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SCALAR PRODUCTS OF CERTAIN HECKE L-SERIES AND MOMENTS OF WEIGHTED NORM-COUNTING FUNCTIONS

BY

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In memory of Robert A. Smith

ABSTRACT. We consider Dirichlet series $R(s)$, constructed by taking scalar products of Hecke L-series with ray-class characters. Using a theorem of G. W. Mackey on tensor products of representations of finite groups we show that $R(s)$ has a meromorphic continuation into $Re(s) > 1/2$ (obtained by more sophisticated methods in [1]–[5]); we then obtain estimates for the growth of $R(s)$ on vertical lines. Via the Mellin transformation we deduce asymptotics for various weighted moment sums involving ideals of given ray-class and norm, in one or several fields simultaneously.

Introduction. Suppose that $\{K_j\}_{j \in J}$ is any finite non-empty set of algebraic number fields (repeats allowed) and that, for each $j \in J$, we are given a ray-class character χ_j corresponding to a conductor $\mathfrak{f}_j$ of K_j. We write, for $n \geqslant 1$ in $\mathbb{Z}$, $r_j(n) = \Sigma\, \chi_j(\mathfrak{a}_j)$, summed over all integral ideals $\mathfrak{a}_j$ of K_j with absolute norm n, with the convention that $\chi_j(\mathfrak{a}_j) = 0$ whenever $\mathfrak{a}_j$ and $\mathfrak{f}_j$ have a common (finite) prime divisor. We now put $r(n) = \Pi_{j \in J}\, r_j(n)$. Then $r(n)$ is clearly multiplicative, while it is well-known that $\#\{\mathfrak{a}_j; N\mathfrak{a}_j = n\} = 0_\epsilon(n^\epsilon)$ as $n \to \infty$ for any fixed $\epsilon > 0$. It follows that the Dirichlet series $\Sigma_{n=1}^{\infty} r(n)n^{-s}$ is absolutely convergent for $\sigma := Re\, s > 1$, in which region it represents an analytic function of s, which we denote by $R(s, \boldsymbol{\chi})$ (or just by $R(s)$ when there is no danger of ambiguity). In the language of [4–6], [8], [9] $R(s)$ is the *scalar product* of the functions $\Sigma_{n=1}^{\infty} r_j(n)n^{-s}$, which clearly coincide (for $\sigma > 1$) with the Hecke L-series $\mathrm{L}(s, \chi_j)$ associated with the ray-class characters χ_j. Our aim in this paper is threefold. In the first place we shall apply known results on the representations of finite groups in order to obtain a meromorphic continuation of $R(s)$ to the half-plane $\sigma > 1/2$. This affords an alternative procedure to those of [4–6], [8], [9], without recourse to deep results in the theory of algebraic groups or of Weil L-series. (It should be pointed out that the methods of [4–6], [8], [9] also yield a meromorphic continuation to $\sigma > 0$, and also work for general Grössencharaktere, not just for ray-class characters). Secondly, we use the continuation of $R(s)$ to $\sigma > 1/2$ in order to obtain the asymptotic behaviour of $\Sigma_{1 \leqslant n \leqslant x}\, r(n)$ and $\Sigma_{1 \leqslant n \leqslant x} |r(n)|^{2h}$ for each positive integral h, as $x \to \infty$. Thirdly, we take the opportunity to point out that our Theorem 1 generalises results in

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[12]. I am greatly indebted to J. Lagarias (MR 80h: 12011) for noticing a gap in the attempted proof in [12] of what is now (0.5) of Theorem 1; we asserted, without proof, that certain Artin L-functions are entire, and this now proved as a consequence of Proposition 2.2.

