A THIRD-ENGEL 5-GROUP

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(Received 20 May 1966)

1. Discussion of results

In this paper a certain group with the third-Engel condition, that is a member of the variety defined by [1]

$$(x, 3y) = 1,$$

will be presented. Reasons for which its properties may be of interest are advanced in the present section.

The theory of third-Engel-groups has been established by Heineken in [3], the main result being:

THEOREM 1. (Heineken). *If G is a group with the third-Engel condition then*

(i) *G is locally nilpotent; and*

(ii) *$\Gamma_5(G) \leqq P_2 \times P_5$ where P_2 and P_5 are respectively the Sylow 2- and 5-subgroups of G.*

The hypothesis of this theorem is implied by the condition that the subgroup generated by any two elements of G has class 3.

A related result is:

THEOREM 2. *Let n be a fixed integer greater than 3. If G is a group such that the subgroup generated by any set of $n-1$ elements has class n, then*

(i) *G is locally nilpotent; and*

(ii) *$\Gamma_{n+2}(G) \leqq P_2$ where P_2 is the Sylow 2-subgroup of G.*

The proof of this theorem depends heavily on known results, especially:

THEOREM 3. (Kappe). *If G is a group with no element of order 2 then its second-right-Engel elements lie in $\zeta_3(G)$.*

We note that Theorem 3 is almost (but not quite) a consequence of Kappe's Satz III in [6]. He shows that if e is second-right-Engel then

$$(e, x, y) = (e, y, x)^{-1}.$$

If at this point y is replaced by yz and a standard commutator expansion is undertaken, then there results

[1] Our notation is explained in § 2.

https://doi.org/10.1017/S144678870000450X Published online by Cambridge University Press

$$(e, x, y, z)^2 = 1;$$

our statement of Theorem 3 follows.

PROOF OF THEOREM 2:

(i) Since G is a member of the variety

$$(x_1, 3x_2, x_3, \cdots, x_{n-1}) = 1,$$

$G/\zeta_{n-3}(G)$ belongs to the variety

$$(x_1, 3x_2) = 1.$$

This variety is locally nilpotent by Theorem 1, and so G is locally nilpotent.

(ii) This implies that there is a unique Sylow 2-subgroup P_2 of G; we have therefore to show that $\zeta_{n+2}(G/P_2) = P_2$. We shall assume $P_2 = 1$ and then prove $\zeta_{n+2}(G) = 1$, thus avoiding a change of notation. We deal first with the case $n > 4$, in which a rather easier proof is available.

If c is any commutator of weight $n-1$ in the elements $x_1, \cdots, x_{n-2}$ of G, then we have $(c, 2y) = 1$ for all y in G. Since G has no element of order 2, Theorem 3 ensures that $c \in \zeta_3(G)$. Thus $G/\zeta_3(G)$ has the property that any subgroup generated by $n-2$ elements has class $n-2$. If $n > 4$, Heineken's Satz in [4] shows that $G/\zeta_3(G)$ has class $n-2$, that is that G has class $n+1$.

A different approach (which could be used when $n > 4$ also) is necessary when $n = 4$. We find as above that the law $c = 1$ holds in $G/\zeta_3(G)$ where

$$c = (x_1, 2x_2).$$

The theory of second-Engel-groups was developed by Levi in [8]; if H is such a group then H has class 3 and $\Gamma_3(H)$ has exponent 3. Therefore our group G satisfies the pair of laws

$$((x_1, x_2, x_3, x_4), x_5, x_6, x_7) = 1,$$

$$((x_1, x_2, x_3)^3, x_4, x_5, x_6) = 1.$$

Standard commutator calculation shows that these imply

$$(x_1, x_2, x_3, x_4, x_5, x_6)^3 = 1;$$

so G has class 6 and $\Gamma_6(G)$ is a 3-group. But since G also satisfies

$$(x_1, 3x_2, x_3) = 1$$

we deduce that $G/\zeta_1(G)$ is third-Engel. By Theorem 1

$$\Gamma_5(G/\zeta_1(G)) \cong \Gamma_5(G)\zeta_1(G)/\zeta_1(G)$$

is the direct product of a 2-group and a 5-group, which means that $\Gamma_6(G)$ has a similar structure. Since we have already proved that $\Gamma_6(G)$ is a 3-group, it follows that $\Gamma_6(G) = 1$.

That proves Theorem 2.

There is a difference between Theorems 1 and 2 regarding the behaviour of elements of order 5. It is these we intend to examine rather than the elements of order 2, for the simple reason that there are some very unpleasant third-Engel 2-groups. All groups in the non-nilpotent variety $\mathfrak{A}_2\mathfrak{A}_2$, for instance, are third-Engel 2-groups (this is the set of all groups G with normal subgroups N such that both N and G/N have exponent 2; see [1], example 3.4, or [11], example 16.36 for details).

It is natural to ask whether Theorem 2 holds when $n = 3$, that is whether Theorem 1 (ii) can be amended to read "$\Gamma_5(G) \leqq P_2$". We shall answer this in:

THEOREM 4. *There exists a group, of order* 5^{20} *and exponent* 5, *with the properties that every* 2 *elements generate a subgroup of class* 3 *and that the group itself has class precisely* 5.

We also record a

CONJECTURE. *For each positive integer* n *there is a finite* 5*-group with the properties that every* 2 *elements generate a subgroup of class* 3 *and that the group itself has class precisely* n.

