞ topology (*i.e.* uniform convergence over compact sets with all the derivatives). The germ g is said to be *stable* if for every Γ-invariant open neighborhood $\mathcal{U}$ of $(0,0)$ in $\mathbb{R}^m \times \mathbb{R}$, there exists a neighborhood $\mathcal{W}$ of $\tilde{g}$ in $C^\infty(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^n; \Gamma)$ such that for every $\tilde{g}' \in \mathcal{W}$, there exists $(x', \lambda') \in \mathcal{U}^\Gamma$ such that g is equivalent to the germ of $\tilde{g}'$ at (x', λ'). More precisely, there are germs $S\colon (\mathbb{R}^m \times \mathbb{R}, 0) \longrightarrow \mathrm{End}(\mathbb{R}^n)$, $X\colon (\mathbb{R}^m \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^m, x')$ and

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$\Lambda: (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, \lambda')$ satisfying: $S \in \overleftrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$, $X(\gamma . x, \lambda) = \gamma . X(x, \lambda)$ for all x, λ and γ, $\det S(0,0) > 0$, $\det(d_x X)_{(0,0)} > 0$ and $(d\Lambda)_0 > 0$, such that

$$g(x, \lambda) = S(x, \lambda) \circ g'\big(X(x, \lambda), \Lambda(\lambda)\big)$$

where g' is the germ of $\tilde{g}'$ at (x', λ').

We can now state the first main result of this paper:

THEOREM 2.2 (STABILITY OF GERMS). *Let* $g \in \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$. *The following conditions are equivalent:*

i) g is stable.

ii) g is infinitesimally stable.

iii) All unfoldings of g are trivial (i.e., g is its own universal unfolding).

3. **Stability of unfoldings.** For a complete exposition of the unfolding theory in this context we refer the reader to [GSS, 1988], Ch. XV. Let us recall that an unfolding $G \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$ of g is called *versal* if for any unfolding $H(x, \lambda, \beta)$ of g with t parameters (*i.e.* $\beta \in \mathbb{R}^t$) there exist $S \in \overleftrightarrow{\mathcal{E}}_{(x,\lambda,\beta);y}(\Gamma)$, $X \in \overrightarrow{\mathcal{M}}_{(x,\lambda,\beta);x}(\Gamma)$ and smooth germs $\Lambda: (\mathbb{R} \times \mathbb{R}^t, 0) \longrightarrow \mathbb{R}$ and $A: (\mathbb{R}^t, 0) \longrightarrow \mathbb{R}^q$ satisfying $X(x, \lambda, 0) = x$, $\Lambda(\lambda, 0) = \lambda$, and $A(0) = 0$ such that

$$H(x, \lambda, \beta) = S(x, \lambda, \beta) \circ G\big(X(x, \lambda, \beta), \Lambda(\lambda, \beta), A(\beta)\big).$$

The *unfolding theorem*, similar to ([GSS, 1988], Theorem 2.1, p. 211), provides a necessary and sufficient condition for G to be versal

$$\text{(I. V.)} \qquad \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma) = T(g, \Gamma) + \mathbb{R}\left\{ d_{\alpha_i} G(x, \lambda, 0), i = 1, \ldots, q \right\}.$$

Here (I.V.) stands for *infinitesimally versal.*

DEFINITION 3.1. Let $G \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$. Assume G is the germ at 0 of the Γ-equivariant mapping $\tilde{G}: \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$. Denote the space of such mappings by $\mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, \mathbb{R}^n; \Gamma)$. G is said to be *stable* if for every Γ-invariant open neighborhood $\mathcal{U}$ of $(0,0,0)$ in $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q$, there exists a neighborhood $\mathcal{W}$ of $\tilde{G}$ in $\mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, \mathbb{R}^n; \Gamma)$ such that for every $\tilde{G}' \in \mathcal{W}$ there exists $(x', \lambda', \alpha') \in \mathcal{U}^\Gamma$ such that G is equivalent to the germ of $\tilde{G}'$ at (x', λ', α'). More precisely, there are germs $S: (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, 0) \longrightarrow \mathrm{End}(\mathbb{R}^n)$, $X: (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, 0) \longrightarrow (\mathbb{R}^m, x')$, $\Lambda: (\mathbb{R} \times \mathbb{R}^q, 0) \longrightarrow (\mathbb{R}, \lambda')$ and $A: (\mathbb{R}^q, 0) \longrightarrow (\mathbb{R}^q, \alpha')$; satisfying: $S \in \overleftrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$, $X(\gamma . x, \lambda, \alpha) = \gamma . X(x, \lambda, \alpha)$ for all x, λ, γ, α, $\det S(0,0,0) > 0$, $\det(d_x X)_{(0,0,0)} > 0$, $(d_\lambda \Lambda)_{(0,0)} > 0$ and $\det(dA)_0 > 0$; such that

$$\text{(3.1.1)} \qquad G(x, \lambda, \alpha) = S(x, \lambda, \alpha) \circ G'\big(X(x, \lambda, \alpha), \Lambda(\lambda, \alpha), A(\alpha)\big),$$

where G' is the germ of $\tilde{G}'$ at (x', λ', α').