We now briefly outline the main ideas used in this paper. Since $r(n)$ is multiplicative, and $\Sigma_{n=1}^{\infty} r(n)n^{-s}$ is absolutely convergent for $\sigma > 1$, we have an Euler product expansion

$$(0.1) \qquad R(s) = \prod_{p \text{ prime}} \left\{1 + \sum_{k=1}^{\infty} p^{-sk} r(p^k)\right\} \qquad (\sigma > 1).$$

As $r(n) = 0_\epsilon(n^\epsilon)$ it is clear that any singularity of $R(s)$ as $\sigma \to 1+$ must be due to the behaviour of the $r(p)$ when p is prime. In §2 we prove the existence of a finite Galois extension $F/\mathbb{Q}$ such that, for any prime $p \geqslant 2$ of $\mathbb{Z}$, unramified in $F/\mathbb{Q}$, the value of $r(p)$ depends only on the Frobenius conjugacy class Frob $p = ((F/\mathbb{Q})/p)$ of p in $G = \text{Gal } F/\mathbb{Q}$. Indeed, we show that $r(p) = \psi(\text{Frob } p)$, where ψ is a monomial character of G, i.e. a (non-empty) sum of characters induced from one-dimensional representations of subgroups of G. (For this we use some standard results of classical representation theory, assembled in §1 for easy reference). From this it is straightforward to show that, for $\sigma > 1$, $R(s) = E(s)\mathrm{L}(s, F/\mathbb{Q}, \psi)$, where L is the Artin L-function and $E(s)$ is analytic and uniformly bounded for $\sigma \geqslant 1/2 + \epsilon$ ($\epsilon > 0$ arbitrary). Moreover, the structure of ψ shows that $\mathrm{L}(s, F/\mathbb{Q}, \psi)$ is a product of Hecke L-series with ray-class characters, so that it is entire, apart from a possible multiple pole at $s = 1$. This then gives the continuation of $R(s)$ to $\sigma > 1/2$.

Using known results [11] on the growth of Hecke L-series on vertical lines, together with the formula $R(s) = E(s)\, \mathrm{L}(s, F/\mathbb{Q}, \psi)$, we are then able (in §3) to tackle the behaviour of $\Sigma_{1 \leqslant n \leqslant x}\, r(n)$. Specifically, we prove

THEOREM 1. (i) *Let* $\psi = \Pi_{j \in J}\, \chi_j^G$ *(see §2 for notation). Then there exists a positive integer k, depending only on the* χ_j*, such that*

$$(0.2) \qquad \sum_{1 \leqslant n \leqslant x} r(n)(1 - n/x)^k = xP(\log x) + 0_\epsilon(x^{\epsilon + 1/2}), \qquad (x \to \infty),$$

where P is a polynomial of degree not exceeding $\langle \psi, 1_G \rangle_G - 1$ *(where* $\langle \cdot, \cdot \rangle_G$ *is the standard inner product of class functions of G);*

(ii) *if* $r(n) \geqslant 0$ *for all n there is a constant* $\theta = \theta(\chi) < 1$ *such that*

$$(0.3) \qquad \sum_{1 \leqslant n \leqslant x} r(n) = xP^{+}(\log x) + 0(x^{\theta}),$$

where P^{+} *is a polynomial of degree exactly* $\langle \psi, 1_G \rangle_G - 1$;

(iii) *whether or not* $r(n) \geqslant 0$ *for all* $n > 0$, *for each fixed positive integer h there exists a constant* $\varphi = \varphi(h, \chi) < 1$ *such that*

$$(0.4) \qquad \sum_{1 \leqslant n \leqslant x} |r(n)|^{2h} = xQ_h(\log x) + 0(x^{\varphi}),$$

where Q_h *is a polynomial of exact degree* $\langle \psi^h, \psi^h \rangle_G - 1$;

(iv) *for each* $j \in J$ *let* $\mathscr{C}_j$ *be a ray-class* (mod $\mathfrak{f}_j$) *in* K_j, and let $f_j(n) = \#\{\mathfrak{a}_j \in \mathscr{C}_j; N\mathfrak{a}_j = n\}$. *We put* $f(n) = \prod_{j \in J} f_j(n)$. *Then*

$$\sum_{1 \leq n \leq x} f(n) = xP(\underline{\mathscr{C}}, \log x) + 0(x^\delta), \tag{0.5}$$

where $\delta = \delta(\underline{\mathscr{C}}) < 1$ *and* $P(\underline{\mathscr{C}}, -)$ *is a polynomial of degree exactly* $\langle \prod_{j \in J} | r_j, 1_G \rangle_G - 1$ (*where* $\Gamma_j = \text{Gal } F/K_j$).