Thus we prove that it is false that $\Gamma_5(G) \leqq P_2$ and we conjecture that it would be false that $\Gamma_n(G) \leqq P_2$ for any n, in Theorem 1 (ii). This is at variance with a "folk-lore" belief that the prime 5 should not be exceptional in Theorem 1. Perhaps a better way of looking at the facts is to regard 3 as being the really exceptional prime.

Another interesting question is whether there exists, for each integer n greater than 2 and each odd prime p, a finite p-group of class precisely $n+1$ with the property that every $n-1$ elements generate a subgroup of class n. We note that such a group is known to exist in two cases:

(i) $n = 3$, any p; see example 4.1 of C.K. Gupta's paper [2].

(ii) $n = 4$, $p = 5$; see Lazard [7], pp. 187–189.

2. Conventions

Our commutator notation is left-normed. We define the commutator $(x_1, m_2x_2, \cdots, m_nx_n)$, in which $m_2, \cdots, m_n$ are positive integers, recursively as follows. If $m > 1$ then

$$(x, my) = ((x, (m-1)y), y),$$

where $(x, 1y) = (x, y) = x^{-1}y^{-1}xy$; and if $n > 2$ then

$$(x_1, m_2x_2, \cdots, m_nx_n) = ((x_1, m_2x_2, \cdots, m_{n-1}x_{n-1}), m_nx_n).$$

A group satisfies the m-th Engel condition if, and only if, it belongs to the variety $(x, my) = 1$. More generally, we say that the element e of the group G is m-right-Engel if, and only if, $(e, my) = 1$ for all y in G. Since we have quoted results of Heineken which are in right-normed notation, we are morally bound to record:

LEMMA 1. *The third-Engel conditions in the right-normed and the left-normed notations are equivalent.*

The proof follows from the commutator identity

$$(y, (y, x))^{(y,x)}(x^{-1}, y, y)^x = 1$$

when x is replaced by (y, x).

The terms $\Gamma_n(G)$ of the lower central series of G are defined thus: $\Gamma_n(G)$ is the subgroup generated by all commutators of the form $(x_1, x_2, \cdots, x_n)$ in G. If $\Gamma_{n+1}(G)$ is 1 then G is said to be (nilpotent) of class n; if in addition $n > 1$ and $\Gamma_n(G) \neq 1$ then G is said to be of class precisely n. The upper central series of G is defined by putting $\zeta_0(G) = 1$, $\zeta_1(G)$ equal to the centre of G, and $\zeta_n(G)/\zeta_{n-1}(G)$ equal to $\zeta_1(G/\zeta_{n-1}(G))$ for $n > 1$. It is a well-known theorem that $\Gamma_n(G) = 1$ if and only if $\zeta_n(G) = G$.

We state the usual list of commutator identities without proof:

(2.1) $$(x^{-1}, y)^{-1} = (x, y)^{x^{-1}},\ (x, y^{-1})^{-1} = (x, y)^{y^{-1}};$$

(2.2) $$(xy, z) = (x, z)^y(y, z),\ (x, yz) = (x, z)(x, y)^z;$$

(2.3) $$(xy, uv) = (x, v)^y(y, v)(x, u)^{yv}(y, u)^v;$$

(2.4) $$(x, y, z^x)(z, x, y^z)(y, z, x^y) = 1.$$

Frequently we shall calculate in $\Gamma_n(G)$ with the group G having class n. It is a consequence of the identities that we then have multilinearity when expanding commutators, in the sense that $(x_1, \cdots, x_{i-1}, yz, x_{i+1}, \cdots, x_n)$ is the product of $(x_1, \cdots, x_i, \cdots, x_n)$ with $x_i = y$ and $x_i = z$; this holds for each i. In particular if $c = (x_1, \cdots, x_{n-2})$ and G has class n then (2.4) gives

$$(c, (y, z)) = (c, y, z)(c, z, y)^{-1}.$$

We use the following abbreviations: x^y means $y^{-1}xy$, x^{-y} means $(x^y)^{-1}$, and $(x, y; u, v)$ means $((x, y), (u, v))$. Some obvious extensions of the last may appear from time to time.

Mappings from a subset of the group G into G will be denoted by the Greek letters α, β, γ, θ. The image of x under θ is x^θ, and we make the following definitions:

$$(x, \theta) = x^{-1}x^\theta,\ (\theta, x) = (x, \theta)^{-1}.$$

Considerable use will be made of the result which follows.

LEMMA 2. *Let θ be a mapping of the group G into G, and let u, v be two fixed elements of G. Then $(u, v)^\theta = (u^\theta, v^\theta)$ if and only if*

$$(v, u; \theta, v)(v, u, \theta)^{(\theta, v)}(\theta, v, u^\theta)(u, \theta, v^u) = 1.$$

REMARK. The abbreviations just used are, explicitly:

$$(v, u; \theta, v) = ((v, u), (\theta, v)),$$
$$(v, u, \theta) = ((v, u), \theta),$$
$$(\theta, v, u^\theta) = ((\theta, v), u^\theta),$$
$$(u, \theta, v^u) = ((u, \theta), v^u).$$

Note also that Lemma 2 becomes just (2.4) when θ is taken to be an inner automorphism of G.