Note that if G is an unfolding of g, setting $\alpha = 0$ gives the definition of stability for g.

The second result of this paper is:

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THEOREM 3.2 (STABILITY OF UNFOLDINGS). *Let* $G \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$ *be an unfolding of* $g \in \overrightarrow{\mathcal{E}}_{(x,\lambda);y}(\Gamma)$. *The following conditions are equivalent:*

i) G *is stable.*

ii) G *is versal.*

COMMENT. This is Theorem 2.3 of [G, 1988]. We propose a new proof for it below.

4. **Proof of the stability theorems.** Let us first give

THEOREM 4.1 (THE MAIN STABILITY THEOREM). *Let* $G \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$. *The following conditions are equivalent:*

i) $T_{\mathrm{un}}(G,\Gamma) = \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$.

ii) G *is stable.*

Before we prove this theorem let us see how it can yield Theorems 2.2 and 3.2. In Theorem 2.2, to obtain the equivalence of (i) and (ii) set $\alpha = 0$ in 4.1; on the other hand (ii) and (iii) are equivalent by the unfolding theorem stated in § 3. To obtain Theorem 3.2 note that the versality condition (I.V.) is equivalent to 4.1.i) by a corollary of the preparation theorem (*cf.* [GSS, 1988]), Corollary 7.2, p. 235).

The essence of the argument in the proof of Theorem 4.1 lies in the passage from the algebraic criterion of infinitesimal stability to a geometric one involving the transversality of J^rG to $\mathcal{K}^r_{\mathrm{un}}(\Gamma).G$. To establish this first we need

LEMMA 4.2. *There exists a number* ω *so that the following conditions are equivalent:*

i) $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \subseteq T_{\mathrm{un}}(G,\Gamma) + [(\mathcal{M}_{x,\lambda,\alpha})^{\omega}.\,\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}] \cap \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$.

ii) $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) = T_{\mathrm{un}}(G,\Gamma)$.

This lemma is a generalization of Mather's lemma to systems of DA-algebras as developed by Damon in [D, 1984], p. 35.

PROOF. (i) implies (ii) is the non-trivial part of the proof. For this apply [D, 1984], Lemma 7.3, (1), p. 35, with $N = T\mathcal{K}_{\mathrm{un}}(\Gamma)$, $M_0 = M = \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$, $\{R_\alpha\}$ equal to the system of DA-algebras $\{\mathcal{E}_{x,\lambda,\alpha}(\Gamma), \mathcal{E}_{\lambda,\alpha}, \mathcal{E}_\alpha\}$ and $a = da_G$. As we saw in 1.3, $a\colon N \longrightarrow M$ is a homomorphism of finitely generated $\{R_\alpha\}$-modules. Therefore, the lemma can be applied. ■

LEMMA 4.3. *If* P *is a subspace of* $m, J^r(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, \mathbb{R}^n; \Gamma)$, *then* $J^rG \pitchfork (\pi^r)^{-1}(P)$ *if and only if* $m, J^rG \pitchfork P$.

Note that we are using here the usual notion of transversality on equivariant jet spaces. The proof of this lemma is obvious since $J^r(x,\lambda,\alpha)G = \big((x,\lambda,\alpha), m, J^r(x,\lambda,\alpha)G\big)$.

The fundamental geometric characterization of infinitesimal stability is

THEOREM 4.4. $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) = T_{\mathrm{un}}(G,\Gamma)$ *if and only if for some (and then any)* $r \geq \omega$, $J^r G \pitchfork \mathcal{K}^r_{\mathrm{un}}(\Gamma).G$.

PROOF. By virture of Lemma 4.3 it suffices to establish this theorem for the medium orbit and jet extension of G. First we need the following lemma. Identify the medium jet space and its tangent space.