By choosing special χ and $\underline{\mathscr{C}}$ in Theorem 1 we recover results obtained by previous authors. For example, if all K_j are equal (to K, say) and all $\mathfrak{f}_j = (1)$ we have $r(n) = r_1(n)^q$ ($q = \#J$) where $r_1(n) = \#\{\mathfrak{a}; N\mathfrak{a} = n\}$. It appears from the literature that even this special case has only been treated in detail when $K/\mathbb{Q}$ is Galois — see e.g. [1], [2], [10]. We remark that, even when $K/\mathbb{Q}$ is Galois, the precise value of φ is unknown, although some progress in this direction is made in [1]. It should also be noted that, by suitable modifications of our method here, we can also treat $\Sigma |r(n)|^{2h}$ when $h > 0$ is not an integer. In this case the function $\Sigma |r(n)|^{2h} n^{-s}$ has a branch point at $s = 1$, corresponding to a "pole of non-integral order". By imitating, e.g., the procedure in ([13], p. 233–235) we find

$$\sum_{1 \leq n \leq x} |r(n)|^{2h} \sim C_h x (\log x)^{e_h - 1} \left\{ 1 + \sum_{n=1}^{\infty} c_{hn} (\log x)^{-n} \right\} \tag{0.6}$$

where $C_h > 0$, $e_h = \langle |\psi|^{2h}, 1_G \rangle_G$, and the c_{hn} are constants. The proof is omitted.

1. **Results on characters of finite groups**. We refer to [3] for all results quoted in this section. Let G be a finite group; if θ and φ are ($\mathbb{C}$-valued) class-functions on a subgroup H of G we write

$$\langle \theta, \varphi \rangle_H = (\#H)^{-1} \sum_{h \in H} \bar{\theta}(h) \varphi(h) \tag{1.1}$$

Then $\langle \, , \rangle_H$ is a (positive definite) inner product on the $\mathbb{C}$-vector space of class-functions: $H \to \mathbb{C}$, and the irreducible characters of H form an orthonormal basis of this space under $\langle \, , \rangle_H$. The restriction to H of a character χ of G is denoted by χ_H, while, if φ is a character of H, we denote by φ^G the induced character of G, with values $\varphi^G(g) = (\#H)^{-1} \Sigma \varphi(xgx^{-1})$, the sum taken over all $x \in G$ with $xgx^{-1} \in H$. Then we have the celebrated Frobenius reciprocity formula

$$\langle \chi, \varphi^G \rangle_G = \langle \chi_H, \varphi \rangle_H \tag{1.2}$$

When χ and ψ are characters of G the function $g \mapsto \chi(g)\psi(g)$ is also a character of G, being the trace of the tensor product of the corresponding representations. In this paper we shall be much concerned with products of characters. Before stating the fundamental results required we consider first the conjugation of characters of subgroups. If χ is a character of H then, for all $u \in G$, we can construct a character ${}^u\chi$ of $u^{-1}Hu$ by putting $({}^u\chi)(u^{-1}hu) = \chi(h)$ for all $h \in H$. We have $\deg {}^u\chi = \deg \chi$ and $\langle {}^u\chi, {}^u\chi \rangle_{u^{-1}Hu} = \langle \chi, \chi \rangle_H$. In particular, irreducibility is conserved by conjugation.