PROOF. We outline the calculations, starting with

$$(v, u; \theta, v) = (v, u)^{-1}(v, u)^{(v^\theta)^{-1}v},$$
$$(v, u, \theta)^{(\theta, v)} = (v, u)^{-(v^\theta)^{-1}v}\{(v, u)^\theta\}^{(v^\theta)^{-1}v},$$
$$(\theta, v, u^\theta) = (v^\theta, u^\theta)^{-(v^\theta)^{-1}v}(v, u^\theta),$$
$$(u, \theta, v^u) = (u, v)^{-u^\theta}(u^\theta, v^u).$$

It is easy to show by direct expansion that

$$(v, u^\theta)(u, v)^{-u^\theta}(u^\theta, v^u)(v, u)^{-1} = 1.$$

The result follows at once.

3. Construction of the group

Information about the structure of the group required in Theorem 4 was acquired in the course of calculations aimed at investigating its existence or non-existence. When existence was suspected this information was used to construct a group G by means of three cyclic splitting extensions, starting from a group isomorphic to $\Gamma_2(G)$; this process is familiar, having already been used to obtain examples in the papers [2], [9], [10]. The details follow.

We start by defining a group A. This shall have the following set of 17 generators:

$$\{c_i, d_j, e_k, f_l\colon\ 1 \leqq i \leqq 3,\ 1 \leqq j \leqq 8,\ 1 \leqq k \leqq 3,\ 1 \leqq l \leqq 3\}.$$

Next we specify some commutation relations by means of a table; if x stands at the beginning of a row and y at the top of a column then (x, y) stands where that row and column intersect.

https://doi.org/10.1017/S144678870000450X Published online by Cambridge University Press

(3.1)

	c_1	c_2	c_3	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
c_1	1	$e_3^{-4}f_2^2$	$e_2^4f_1^3$	f_1^3	1	1	f_2^2	f_1	1	f_3^2	1
c_2		1	$e_1^{-4}f_3^2$	f_2	f_1^2	1	1	f_2^3	1	1	f_3^2
c_3			1	f_3	1	f_1^2	1	f_3	f_2^2	1	1

We add the further defining relations:

$$
\begin{aligned}
&(c_i, e_k) = (c_i, f_l) = 1,\\
&(d_j, d_{j'}) = (d_j, e_k) = (d_j, f_l) = 1,\\
&(e_k, e_{k'}) = (e_k, f_l) = 1,\\
&(f_l, f_{l'}) = 1,
\end{aligned} \tag{3.2}
$$

where of course

$$1 \leqq i \leqq 3,\ 1 \leqq j, j' \leqq 8,\ 1 \leqq k, k' \leqq 3,\ 1 \leqq l, l' \leqq 3.$$

We also put

$$f_l^5 = 1 \ \ (1 \leqq l \leqq 3). \tag{3.3}$$

That completes the definition of A, which clearly has class precisely 2.

Several interpretations may be put on A. The exponents in the obvious normal form for its elements may be taken as integers, or alternatively as integers modulo 5^n for any $n > 0$; when $n = 1$ the order of A is 5^{17}. Normally we use integers modulo 5.

Next we prepare to construct an extension B of A. Define a mapping α from the given generators of A into A as follows:

$$
\begin{aligned}
&(c_1, \alpha) = d_1, && (c_2, \alpha) = d_4, && (c_3, \alpha) = d_7;\\
&(d_1, \alpha) = e_1^{-2}f_2^4f_3^2, && (d_2, \alpha) = e_2^{-3}f_1^2f_3, &&\\
&(d_3, \alpha) = e_3^{-3}f_1^2f_2^3, && (d_4, \alpha) = 1, && (d_5, \alpha) = e_1f_2^2f_3^2,\\
&(d_6, \alpha) = 1, && (d_7, \alpha) = 1, && (d_8, \alpha) = 1;\\
&(e_1, \alpha) = 1, && (e_2, \alpha) = f_3^4, && (e_3, \alpha) = f_2;\\
&(f_l, \alpha) = 1 \text{ for } 1 \leqq l \leqq 3. && &&
\end{aligned} \tag{3.4}
$$

There is an obvious way of extending α to a mapping of A into A, since the elements of A have a normal form. We denote the extended mapping by α. To prove that α is an automorphism we need to show that α is onto A, which is easy, and to show that α is a homomorphism. For the latter purpose we use Lemma 2 on (3.1), (3.2) in a simplified form; since A has class 2 we see that $(u, v)^\alpha = (u^\alpha, v^\alpha)$, for any pair u, v of the given generators, if and only if

$$(v, u, \alpha)(\alpha, v, u)(u, \alpha, v) = 1.$$

The verification of the equation $(u, v)^\alpha = (u^\alpha, v^\alpha)$ is negligible except in the following cases:

$$(c_3, c_2, \alpha)(\alpha, c_3, c_2)(c_2, \alpha, c_3) = 1 \cdot 1 \cdot 1 = 1,$$
$$(c_1, c_3, \alpha)(\alpha, c_1, c_3)(c_3, \alpha, c_1) = f_3 \cdot f_3 \cdot f_3^3 = 1,$$
$$(c_2, c_1, \alpha)(\alpha, c_2, c_1)(c_1, \alpha, c_2) = f_2^4 \cdot f_2^2 \cdot f_2^4 = 1.$$

The relations (3.3) present no difficulty. Thus α is an automorphism, and when exponents modulo 5 are used its order is 5.

