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RESIDUAL FINITENESS AND 'FREE' DISTRIBUTIVELY GENERATED NEAR-RINGS

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Abstract

Let **V** be a variety of groups in which the free group is residually finite, and let S be a residually finite semigroup. Let $N_{\mathbf{V}}(S)$ be the 'free' distributively generated near-ring constructed from S and **V**. *Theorem;* $N_{\mathbf{V}}(S)$ is residually finite.

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A near-ring N is a set with two binary operations $+$ and $\cdot$, such that $\{N, +\}$ is a group, $\{N, \cdot\}$ is a semigroup, and $\cdot$ is left distributive over $+$. A distributively generated near-ring is a near-ring N which is additively generated by a set of right (and left) distributive elements.

Given a variety **V** of groups and a semigroup S we define a new distributively generated near-ring, $N_{\mathbf{V}}(S)$. $\{N_{\mathbf{V}}(S), +\}$ is the free group in **V** on free generators of S. We define $\{N_{\mathbf{V}}(S), \cdot\}$ inductively on the words of $\{N_{\mathbf{V}}(S), +\}$. If $s, t \in S$ and $u, v, w \in N_{\mathbf{V}}(S)$, then $s \cdot t = st$ (the product of s and t in S), and

$$(-u)\cdot t = -(u\cdot t), \text{ and } (w+u)\cdot s = w\cdot s + u\cdot s, \text{ and } w\cdot(u+v) = w\cdot u + w\cdot v,$$
$$\text{and } w\cdot(-u) = -(w\cdot u), \quad \text{and} \quad w\cdot 0 = 0.$$

It has been shown in Fröhlich (1960) and Evans and Neff (1964) that

$$\{N_{\mathbf{V}}(S), +, \cdot\}$$

is a distributively generated near-ring. When **V** is the variety of all groups and S is the free semi-group in the variety of all semigroups, $N_{\mathbf{V}}(S)$ is the 'free' distributively generated near-ring.

A near-ring N is residually finite if for any $n \neq 0$ belonging to N there exists a finite near-ring N_f and a near-ring homomorphism $\theta\colon N \to N_f$ such that $n\theta \neq 0$.

More generally, an algebra A is residually finite if for any distinct $u, v \in A$, there exists a finite algebra F in the variety generated by A and a homomorphism $\beta: A \to F$ such that $u\beta \neq v\beta$.

We will show that if $\mathbf{V}$ is a variety of groups in which all free groups are residually finite, and if S is a residually finite semigroup, then $N_{\mathbf{V}}(S)$ is a residually finite distributively generated near-ring.

THEOREM. *If* $\mathbf{V}$ *is a variety of groups in which all free groups are residually finite, and if* S *is a residually finite semigroup, then* $N_{\mathbf{V}}(S)$ *is residually finite.*

PROOF. Let $w_0 \in N_{\mathbf{V}}(S)$, w_0 different from zero. We can write w_0 as an additive word in terms of distinct $s_i \in S$, say $w_0(s_1, \ldots, s_n)$. Since $\{N_{\mathbf{V}}(S), +\}$ is a free group in $\mathbf{V}$, and by hypothesis a free group $\mathbf{V}$ is residually finite, there exists a finite group G in $\mathbf{V}$ and elements $x_1, \ldots, x_n$ of G such that $w_0(x_1, \ldots, x_n)$ is not zero. Let $\mathbf{V}(G)$ be the variety generated by G.

Since S is residually finite, for any $s_i, s_j \in S$, $i \neq j$, there exists a finite semigroup R_{ij} and a homomorphism $f_{ij}: S \to R_{ij}$ such that $s_i f_{ij} \neq s_j f_{ij}$. Let $R = \prod_{i \neq j} R_{ij}$, and define $f: S \to \prod_{i \neq j} R_{ij}$ by $tf = (tf_{ij})$. f is a homomorphism and $s_i f \neq s_j f$ for $i \neq j$. Suppose R contains m elements, $r_1, \ldots, r_m$.

Construct $N_{\mathbf{V}(G)}(R)$, the distributively generated near-ring with distributive generating set R such that $\{N_{\mathbf{V}(G)}(R), +\}$ is free in $\mathbf{V}(G)$ on generators R. Since G is a finite group and $\{N_{\mathbf{V}(G)}(R), +\}$ is a finitely generated free group in $\mathbf{V}(G)$, $\{N_{\mathbf{V}(G)}(R), +\}$ is finite. Thus $\{N_{\mathbf{V}(G)}(R), +, \cdot\}$ is a finite distributively generated near-ring. Let $\theta: N_{\mathbf{V}}(S) \to N_{\mathbf{V}(G)}(R)$ be the near-ring homomorphism determined by

$$g_i\theta = \begin{cases} g_i f & \text{if } g_i \in \{s_1, \ldots, s_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we have $w_0(s_1, \ldots, s_n)\theta = w_0(s_1\theta, \ldots, s_n\theta) = w_0(s_1 f, \ldots, s_n f)$. f was chosen so that $s_i f \neq s_j f$ for $i \neq j$, so $\{s_1 f, \ldots, s_n f\}$ is a set of n distinct elements of R, that is n distinct free generators of $\{N_{\mathbf{V}(G)}(R), +\}$. G was chosen with the restriction that $w_0(x_1, \ldots, x_n)$ was not zero. Since $N_{\mathbf{V}(G)}(R)$ is free in $\mathbf{V}(G)$ the map $s_i \to x_i$ can be extended to a homomorphism, hence $w_0(s_1 f, \ldots, s_n f)$ is not zero.

We now have that $N_{\mathbf{V}}(S)$ is residually finite.

In John and Neff (1979) it is proved that the free near-ring N_0 in $\mathcal{N}_0$, the variety of near-rings with the additional identity $0x = 0$, is a subnear-ring of the 'free' distributively-generated near-ring. Hence, we have that N_0 is residually finite.

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