PROPOSITION 1.1. (Mackey's tensor product theorem, [3], p. 325). *Let H and K be subgroups of G, and let $G = \bigcup_{u \in U} HuK$ be an irredundant decomposition of G into $H - K$ double cosets. For $u \in U$ we put $D_u := u^{-1}Hu \cap K$. Then, if χ and ψ are characters of H and K respectively, we have*

$$(1.3) \qquad \chi^G \psi^G = \sum_{u \in U} ((^u\chi)_{D_u} \psi_{D_u})^G$$

COROLLARY 1.2. *If, in Proposition 1.1, $\deg \chi = \deg \psi = 1$, then $\chi^G\psi^G$ is a (non-empty) sum of characters $\varphi_i{}^G$, where each φ_i is a character of degree 1 on a subgroup S_i of G (repeats allowed). More generally any finite product of characters (of G) induced from one-dimensional representations is again a non-empty finite sum of $\varphi_i{}^G$. ($\deg \varphi_i = 1$).*

PROOF. For a product of two characters we use (1.3), observing that, in this case, $\deg (^u\chi)_{D_u} = \deg {}^u\chi = \deg \chi = 1 = \deg \psi = \deg \psi_{D_u}$. For a product of $n \geqslant 2$ characters we use induction on n.

COROLLARY 1.3. *If $\chi = 1_H$ and $\psi = 1_K$ then $1_H{}^G 1_K{}^G$ is a (non-empty) sum of $1_{S_i}{}^G$ over subgroups S_i of G (repeats allowed). More generally, if $H_1, \ldots, H_n$ are subgroups of G (repeats allowed), then $1_{H_1}{}^G \ldots 1_{H_n}{}^G$ is a (non-empty) sum of $1_{S_i}{}^G \cdot (n \geqslant 1)$. In particular, if $H_1 = \ldots = H_N = H$,* the *S_i are intersections of conjugates of H.*

PROOF. (i) Using (1.3), and noting that $(^u1_H)_{D_u} = 1_{D_u} = 1^2_{D_u}$, we have the required result for $1_H{}^G 1_K{}^G$;

(ii) the result for $\Pi_{i=1}^n 1_{H_i}{}^G$ now follows by induction on $n \geqslant 2$;

(iii) when $H_1 = H_2 = \ldots = H$ the result follows from the decomposition $G = \bigcup_{u \in U} HuH$, and the fact that intersections involving the D_u and/or H are intersections of conjugates of H.

PROPOSITION 1.4. *Let H be a subgroup of G. Then $1_H{}^G$ takes integer values between 0 and $d = (G:H)$. Let $N_r = \#\{g \in G;\ 1_H{}^G = r\}$, $0 \leqslant r \leqslant d$. Then, for all $k \geqslant 1$ in $\mathbb{Z}$, $c_k := (\#G)^{-1} \Sigma_{r=0}^d N_r r^k = \langle (1_H{}^G)^k, 1_G \rangle_G$ is a positive integer. We have $c_1 = 1$, while c_2 is the number of $H - H$ double cosets in G. Further $N_d = \# \bigcap_{x \in G} x^{-1}Hx$, while, if H is normal, we have $c_k = d^{k-1}$ for all $k \geqslant 1$.*

PROOF. (i) We have $1_H{}^G(g) = (\#H)^{-1} \#\{x \in G;\ xgx^{-1} \in H\}$. For fixed g the set of $x \in G$ with $xgx^{-1} \in H$ is a (possibly empty) union of cosets Ht so that $1_H{}^G$ takes non-negative values in $\mathbb{Z}$. Since $1_H{}^G$ is a character we have $|1_H{}^G(g)| \leqslant \deg 1_H{}^G = (G:H) = d$, as required.

(ii) It is clear that $\langle (1_H{}^G)^k, 1_G \rangle_G$ is a non-negative integer, being an inner product of characters. Using (1.1) we see that $\langle (1_H{}^G)^k, 1_G \rangle_G = (\#G)^{-1} \Sigma_{r=0}^d N_r r^k = c_k$. Moreover $1_H{}^G(1) = d$, so that $N_d \geqslant 1$ and hence $c_k > 0$.