We form the splitting extension B of A by a group of order 5 which is generated by the element a inducing α in A. Thus $B = gp\{A, a\}$, and a set of defining relations for B can be obtained from (3.1)—(3.4) and

$$a^5 = 1. \tag{3.5}$$

Now consider the following mapping β in B:

$$\begin{aligned}
&(a, \beta) = c_3; \\
&(c_1, \beta) = d_2, \quad (c_2, \beta) = d_5, \quad (c_3, \beta) = d_8; \\
&(d_1, \beta) = e_2, \quad (d_2, \beta) = 1, \quad (d_3, \beta) = 1, \\
&(d_4, \beta) = e_1^{-3} f_2^2 f_3^3, \quad (d_5, \beta) = e_2^{-2} f_1^2 f_3^4, \\
&(d_6, \beta) = e_3^{-3} f_1 f_2^2, \quad (d_7, \beta) = 1, \quad (d_8, \beta) = 1; \\
&(e_1, \beta) = f_3, \quad (e_2, \beta) = 1, \quad (e_3, \beta) = f_1^4; \\
&(f_l, \beta) = 1 \text{ for } 1 \leqq l \leqq 3.
\end{aligned} \tag{3.6}$$

Extend β to a mapping (again denoted by β) of B into B in the now obvious manner, and verify that β is onto B. We shall show that β is a homomorphism and so an automorphism. Proofs involving (3.1) may be carried out using the simplified form of Lemma 2 introduced above:

$$(c_3, c_2, \beta)(\beta, c_3, c_2)(c_2, \beta, c_3) = f_3^4 \cdot f_3^2 \cdot f_3^4 = 1,$$
$$(c_1, c_3, \beta)(\beta, c_1, c_3)(c_3, \beta, c_1) = 1 \cdot 1 \cdot 1 = 1,$$
$$(c_2, c_1, \beta)(\beta, c_2, c_1)(c_1, \beta, c_2) = f_1 \cdot f_1 \cdot f_1^3 = 1.$$

The other relations in (3.1) and all in (3.2), (3.3) give no trouble. For some of (3.4) Lemma 2 is used:

$$\begin{aligned}
&(a, c_1; \beta, a)(a, c_1, \beta)^{(\beta, a)}(\beta, a, c_1^\beta)(c_1, \beta, a^{c_1}) \\
&\qquad = f_3^4 \cdot e_2^{-1} \cdot e_2^4 f_1^3 \cdot e_2^{-3} f_1^2 f_3 = 1, \\
&(a, c_2; \beta, a)(a, c_2, \beta)^{(\beta, a)}(\beta, a, c_2^\beta)(c_2, \beta, a^{c_2}) \\
&\qquad = 1 \cdot e_1^3 f_2^3 f_3^2 \cdot e_1^{-4} f_3 \cdot e_1 f_2^2 f_3^2 = 1, \\
&(a, c_3; \beta, a)(a, c_3, \beta)^{(\beta, a)}(\beta, a, c_3^\beta)(c_3, \beta, a^{c_3}) \\
&\qquad = 1 \cdot 1 \cdot 1 \cdot 1 = 1.
\end{aligned}$$

The rest of (3.4) require no more than the simplified form of the lemma, and we summarise the calculations as follows. If we put

$$p_i = (a, d_i, \beta)(\beta, a, d_i)(d_i, \beta, a)$$

for $1 \leqq i \leqq 8$ then some applications of Lemma 2 are described in the following table:

i	(a, d_i, β)	(β, a, d_i)	(d_i, β, a)	p_i
1	f_3^2	f_3^4	f_3^4	1
2	1	1	1	1
3	f_1^2	f_1^3	1	1
4	1	1	1	1
5	f_3^4	f_3^4	f_3^2	1
6	1	f_2^3	f_2^2	1
7	1	1	1	1
8	1	1	1	1

The remainder of (3.4) present no difficulty. For (3.5) we have

$$(a^\beta)^5 = (ac_3)^5 = a^5 c_3^5 d_7^{10} = 1 = (a^5)^\beta.$$

It follows that β is a homomorphism, and so an automorphism, of B. In the case of exponents modulo 5 some calculations, of which the hardest is

$$a^{\beta^5} = ac_3^5 d_8^{10} = a,$$

show that its order is 5.

The next step is to form the splitting extension C of B by a group of order 5 which is generated by the element b inducing β in B. Thus $C = gp\{B, b\}$, and a set of defining relations for C can be obtained from (3.1)—(3.6) and

(3.7) $$b^5 = 1.$$

We define a mapping γ in C:

(3.8)
$$\begin{aligned}
&(a, \gamma) = c_2^{-1}, \quad (b, \gamma) = c_1; \\
&(c_1, \gamma) = d_3, \quad (c_2, \gamma) = d_6, \\
&(c_3, \gamma) = d_1^{-1} d_5^{-1} e_1^{-4} e_2^{-4} e_3^{-4} f_1^3 f_2^3 f_3^3; \\
&(d_1, \gamma) = e_3 f_1^2 f_2^2, \quad (d_2, \gamma) = 1, \quad (d_3, \gamma) = 1, \\
&(d_4, \gamma) = 1, \quad (d_5, \gamma) = e_3, \quad (d_6, \gamma) = 1, \\
&(d_7, \gamma) = e_1^{-3} f_2 f_3^2, \quad (d_8, \gamma) = e_2^{-3} f_1^3 f_3^2; \\
&(e_1, \gamma) = f_2^4, \quad (e_2, \gamma) = f_1, \quad (e_3, \gamma) = 1; \\
&(f_l, \gamma) = 1 \text{ for } 1 \leqq l \leqq 3.
\end{aligned}$$

