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THE LONG ANNULUS THEOREM

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ABSTRACT. Given a properly embedded incompressible surface F in a Haken manifold M, there is an integer n depending only on M and F with the following property: If there is a singular annulus in M that meets F in more then n nontrivial loops that are not freely homotopic on F then M contains an essential torus or annulus, or M is a bundle with fiber F, or M is a doubled twisted I-bundle with doubling surface F.

There are a number of Theorems in print that give conditions under which a singular torus (or proper singular annulus) in a 3-manifold can be replaced by an embedding. The main Theorem of this paper is of a somewhat different flavor. Let F be an incompressible surface in a 3-manifold and consider a map of an annulus into M that meets F in a number of loops. If some pair of these loops are freely homotopic on F then the existing Theorems may be sufficient to conclude the existence of an embedded torus or annulus in M. Of interest here is the case that no pair of the loops are freely homotopic on F. (The ends of the annulus can't be altered to produce a singular torus.) The long annulus Theorem (Theorem 2.7) provides an N depending only on F such that if the singular annulus meets F essentially more than N times, then there is an embedded torus or annulus in M, or else M is a bundle with fiber F or a double twisted I-bundle.

This Theorem (and its companion Theorem 2.8) have been used in the calculation of Frattini subgroups of 3-manifold groups [1]. They also have important algorithmic applications (e.g. the conjugacy problem for 3-manifold groups.) In this context, it is crucial that the number N actually be calculable from algebraic invariants of F. (The existence alone of N is not sufficient.) A map of a torus (resp. a proper map of an annulus) into a manifold M is *essential* if the induced map at the fundamental group level is monic and the map does not deform (resp. relative to the boundary) into the boundary of M. A 3-manifold is *simple* if it contains no essentially embedded torus or annulus. Let N denote an orientable I bundle over a compact nonorientable 2-manifold F, and let M denote N doubled along the $\{0, 1\}$ level of N. M will be termed a *double twised* I *bundle* with *doubling surface* the $\{0, 1\}$ level of N.

A two sided, incompressible, boundary incompressible surface F properly embedded in a 3-manifold M will be termed a *cutting surface* for M. The closure of the manifold

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obtained by removing a regular neighborhood of F from M will be referred to as *M cut at F* and will be denoted by $\bar{M}$. In the boundary of $\bar{M}$ are two copies of F. These will be denoted by $+F$ and $-F$.

The single most important ingredient in the proof of the long annulus Theorem is the decomposition Theorems of W. Jaco and P. Shalen (4).

A pair (S, T) of compact orientable manifolds is a *Seifert Pair* if:

i) S is either a 3-dimensional Seifert Fiber space or an orientable I bundle over a 2-manifold.

ii) T is a compact (possibly disconnected) incompressible submanifold of $bd\ S$.

iii) If S is Seifert fibered, then T is a union of fibers. If S is an I bundle then T is the $\{0, 1\}$ level of the bundle.

The Seifert pair (S, T) is *degenerate* if $\pi_1(S)$ is trivial or cyclic, or if $T = \phi$ Otherwise (S, T) is nondegenerate. If F is an incompressible submanifold of the boundary of a 3-manifold M, then a map f of a nondegenerate Seifert pair (S, T) into (M, F) is *essential* if

i) $f_*: \pi_1(S) \to \pi_1(M)$ is monic.

ii) f cannot be deformed as a map of pairs to a map g such that $g(S) \subset F$.

1.1 THEOREM (Jaco-Shalen [JS]).* *Let F denote an incompressible submanifold of the boundary of a compact, orientable, irreducible 3-manifold M. Then there is a mutually exclusive collection $\mathbf{S} = \{(S_1, T_1,), (S_2, T_2), \ldots, (S_k, T_k)\}$ of Seifert pairs essentially embedded in (M, F) with the following properties.*

i) *Each essential map f of a Seifert pair (S, T) into (M, F) deforms as a map of pairs into (S_i, T_i) for some $i(1 \leq i \leq k)$.*

ii) *For $i \neq j$, (S_i, T_i) does not deform as a map of pairs into (S_j, T_j).*

The collection $\mathbf{S}$ of Theorem 1.1 will be referred to as the *Seifert set* of (M, F). It is proved in [4] that the Seifert set is unique up to ambient isotopy. The reader is asked to observe for future reference that each component of the frontier of $\mathbf{S}$ in M is an incompressible torus or annulus in M.

2. THE MAIN THEOREM.

2.1 LEMMA. *Let M denote a compact, orientable, irreducible 3-manifold with incompressible boundary. Suppose T is a torus component of $bd\ M$. Let $f_1, f_2 : S^1 \times I \to M$ denote essential maps into M such that $f_i(S^1 \times 0) \subset T (i = 1, 2)$, but no power of $f_1(S^1 \times 0)$ is freely homotopic in T to any power of $f_2(S^1 \times 0)$. Then M is homeomorphic with $S^1 \times S^1 \times I$ or twisted I-bundle over the Klein bottle.*

PROOF: Let $\mathbf{S}$ be the (M, T) Seifert set. Let $(S, F) \in \mathbf{S}$ such that $T \cap F \neq \phi$. Now $f_1(S^1 \times 0)$ and $f_2(S^1 \times 0)$ can neither lie in a single annulus in F nor in a disjoint pair of annuli in F. It follows that $F = T$. If S is an I-bundle, then it is orientable with a torus boundary component. S must then be either $S^1 \times S^1 \times I$ or the orientable I-bundle over the Klein bottle. In either case, the result is immediate.