(iii) We have $c_1 = \langle 1_H{}^G, 1_G \rangle_G = \langle 1_H, 1_H \rangle_H$ (by (1.2)) $= 1$. Using (1.3) we have $c_2 = \langle (1_H{}^G)^2, 1_G \rangle_G = \Sigma_{u \in U} \langle 1_{D_u}{}^G, 1_G \rangle_G = \Sigma_{u \in U} \langle 1_{D_u}, 1_{D_u} \rangle_{D_u} = \#U$, again using (1.2). (Here $G = \bigcup_{u \in U} HuH$ in the notation of Proposition 1.1).

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(iv) $1_H{}^G$ is the character of the permutation representation π of G on the right cosets of H, given by $\pi: g \mapsto (Hx \mapsto Hxg)$. The kernel of π is clearly $\bigcap_{x \in G} x^{-1}Hx$, while $1_H{}^G(g) = d$ if and only if $g \in \ker \pi$. Hence $N_d = \# \bigcap_{x \in G} x^{-1}Hx$.

(v) If $H \lhd G$ we have $1_H{}^G(g) = d$ if $g \in H$ and 0 otherwise. Hence $c_k = (\#G)^{-1} \Sigma_{h \in H} d^k = d^{k-1}$, as required.

2. **Meromorphic continuation of $R(s)$**. We revert now to the notation of §0. For each $j \in J$ let H_j/K_j be the maximal (finite) abelian extension of K_j with conductor f_j. Then, by Artin's reciprocity theorem ([7], p. 205–206) we may regard χ_j as a character of Gal H_j/K_j. Let $F/\mathbb{Q}$ be any finite Galois extension extension with $H_j \subseteq F$ for all $j \in J$. Then each χ_j lifts to a character of Gal F/K_j, since Gal H_j/K_j is a quotient of Gal F/K_j. By abuse of notation we still denote the lifted character by χ_j. Now let $G =$ Gal $F/\mathbb{Q}$. Then each χ_j yields an induced character $\chi_j{}^G$ of G.

PROPOSITION 2.1. *If p is any prime $\geqslant 2$ unramified in $F/\mathbb{Q}$ then $r(p) = \Pi_{j \in J} \chi_j{}^G$* (Frob p), *where* Frob $p = ((F/\mathbb{Q})/p)$ *is the Frobenius class of p in G.*

PROOF. It suffices to prove that $r_j(p) = \chi_j^G$ (Frob p) for all $j \in J$. The latter is clear, however, from the standard proof that $\mathrm{L}(s, F/\mathbb{Q}, \chi_j{}^G) = \mathrm{L}(s, F/K_j, \chi_j)$ for Artin L-functions ([7], p. 236–239).

We now exploit Proposition 2.1 in order to find the singularities of $R(s)$ as $\sigma \to 1+$. Apart from the (finitely many) primes which ramify in $F/\mathbb{Q}$, the coefficients of p^{-s} in the Euler products for $R(s)$ and $\mathrm{L}(s, F/\mathbb{Q}, \Pi_j\, \chi_j{}^G)$ ($\sigma > 1$) are equal, while, by Corollary 1.2, $\Pi_j\, \chi_j{}^G$ is a (non-empty) sum of characters $\phi_i{}^G$, ϕ_i a degree-one character of S_i, the latter being various subgroups of G. Using the result $\mathrm{L}(s, F/T_i, \phi_i) = \mathrm{L}(s, F/\mathbb{Q}, \phi_i{}^G)$ ([7], p. 236–239), where T_i is the fixed field of S_i, we see that $\mathrm{L}(s, F/\mathbb{Q}, \Pi_j\, \chi_j{}^G) = \Pi_i\, \mathrm{L}(s, F/T_i, \phi_i)$ is a product of Hecke L-series ($\sigma > 1$). Thus it cannot vanish when $\sigma \geqslant 1$, and, in this region, $E(s) := R(s)/\mathrm{L}(s, F/\mathbb{Q}, \Pi_j\, \chi_j{}^G)$ has an Euler product in which the unramified primes contribute factors $1 + c_2(p)p^{-2s} + \ldots$ with $|c_k(p)| = 0(k^a)$ for some fixed a, all p and all $k \leqslant k_0$. The ramified p contribute factors which are analytic for $\sigma > 0$. It follows that $E(s)$ is analytic and uniformly bounded for $\sigma \geqslant \epsilon + 1/2$ for any fixed $\epsilon > 0$. (In fact $E(s)$ can only have finitely many zeros in $\sigma \geqslant \epsilon + 1/2$, but it is not clear whether any of them lie in $\sigma \geqslant 1$). This establishes the existence of a meromorphic continuation of $R(s)$ to $\sigma > 1/2$. The question of continuation of $R(s)$ into $\sigma \leqslant 1/2$ depends more on $E(s)$ than on L, in view of