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Next let γ denote the mapping of C into C which is the obvious extension of the above mapping. A little effort shows that γ is indeed onto C, and we shall give some details of the proof that γ is a homomorphism. For (3.1) use the simplified form of the lemma:

$$
\begin{aligned}
(c_3, c_2, \gamma)(\gamma, c_3, c_2)(c_2, \gamma, c_3) &= f_2 \cdot f_2 \cdot f_2^3 = 1, \\
(c_1, c_3, \gamma)(\gamma, c_1, c_3)(c_3, \gamma, c_1) &= f_1^4 \cdot f_1^2 \cdot f_1^4 = 1, \\
(c_2, c_1, \gamma)(\gamma, c_2, c_1)(c_1, \gamma, c_2) &= 1 \cdot 1 \cdot 1 = 1.
\end{aligned}
$$

The other relations in (3.1) and all in (3.2), (3.3) present no problems. Next use Lemma 2 in the case of (3.4):

$$
\begin{aligned}
&(a, c_1; \gamma, a)(a, c_1, \gamma)(\gamma, a, c_1^\gamma)(c_1, \gamma, a) \\
&\qquad = f_2 \cdot e_3^{-1} f_1^3 f_2^3 \cdot e_3^4 f_2^3 \cdot e_3^{-3} f_1^2 f_2^3 = 1, \\
&(a, c_2; \gamma, a)(a, c_2, \gamma)(\gamma, a, c_2^\gamma)(c_2, \gamma, a) \\
&\qquad = 1 \cdot 1 \cdot 1 \cdot 1 = 1, \\
&(a, c_3; \gamma, a)(a, c_3, \gamma)(\gamma, a, c_3^\gamma)(c_3, \gamma, a) \\
&\qquad = 1 \cdot e_1^3 f_2^4 f_3^3 \cdot e_1^{-4} f_2 f_3^2 \cdot e_1 = 1.
\end{aligned}
$$

Next we put

$$q_i = (a, d_i, \gamma)(\gamma, a, d_i)(d_i, \gamma, a)$$

for $1 \leqq i \leqq 8$, and find as a result of calculation:

i	(a, d_i, γ)	(γ, a, d_i)	(d_i, γ, a)	q_i
1	f_2^3	f_2	f_2	1
2	f_1^3	f_1^2	1	1
3	1	1	1	1
4	1	1	1	1
5	f_2	f_2^3	f_2	1
6	1	1	1	1
7	1	1	1	1
8	1	f_3^2	f_3^3	1

In this way the simplified form of Lemma 2 takes care of eight more relations in (3.4), and the rest are easily checked. Consider (3.5) next:

$$(a^\gamma)^5 = (ac_2^{-1})^5 = a^5 c_2^{-5} d_4^{-10} = 1 = (a^5)^\gamma.$$

Turning to (3.6), we use Lemma 2 as follows:

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$$(a, b; \gamma, a)(a, b, \gamma)^{(\gamma, a)}(\gamma, a, b^\gamma)(b, \gamma, a^b)$$
$$= e_1^4 f_3^3 \cdot d_1^{-1} d_5^{-1} e_1^{-4} e_2^{-4} e_3^{-4} f_1^3 f_2^2 f_3^3 \cdot d_5 e_3^4 f_1^4 f_2^3 \cdot d_1 e_2^4 f_1^3 f_3^4 = 1,$$
$$(b, c_1; \gamma, b)(b, c_1, \gamma)^{(\gamma, b)}(\gamma, b, c_1^\gamma)(c_1, \gamma, b^{c_1})$$
$$= 1 \cdot 1 \cdot 1 \cdot 1 = 1,$$
$$(b, c_2; \gamma, b)(b, c_2, \gamma)^{(\gamma, b)}(\gamma, b, c_2^\gamma)(c_2, \gamma, b^{c_2})$$
$$= f_1^4 \cdot e_3^{-1} \cdot e_3^4 f_2^3 \cdot e_3^{-3} f_1 f_2^2 = 1,$$
$$(b, c_3; \gamma, b)(b, c_3, \gamma)^{(\gamma, b)}(\gamma, b, c_3^\gamma)(c_3, \gamma, b^{c_3})$$
$$= 1 \cdot e_2^3 f_1^2 f_3^3 \cdot e_2^{-4} f_1 \cdot e_2 f_1^2 f_3^2 = 1.$$

Then we put

$$r_i = (b, d_i, \gamma)(\gamma, b, d_i)(d_i, \gamma, b)$$

for $1 \leqq i \leqq 8$, and find that:

i	(b, d_i, γ)	(γ, b, d_i)	(d_i, γ, b)	r_i
1	f_1^4	f_1^2	f_1^4	1
2	1	1	1	1
3	1	1	1	1
4	f_2^2	f_2^3	1	1
5	f_1^2	f_1^4	f_1^4	1
6	1	1	1	1
7	1	f_3^3	f_3^2	1
8	1	1	1	1

Thus the simplified form of Lemma 2 deals with the remaining non-trivial relations in (3.6). Finally we have (3.7):

$$(b^\gamma)^5 = (bc_1)^5 = b^5 c_1^5 d_2^{10} = 1 = (b^5)^\gamma.$$

This completes the proof that γ is an automorphism of C. In the case of exponents modulo 5 it will be found that γ has order 5.

We form the splitting extension G of C by a group of order 5 which is generated by the element c inducing γ in C. Thus $G = gp\{C, c\}$, and a set of defining relations can be obtained from (3.1)—(3.8) and

(3.9) $$c^5 = 1.$$

This group G is an example which enables Theorem 4 to be proved. Its order is of course 5^{20}, exponents modulo 5 being used, and it is generated by $\{a, b, c\}$, with defining relations in these generators given by **(3.1)—(3.9)**.