LEMMA 4.5. $d(m, J^r G)_{(0,0,0)}\colon (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R})^\Gamma \longrightarrow m, J^r(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, \mathbb{R}^n; \Gamma)$

$$(\xi_1, \xi_2, \xi_3) \longmapsto J^r_0[d_x G(\xi_1) + d_\lambda G(\xi_2) + d_\alpha G(\xi_3)]$$

i.e., $d(m, J^r G)_{(0,0,0)}(\xi_1, \xi_2, \xi_3) = d_x G(\xi_1) + d_\lambda G(\xi_2) + d_\alpha G(\xi_3)$ *modulo* $[(\mathcal{M}_{x,\lambda,\alpha})^{r+1}.\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}] \cap \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$.

PROOF. Recall that $d(m, J^r G)_{(0,0,0)}(\xi_1, \xi_2, \xi_3) =$ linear part with respect to (ξ_1, ξ_2, ξ_3) of $m, J^r(\xi_1, \xi_2, \xi_3)G - m, J^r(0,0,0)G$. Since $G(x + \xi_1, \lambda + \xi_2, \alpha + \xi_3) - G(x, \lambda, \alpha) = d_x G(\xi_1) + d_\lambda G(\xi_2) + d_\alpha G(\xi_3) + O(\xi_1, \xi_2, \xi_3)$ it follows $m, J^r(\xi_1, \xi_2, \xi_3)G - m, J^r(0,0,0)G = J^r_0[d_x G(\xi_1) + d_\lambda G(\xi_2) + d_\alpha G(\xi_3)] + O(\xi_1, \xi_2, \xi_3)$. Taking the linear part of this with respect to (ξ_1, ξ_2, ξ_3) yields the result. ■

LEMMA 4.6. $m, J^r G \pitchfork m, \mathcal{K}^r_{\mathrm{un}}(\Gamma).G$ *can be expressed by*

$(T.r.)$ *any* $\sigma \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$ *admits a decomposition*

$\sigma = S \circ G + d_x G(X) + d_\lambda G(\Lambda) + d_\alpha G(A) + d_x G(\xi_1) + d_\lambda G(\xi_2) + d_\alpha G(\xi_3)$, *modulo* $[(\mathcal{M}_{x,\lambda,\alpha})^{r+1}.\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}] \cap \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$; *where* $S \in \overleftarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$, $X \in \overrightarrow{\mathcal{M}}_{(x,\lambda,\alpha);x}(\Gamma)$, $\Lambda \in \overrightarrow{\mathcal{M}}_{(\lambda,\alpha);\lambda}$, $A \in \overrightarrow{\mathcal{M}}_{\alpha;\alpha}$ *and* $(\xi_1, \xi_2, \xi_3) \in (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q)^\Gamma$.

PROOF. This follows at once from (1.4.1) and Lemma 4.5. ■

We can now complete the proof of Theorem 4.4. Assume $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) = T_{\mathrm{un}}(G,\Gamma)$. Then for any $\sigma \in \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma)$ there exist $(\tilde{S}, \tilde{X}, \tilde{\Lambda}, \tilde{A}) \in \overleftarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) \times \overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);x}(\Gamma) \times \overrightarrow{\mathcal{E}}_{(\lambda,\alpha);\lambda} \times \overrightarrow{\mathcal{E}}_{\alpha;\alpha}$ such that

$$\sigma(x,\lambda,\alpha) = \tilde{S}(x,\lambda,\alpha) \circ G(x,\lambda,\alpha) + d_x G\big(\tilde{X}(x,\lambda,\alpha)\big) + d_\lambda G\big(\tilde{\Lambda}(\lambda,\alpha)\big) + d_\alpha G\big(\tilde{A}(\alpha)\big).$$

Separate off the constant term in $\tilde{X}$ to get $\tilde{X}(x,\lambda,\alpha) = X(x,\lambda,\alpha) + \xi_1$, where $X(0,0,0) = 0$; furthermore, $\tilde{X}(\gamma.x,\lambda,\alpha) = \gamma.\tilde{X}(x,\lambda,\alpha)$ for all $\gamma \in \Gamma$, $x \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$, and $\alpha \in \mathbb{R}^q$. Hence $X(\gamma.x,\lambda,\alpha) + \xi_1 = \gamma.[X(x,\lambda,\alpha) + \xi_1]$; thus, $X \in \overrightarrow{\mathcal{M}}_{(x,\lambda,\alpha);x}(\Gamma)$ and $\xi_1 \in (\mathbb{R}^m)^\Gamma$. Doing the same with $\tilde{\Lambda}$ and $\tilde{A}$, one obtains $\tilde{\Lambda}(\lambda,\alpha) = \Lambda(\lambda,\alpha) + \xi_2$ and $\tilde{A}(\alpha) = A(\alpha) + \xi_3$ with Λ and A vanishing at the origin. Since Γ acts trivially on $\mathbb{R}$ and $\mathbb{R}^q$, $\xi_2 \in (\mathbb{R})^\Gamma$ and $\xi_3 \in (\mathbb{R}^q)^\Gamma$. Therefore we have