PROPOSITION 2.2. *$\mathrm{L}(s, F/\mathbb{Q}, \Pi_j\, \chi_j{}^G)$ is analytic in $\mathbb{C}$, except for a pole of order precisely $\langle \Pi_j\, \chi_j{}^G, 1_G \rangle_G$ at $s = 1$.*

PROOF. $\mathrm{L}(s, F/\mathbb{Q}, \Pi\, \chi_j{}^G)$ is a product of Hecke L-series $\mathrm{L}(s, F/T_i, \varphi_i)$. These are entire functions except when $\varphi_i \neq 1_{S_i}$, when there is a simple pole at $s = 1$ and no other singularity. Moreover, when $\varphi_i \neq 1_{S_i}$, $\mathrm{L}(s, F/T_i, \varphi_i)$ does not vanish for $\sigma \geqslant 1$. Consequently $\mathrm{L}(s, F/\mathbb{Q}, \Pi_j\, \chi_j{}^G)$ has a pole of order precisely $\Sigma_i \langle \varphi_i, 1_{S_i} \rangle_{S_i} = \Sigma_i \langle \varphi_i{}^G, 1_G \rangle_G = \langle \Pi_j\, \chi_j{}^G, 1_G \rangle_G$ at $s = 1$, and is analytic elsewhere.

Although it is not needed for our proof of Theorem 1, we remark that a detailed examination of $E(s)$ will yield the continuation of $R(s)$ to $\sigma > 0$ (possibly modulo the extended Riemann hypothesis); for this see [4–6], [8], [9]. In [9] Moroz has shown how certain aspects of Kurokawa's analysis can be simplified by applying the analogue of Proposition 1.1 for finite dimensional representations of Weil groups. We shall not pursue this topic any further in this paper.

3. **Proof of Theorem 1**. In order to prove Theorem 1 it is tempting to apply Perron's summation formula

$$\sum_{1 \leq n \leq x} r(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s}{s} R(s)\, ds, \tag{3.1}$$

and then to move the vertical contour to the left, using Cauchy's residue theorem. Unfortunately $R(s)$ appears to grow too rapidly as $|\mathrm{Im}\, s| \to \infty$ to allow this direct approach to work. However, as $E(s)$ is uniformly bounded for $\sigma \geqslant \epsilon + 1/2$, we can apply the estimate

$$|\mathrm{L}(\sigma + i\tau, \varphi)| = O(|\tau|^{C(\sigma, \varphi)})\,(0 < \sigma < 3 \text{ fixed, } |\tau| \to \infty, \tau \text{ real}) \tag{3.2}$$

for Hecke L-series with ray-class characters [11] to deduce that

$$|R(\sigma + i\tau)| = O_\epsilon(|\tau|^q) \qquad (|\tau| \to \infty, \tau \text{ real}) \tag{3.3}$$

uniformly for $\sigma \geqslant \epsilon + 1/2$ (ϵ arbitrary fixed $> O$), where $q = q(\epsilon, \chi)$. Now choose k to be any positive integer greater than q. We use the weighted Perron summation formula

$$\sum_{1 \leq n \leq x} r(n)(1 - n/x)^k = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s R(s)\, ds}{s(s+1)\ldots(s+k)} \tag{3.4}$$

in place of (3.1). The estimate (3.3) suffices to yield the bound

$$\int_{1/2+\epsilon-i\infty}^{1/2+\epsilon+i\infty} \frac{x^s R(s)\, ds}{s(s+1)\ldots(s+k)} = O(x^{1/2+\epsilon}). \tag{3.5}$$