4. Proof of Theorem 4

It is easily verified from the defining relations of G that

$$\begin{aligned}
\Gamma_2(G) &= gp\{\Gamma_3(G),\, c_i : 1 \leqq i \leqq 3\},\\
\Gamma_3(G) &= gp\{\Gamma_4(G),\, d_j : 1 \leqq j \leqq 8\}\\
\Gamma_4(G) &= gp\{\Gamma_5(G),\, e_k : 1 \leqq k \leqq 3\},\\
\Gamma_5(G) &= gp\{f_l : 1 \leqq l \leqq 3\},\\
\Gamma_6(G) &= 1,
\end{aligned}$$

from which it follows that G has class precisely 5. The further consequence that $\Gamma_5(G)$ has exponent 5 will be used without explicit reference from now on. The hard part of the proof of Theorem 4 lies in showing that any pair of elements of G generate a subgroup of class 3. If this is the case (and if G is finite) then G is regular. Therefore when exponents modulo 5 are being used G has exponent 5, for each of its generators a, b, c has order 5.

It is necessary to derive a number of laws in G before showing that all two-generator subgroups have class 3. First we show that all such subgroups are metabelian. By Theorem 2.1 of [5], this is equivalent to showing that G belongs to the variety $(x, y; x^{-1}, y) = 1$, which is clearly equivalent to $(x, y, y; x, y) = 1$. Let z_i, for $1 \leqq i \leqq 3$, be arbitrary elements in G and let

$$z_i\Gamma_2(G) = a^{\xi_i} b^{\eta_i} c^{\zeta_i}\Gamma_2(G).$$

It follows that

$$(z_1, z_2)\Gamma_3(G) = c_1^{\varphi_1} c_2^{\varphi_2} c_3^{\varphi_3}\Gamma_3(G)$$

where

$$\varphi_1 = \eta_1\zeta_2 - \eta_2\zeta_1, \quad \varphi_2 = \zeta_1\xi_2 - \zeta_2\xi_1, \quad \varphi_3 = \xi_1\eta_2 - \xi_2\eta_1.$$

Next we find that the value of $(z_1, z_2, z_2)\Gamma_4(G)$ is

$$d_1^{\xi_2\varphi_1-\zeta_2\varphi_3} d_2^{\eta_2\varphi_1} d_3^{\zeta_2\varphi_1} d_4^{\xi_2\varphi_2} d_5^{\eta_2\varphi_2-\zeta_2\varphi_3} d_6^{\zeta_2\varphi_2} d_7^{\xi_2\varphi_3} d_8^{\eta_2\varphi_3}\Gamma_4(G).$$

Therefore we have

$$(z_1, z_2, z_2; z_1, z_2) = f_1^{\psi_1} f_2^{\psi_2} f_3^{\psi_3},$$

where

$$\begin{aligned}
\psi_1 &= 2\varphi_1(\xi_2\varphi_1 + 4\zeta_2\varphi_3) + 4\varphi_1(\eta_2\varphi_2 + 4\zeta_2\varphi_3) + 3\varphi_2\eta_2\varphi_1 + 3\varphi_3\zeta_2\varphi_1,\\
\psi_2 &= 3\varphi_1\xi_2\varphi_2 + 4\varphi_2(\xi_2\varphi_1 + 4\zeta_2\varphi_3) + 2\varphi_2(\eta_2\varphi_2 + 4\zeta_2\varphi_3) + 3\varphi_3\zeta_2\varphi_2,\\
\psi_3 &= 3\varphi_1\xi_2\varphi_3 + 3\varphi_2\eta_2\varphi_3 + 4\varphi_3(\xi_2\varphi_1 + 4\zeta_2\varphi_3) + 4\varphi_3(\eta_2\varphi_2 + 4\zeta_2\varphi_3).
\end{aligned}$$

It will thus be found that

$$(x, y, y; x, y) = 1. \tag{4.1}$$

Next we consider the commutators (c_j, z_1, z_2, z_3) for $1 \leqq j \leqq 3$. By the usual expansion process such a commutator may be expressed as the

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product of 27 elements, each of which is a power of a commutator of weight 5 and with entries from $\{a, b, c\}$. These may be evaluated by means of the defining relations in G, with the following results:

$$\begin{aligned}
(c_j, z_1, z_2, z_3) &= f_1^{\varphi_{j1}} f_2^{\varphi_{j2}} f_3^{\varphi_{j3}};\\
\varphi_{11} &= \xi_1\eta_2\zeta_3+4\xi_1\zeta_2\eta_3+2\eta_1\xi_2\zeta_3+3\zeta_1\xi_2\eta_3,\\
\varphi_{12} &= 2\xi_1\xi_2\zeta_3+\xi_1\zeta_2\xi_3+2\zeta_1\xi_2\xi_3,\\
\varphi_{13} &= 3\xi_1\xi_2\eta_3+4\xi_1\eta_2\xi_3+3\eta_1\xi_2\xi_3,\\
\varphi_{21} &= 3\eta_1\eta_2\zeta_3+4\eta_1\zeta_2\eta_3+3\zeta_1\eta_2\eta_3,\\
\varphi_{22} &= 3\xi_1\eta_2\zeta_3+4\eta_1\xi_2\zeta_3+\eta_1\zeta_2\xi_3+2\zeta_1\eta_2\xi_3,\\
\varphi_{23} &= 2\xi_1\eta_2\eta_3+\eta_1\xi_2\eta_3+2\eta_1\eta_2\xi_3,\\
\varphi_{31} &= \zeta_1\eta_2\zeta_3+2\zeta_1\zeta_2\eta_3+2\eta_1\zeta_2\zeta_3,\\
\varphi_{32} &= 3\xi_1\zeta_2\zeta_3+4\zeta_1\xi_2\zeta_3+3\zeta_1\zeta_2\xi_3,\\
\varphi_{33} &= \zeta_1\xi_2\eta_3+2\xi_1\zeta_2\eta_3+3\eta_1\zeta_2\xi_3+4\zeta_1\eta_2\xi_3.
\end{aligned}$$