$$\sigma = d_x G(\xi_1) + d_\lambda G(\xi_2) + d_\alpha G(\xi_3) + S \circ G + d_x G(X) + d_\lambda G(\Lambda) + d_\alpha G(A)$$

i.e., (T.r.) is true for all r. Conversely, assume (T.r.) is true for some $r \geq \omega$. In particular (T.ω.) is true. Since $(\xi_1, \xi_2, \xi_3) \in (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q)^\Gamma$, in (T.$\omega$.), we can incorporate (as we did above) ξ_1 into X, ξ_2 into Λ and ξ_3 into A. Now by Lemma 4.2, it follows $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) = T_{\mathrm{un}}(G,\Gamma)$. ■

4.7 PROOF OF THEOREM 4.1. Since $\overrightarrow{\mathcal{E}}_{(x,\lambda,\alpha);y}(\Gamma) = \mathcal{T}_{\text{un}}(G,\Gamma)$, by the finite determinacy theorem (*cf.* [D, 1984], Theorem 10.2, p. 49) G is r_0 $\mathcal{K}_{\text{un}}(\Gamma)$-determined for some integer r_0. Let $\mathcal{U}$ be a Γ-invariant open neighborhood of $(0,0,0)$ in $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q$ and consider $\tilde{G}\colon \mathcal{U} \longrightarrow \mathbb{R}^n$. Let $\tilde{G}'\colon \mathcal{U} \longrightarrow \mathbb{R}^n$ be sufficiently close to $\tilde{G}$ in the C^∞ topology. By Theorem 4.4, $J^{r_0}G \pitchfork \mathcal{K}_{\text{un}}^{r_0}(\Gamma).G$ at $J^{r_0}_{(0,0,0)}G$. Since $\tilde{G}'$ is close to $\tilde{G}$, $J^{r_0}G' \pitchfork \mathcal{K}_{\text{un}}^{r_0}(\Gamma).G$ at some jet Z close to $J^{r_0}_{(0,0,0)}G$. Let (x',λ',α') be the source of Z. Then $J^{r_0}(x',\lambda',\alpha')G' \in \mathcal{K}_{\text{un}}^{r_0}(\Gamma).G$; whence, there exist S, X, Λ and A as in Definition 3.1, such that the following diagram commutes:

$$
\begin{array}{ccc}
(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, 0) & \xrightarrow{J^{r_0}(0,0,0)G} & \mathbb{R}^n \\
{\scriptstyle J_0^{r_0}((X,\Lambda,A))}\big\downarrow & & \big\uparrow{\scriptstyle J_0^{r_0}(S)} \\
\big(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^q, (x',\lambda',\alpha')\big) & \xrightarrow{J^{r_0}(x',\lambda',\alpha')G'} & \mathbb{R}^n.
\end{array}
\tag{4.7.1}
$$

Now define the germ H by

$$H = S \circ G'(X,\Lambda,A), \tag{4.7.2}$$

where S, X, Λ and A are the same as in (4.7.1). Taking the r_0-jet of (4.7.2) and comparing it to (4.7.1) yields $J_0^{r_0}G' = J_0^{r_0}H$. But G is r_0 $\mathcal{K}_{\text{un}}(\Gamma)$-determined; therefore, G and H are Γ-equivalent at $(0,0,0)$. This together with (4.7.2) yields (3.1.1); *i.e.*, G is stable. Conversely, assume G is stable. For any $r \geq \omega$, since $\mathcal{W}$ is an open neighborhood of $\tilde{G}$, by Thom's transversality theorem there exists $\tilde{G}' \in \mathcal{W}$ such that $J^r\tilde{G}' \pitchfork \mathcal{K}_{\text{un}}^r(\Gamma).G$. On the other hand, since $\tilde{G}' \in \mathcal{W}$, the germ G' at (x',λ',α') is Γ-equivalent to the germ G at $(0,0,0)$; hence, $J^rG \pitchfork \mathcal{K}_{\text{un}}^r(\Gamma).G$ at $J^r(0,0,0)G$. Now Theorem 4.4 concludes the proof.

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