Combining (3.4), (3.5) and Cauchy's residue theorem, we obtain

$$\sum_{1 \leq n \leq x} r(n)(1 - n/x)^k = xP(\log x) + O(x^{1/2+\epsilon}), \tag{3.6}$$

on calculating the residue at $s = 1$ (the only relevant pole). Here P is a polynomial; since $E(s)$ may (conceivably) have a zero at $s = 1$ we can only say, in general, that the order of the pole of $R(s)$ at $s = 1$ does not exceed $\langle \Pi_j \chi_j{}^G, 1_G \rangle_G$, so that $\deg P \leqslant \langle \Pi_j \chi_j{}^G, 1_G \rangle_G - 1$. This gives (0.2) of Theorem 1.

Now consider the case where $r(n) \geqslant 0$ for all $n > 0$. We shall apply the following simple Tauberian

LEMMA. *Let $S_n \uparrow \infty$, and suppose that*

$$\sum_{1 \leq n \leq N} S_n = N^2 P(\log N) + O(N^{\alpha+1}) \qquad (N \to \infty), \tag{3.7}$$

where P is a polynomial and $\alpha < 1$. Then

$$(3.8) \qquad S_N = 2NP(\log N) + NP^1(\log N) + O(N^{(\alpha+1)/2}(\log N)^{\deg P/2}),$$

where $P^1(y) = dP(y)/dy$.

This is easily proved on combining (3.7) with the inequalities $(1 + k)S_N \leqslant S_N + \ldots + S_{N+k} \leqslant (1 + k)S_{N+k}$, taking $k \sim N^{(1+\alpha)/2}(\log N)^{-\deg P/2}$. The error term (3.8) is near-optimal at this level of generality.

Since we are assuming that $r(n) \geqslant 0$ we can apply (3.8) with $S_N = \Sigma_{1 \leqslant n \leqslant N} r(n)(1 - n/N)^{k-1}$ and $\alpha = \epsilon + 1/2$ (in view of 3.6). We find that

$$(3.9) \qquad \sum_{1 \leqslant n \leqslant x} r(n)(1 - n/x)^{k-1} = 2xP(\log x) + xP^1(\log x) + O(x^{3/4+\epsilon}).$$

On repeating the process $k - 1$ more times we have

$$(3.10) \qquad \sum_{n \leqslant x} r(n) = xP^*(\log x) + O(\chi^\theta),$$

where we may take $\theta = 1 - 2^{-k} + \epsilon$ (ϵ arbitrary > 0). We also observe that, for $s > 1$, we have $R(s) \geqslant \Pi_p(1 + r(p)p^{-s})$ when $r(n) \geqslant 0$, so that $\log R(s) \geqslant \Sigma_p\, r(p)p^{-s} + C$, and this is enough to ensure that $R(s)$ has a pole of order precisely $\langle \Pi_{j \in J}\, \chi_j{}^G, 1_G \rangle_G$, since $\log \mathrm{L}(s, F/\mathbb{Q}, \Pi_{j \in G}\, \chi_j{}^G) = \Sigma_p\, r(p)p^{-s} + O(1)$ as $s \to 1+$. This now gives (0.3). To obtain (0.4) we simply observe that $r^*(n) := |r(n)|^{2h} = \Pi_{j \in J}\, r_j(n)^h \overline{r_j(n)}^h$ is another function of the same type as r, while $r^*(n) \geqslant 0$ for all $n > 0$. Thus we can apply (3.10) with $P^* = Q_h$.