Further calculations show that

$$(c_j, z_1, z_2, z_2) = (c_j, z_2, z_1, z_2)^2 = (c_j, z_2, z_2, z_1)$$

for $1 \leqq j \leqq 3$. Since G has class 5 we arrive at the following laws in G:

$$(x, y, u, v, v) = (x, y, v, u, v)^2 = (x, y, v, v, u), \tag{4.2}$$

$$(x, y, 3z) = 1. \tag{4.3}$$

We deduce from (4.1) that

$$(x, y, y, x, y) = (x, y, y, y, x),$$

and this with (4.2) at once gives

$$(x, y, x, y, y) = (x, y, y, x, y) = 1. \tag{4.4}$$

That is to say, every two-generator subgroup of G has class 4.

Expansion of $(xz, y, xz, y, y) = 1$, together with (4.3) and the law

$$(x, y, z, y, y)(z, x, y, y, y)(y, z, x, y, y) = 1$$

derived from (2.4), gives

$$(x, y, z, y, y) = 1. \tag{4.5}$$

At this point (4.2), (4.3) and (4.5) show that any commutator of weight 5 with three identical entries is trivial.

Now suppose that x_0 and y_0 are third-right-Engel elements in G. Since

$$(x_0y_0, 3z) = (x_0, 3z)(y_0, 3z)(x_0, z, y_0, 2z)$$

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it follows from (4.5) that $x_0 y_0$ is third-right-Engel. So if we can prove that x_0 is third-right-Engel when $x_0 = a, b, c$, it will follow that G is a third-Engel-group. Suppose $z = y_0 g$ where $y_0 = a^\xi b^\eta c^\zeta$ (with $0 \leqq \xi, \eta, \zeta < 5$) and $g \in \Gamma_2(G)$. Since G has class 5,

$$(x_0, 3z) = (x_0, 3y_0)(x_0, g, 2y_0)(x_0, y_0, g, y_0)(x_0, 2y_0, g).$$

So it will suffice to prove the following equations:

$$(4.6) \qquad (x_0, 3y_0) = 1,$$

$$(4.7) \qquad (x_0, g, 2y_0)(x_0, y_0, g, y_0)(x_0, 2y_0, g) = 1,$$

for $x_0 = a, b, c$, for $y_0 = a^\xi b^\eta c^\zeta$, and for all $g \in \Gamma_2(G)$.

However it is now easy to prove (4.7). Calculation with (2.4) gives

$$\begin{aligned}(x_0, 2y_0, g) &= (x_0, y_0; y_0, g)(x_0, y_0, g, y_0)\\ &= (g, y_0; x_0, y_0)(x_0, y_0, g, y_0)\\ &= (g, y_0, x_0, y_0)(g, y_0, y_0, x_0)^{-1}(x_0, y_0, g, y_0);\\ (x_0, y_0, g, y_0) &= (g, x_0, y_0, y_0)^{-1}(g, y_0, x_0, y_0).\end{aligned}$$

Hence (4.7) is equivalent to

$$(g, x_0, y_0, y_0)^2(g, y_0, x_0, y_0)^3(g, y_0, y_0, x_0)^4 = 1,$$

which is an immediate consequence of (4.2).

Next we consider (4.6) with $x_0 = a$. It will be sufficient to prove that $gp\{a, b^\eta c^\zeta\}$ has class 3, and this in turn would follow from these equations:

$$(4.8) \qquad (a, b^\eta c^\zeta, a, a) = 1,$$

$$(4.9) \qquad (a, b^\eta c^\zeta, a, b^\eta c^\zeta) = 1,$$

$$(4.10) \qquad (a, 3b^\eta c^\zeta) = 1.$$

Note that, since every pair of elements generates a metabelian subgroup, (4.9) is equivalent to

$$(a, b^\eta c^\zeta, b^\eta c^\zeta, a) = 1.$$

Consider (4.8). Since a commutator of weight 5 with three identical entries is trivial we have

$$(a, b^\eta c^\zeta, a, a) = (a, b, a, a)^\eta (a, c, a, a)^\zeta.$$

Then (4.8) follows from the defining relations in G.

Consider (4.9). Expansion by the usual means gives

$$(a, b^\eta c^\zeta, a, b^\eta c^\zeta) = W_1 W_2,$$

where

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$$W_1 = (a, b^\eta, a, b^\eta)(a, b^\eta, a, c^\zeta)(a, c^\zeta, a, b^\eta)(a, c^\zeta, a, c^\zeta),$$
$$W_2 = (a, b^\eta, c^\zeta, a, b^\eta)(a, b^\eta, c^\zeta, a, c^\zeta)(a, b^\eta, a, b^\eta, c^\zeta)(a, c^\zeta, a, b^\eta, c^\zeta).$$

Simplification of the various terms goes as follows:

$$
\begin{aligned}
(a, b^\eta, a, b^\eta) &= (a, c^\zeta, a, c^\zeta) = 1,\\
(a, b^\eta, a, c^\zeta) &= (a, b, a, c)^{\eta\zeta}(a, b, b, a, c)^{\binom{\eta}{2}\zeta}(a, b, a, c, c)^{\eta\binom{\zeta}{2}}\\
&= (e_1^{-3} f_2 f_3^2)^{\eta\zeta} f_2^{3\eta\binom{\zeta}{2}},\\
(a, c^\zeta, a, b^\eta) &= (a, c, a, b)^{\eta\zeta}(a, c, c, a, b)^{\eta\binom{\zeta}{2}}(a, c, a, b, b)^{\binom{\eta}{2}\zeta}\\
&= (e_1^3 f_2^3 f_3^2)^{\eta\zeta} f_3^{3\binom{\eta}{2}\zeta},\\
(a, b^\eta, c^\zeta, a, b^\eta) &= (a, b, c, a, b)^{\eta^2\zeta} = f_3^{\eta^2\zeta},\\
(a, b^\eta, c^\zeta, a, c^\zeta) &= (a, b, c, a, c)^{\eta\zeta^2} = f_2^{4\eta\zeta^2},\\
(a, b^\eta, a, b^\eta, c^\zeta) &= (a, b, a, b, c)^{\eta^2\zeta} = 1,\\
(a, c^\zeta, a, b^\eta, c^\zeta) &= (a, c, a, b, c)^{\eta\zeta^2} = f_2^{2\eta\zeta^2}.
\end{aligned}
$$

It will be found that $W_1 W_2 = 1$, and (4.9) follows.

Consider (4.10). On expanding $(a, 3b^\eta c^\zeta)$ we obtain terms of weights 4 and 5. Those terms which are non-trivial are listed and evaluated below:

$$
\begin{aligned}
(a, b^\eta, b^\eta, c^\zeta) &= (a, b, b, c)^{\eta^2\zeta}(a, b, b, c, c)^{\eta^2\binom{\zeta}{2}}\\
&= (e_2^{-3} f_1^3 f_3^2)^{\eta^2\zeta} f_1^{2\eta^2\binom{\zeta}{2}},\\
(a, b^\eta, c^\zeta, b^\eta) &= (a, b, c, b)^{\eta^2\zeta}(a, b, c, c, b)^{\eta^2\binom{\zeta}{2}}\\
&= (e_2 f_1^2 f_3^2)^{\eta^2\zeta} f_1^{2\eta^2\binom{\zeta}{2}},\\
(a, c^\zeta, b^\eta, b^\eta) &= (a, c, b, b)^{\eta^2\zeta}(a, c, c, b, b)^{\eta^2\binom{\zeta}{2}}\\
&= (e_2^2 f_1^3 f_3)^{\eta^2\zeta} f_1^{2\eta^2\binom{\zeta}{2}},\\
(a, c^\zeta, c^\zeta, b^\eta) &= (a, c, c, b)^{\eta\zeta^2}(a, c, c, b, b)^{\binom{\eta}{2}\zeta^2}\\
&= (e_3^3 f_1^4 f_2^3)^{\eta\zeta^2} f_1^{2\binom{\eta}{2}\zeta^2}\\
(a, c^\zeta, b^\eta, c^\zeta) &= (a, c, b, c)^{\eta\zeta^2}(a, c, b, b, c)^{\binom{\eta}{2}\zeta^2}\\
&= e_3^{-\eta\zeta^2} f_1^{2\binom{\eta}{2}\zeta^2},\\
(a, b^\eta, c^\zeta, c^\zeta) &= (a, b, c, c)^{\eta\zeta^2}(a, b, b, c, c)^{\binom{\eta}{2}\zeta^2}\\
&= (e_3^{-2} f_1^4 f_2^2)^{\eta\zeta^2} f_1^{2\binom{\eta}{2}\zeta^2},\\
(a, b^\eta, c^\zeta, b^\eta, c^\zeta) &= (a, b, c, b, c)^{\eta^2\zeta^2} = f_1^{\eta^2\zeta^2},\\
(a, b^\eta, c^\zeta, c^\zeta, b^\eta) &= (a, b, c, c, b)^{\eta^2\zeta^2} = f_1^{2\eta^2\zeta^2},\\
(a, b^\eta, b^\eta, c^\zeta, c^\zeta) &= (a, b, b, c, c)^{\eta^2\zeta^2} = f_1^{2\eta^2\zeta^2},\\
(a, c^\zeta, b^\eta, c^\zeta, b^\eta) &= (a, c, b, c, b)^{\eta^2\zeta^2} = f_1^{\eta^2\zeta^2},\\
(a, b^\eta, c^\zeta, b^\eta, c^\zeta) &= (a, b, c, b, c)^{\eta^2\zeta^2} = f_1^{\eta^2\zeta^2},\\
(a, c^\zeta, b^\eta, b^\eta, c^\zeta) &= (a, c, b, b, c)^{\eta^2\zeta^2} = f_1^{2\eta^2\zeta^2}.
\end{aligned}
$$

Some arithmetic at this point will verify (4.10).

We have now proved that a is third-right-Engel in G. The proof that b and c have this property is omitted, being entirely similar. It follows as explained above that every element of G is third-right-Engel. But since G is a 5-group zusatz 2 of [4] shows that every two-generator subgroup of G has class 3. The proof of Theorem 4 is now complete.

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