Now let $\mathscr{C}_j$ $(j \in J)$ be a ray-class (mod* $\mathfrak{f}_j$). We write $f_j(n) := \#\{\mathfrak{a}_j; \mathfrak{a}_j \in \mathscr{C}_j, N\mathfrak{a}_j = n\}$, $f(n) := \Pi_{j \in J}\, f_j(n)$ and $H_j = \{$ray-class characters (mod* $\mathfrak{f}_j$)$\}$. It is then easily seen, by the orthogonality of group characters, that $f(n) = \Sigma_{\boldsymbol{\chi}}\, A(\boldsymbol{\chi})r(n, \boldsymbol{\chi})$, where $\boldsymbol{\chi}$ runs over all vectors of characters chosen from $H_j (j \in J)$, the $A(\boldsymbol{\chi})$ are constants (independent of n), and $r(n, \boldsymbol{\chi}) := \Pi_{j \in J}\, r_j(n, \chi_j)$. We may therefore apply (0.2), obtaining

$$(3.11) \qquad \sum_{1 \leqslant n \leqslant x} f(n)(1 - n/x)^k = xP_f(\log x) + O(x^{\epsilon + 1/2})$$

Since $f(n) \geqslant 0$ for all $n \geqslant 0$ we can now use the Tauberian Lemma, and we find that

$$(3.12) \qquad \sum_{1 \leqslant n \leqslant x} f(n) = xP_f^+(\log x) + O(x^\delta) \qquad (\delta < 1)$$

Let $\mathbf{1}$ be the character-vector corresponding to $\chi_j \equiv 1$ for all $j \in J$. Then it is clear that the dominant contribution to (3.12) comes from $\Sigma\, r(n, \mathbf{1})$, yielding a non-zero leading term $Bx(\log x)^t$, where $t = \langle \Pi_j\, 1_{\Gamma_j}{}^G, 1_G \rangle_G - 1$ (where $\Gamma_j = \mathrm{Gal}\, F/K_j$), while no other $\boldsymbol{\chi}$ can produce a term in $x(\log x)^t$. We have thus proved (0.5).

4. **Some special cases; scalar products of Dedekind zeta functions**. An interesting

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special case of Theorem 1 series when all $f_j = (1)$. Then $R(s)$ is a scalar product of Dedekind zeta functions. Applying Corollary (1.3) we see that $L(s, F/\mathbb{Q}, \Pi_{j\in J}\chi_j{}^G)$ is an (ordinary) product of Dedekind zeta functions, since $\chi_j = 1_{H_j}$ for some subgroup H_j of G. Hence (0.3) applies in this case. If all the K_j are equal then $\Pi_{j\in J}\chi_j{}^G = (1_H{}^G)^k$, where $k = \# J$, and the strength of the pole at $s = 1$ (of L) is $\langle (1_H{}^G)^k, 1_G\rangle_G$.

Hence Proposition 1.4 can be applied. It is a matter of calculating the quantities N_r, $O \leqslant r \leqslant d = (G:H)$, and this is easily accomplished for any given pair G, H, although it would be good to have a simple closed formula for the N_r. The subgroups S_i of Corollary 1.3 are intersections of conjugates of H, and so their fixed fields T_i are composita of conjugates of K over $\mathbb{Q}$. If $K/\mathbb{Q}$ is Galois the situation simplifies considerably; we have $H \lhd G$, so that $\langle (1_H{}^G)^k, 1_G\rangle_G = (G:H)^{k-1}$ for all $k \geqslant 1$ (by Proposition 1.4), while all $S_j = H$ and $T_i = K$. Hence $L(s, F/\mathbb{Q}, \Pi_{j\in J}\chi_j{}^G)$ is the $(G:H)^{k-1}$st power of $\zeta_K(s)$. Theorem 1, in this case, recovers the relevant results in [1], [2], [10], except for the accurate estimation of θ and φ.

It would be interesting to have examples where $E(s)$ (or even $R(s)$) has a zero at $s = 1$. The author is not aware of any examples of this.

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