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*This theorem was also proved independently by Johannson [5].

Suppose on the other hand that S is Seifert fibered but is not an I-bundle. By 1.1., f_1 and f_2 deform into S. This gives a contradiction to lemma 2.8 of (4), and the proof of 2.1 is complete.

If F is a compact, orientable 2-manifold of genus g and first Betti number b, the *complexity* of F is defined to be $c(F) = (g, b)$. The complexities of 2-manifolds are ordered lexicographically. The bounds given in the following lemmas are chosen to facilitate the proofs. They are many orders of magnitude larger than is in fact necessary.

2.2 LEMMA. *Let F be a compact, orientable 2-manifold of complexity (g, b). Let $K = \{k_1, k_2, \ldots, k_n\}$ denote a mutually exclusive collection of noncontractible simple closed curves on F. If $n \geq \exp(g + b)$, then some pair of curves in K cobound an annulus on F.*

PROOF: There is of course nothing to do if $c(F) \leq (0, 1)$. The proof proceeds inductively on $c(F)$, and $c(F)$ is taken larger than $(0, 1)$. Consider first the possibility that some element k_i of K does not separate F. Let k denote a simple closed curve on F that is parallel on F to k_i but $k \cap k_i = \phi$. If G denotes F cut at k, it follows that $k_i \subset G$. Put $c(G) = (g_1, b_1)$. Then $g_1 < g$ and $b_1 \leq b$ so that $b < \exp(g + b) \leq \exp(g_1 + b_1)$. Then since $c(G) < c(F)$, the inductive hypothesis gives the desired result.

Suppose now that each curve in K separates F. Now F has no more than $b + 1$ boundary components, and $n > b + 1$. So it is enough to prove the lemma if some element k_i of K is not parallel to any boundary component of F. Let k_i separate F into nonannular components F_1, F_2. A copy of k_i may be taken to lie in each of F_1 and F_2. Put $k_j = K \cap F_j$ and $c(F_j) = (g_j, b_j)(j = 1, 2)$. Then $g_1 + g_2 = g$ and $b_1 + b_2 - 1 \leq b$. Since neither F_1 nor F_2 has first Betti number smaller than 2, it must be so that $b_i < b$ and $b_2 < b$. In particular, $c(F_1) < c(F)$ and $c(F_2) < c(F)$. Assuming no curves in K_j are parallel on F_j, it must be so that K_j contains fewer than $\exp(g_j + b_j)$ elements $(j = 1, 2)$.

Then

$$
\begin{aligned}
n &< \exp(g_1 + b_1) + \exp(g_2 + b_2) \\
&\leq 2 \exp(b + b - 1) \\
&< \exp(g + b).
\end{aligned}
$$

This contradiction completes the proof of lemma 2.2.

The next lemma is a group theoretic statement of an elementary geometric property of 2-manifolds. The proof is omitted.

2.3. LEMMA. *Let α and β denote loops on a compact, orientable 2-manifold F. If there are nonzero integers p and q such that α^p is freely homotopic on F to β^q, then α and β are freely homotopic to powers of a common loop x on F.*

Let F be an incompressible surface properly embedded in a 3-manifold M. A map $f: S^1 \times I \to M$ is (n, F) *essential* if:

i) f is transverse with respect to F

ii) $f^{-1}(F) = \{S^1 \times x_1, S^1 \times x_2, \ldots, S^1 \times x_n\}$
where $0 = x_1 < x_2 < \ldots < x_n = 1$.

iii) If $i \neq j$, then $f(k_i)$ is not freely homotopic on F to $f(k_j)$.

Observe that if $f: S^1 \times I \to M$ is (n, F) essential, then for each i, $f|S^1 \times [x_i, x_{i+1}]$ is an essential map into $(\bar{M}, \pm F)$.

2.4 LEMMA. *Let F be an incompressible surface properly embedded in a compact, orientable, irreducible 3-manifold M, and let $f: S^1 \times I \to M$ be an (n, F) essential map. Let $f^{-1}(F) = \{k_1, k_2, \ldots, k_n\}$. Then for each i, there is at most one $j \neq i$ such that some nonzero power of $f(k_i)$ is freely homotopic in F to a power of $f(k_j)$.*

PROOF: Suppose a nonzero power of $f(k_i)$ is freely homotopic on F to powers of $f(k_j)$ and $f(k_s)$. Three applications of lemma 2.3 gives a loop x on F such that x^a is freely homotopic on F to $f(k_i)$, x^b to $f(k_j)$, and x^c to $f(k_s)$. Since $f(k_i)$, $f(k_j)$, and $f(k_s)$ are freely homotopic in M, x^a, x^b, and x^c define conjugate elements of $\pi_1(M)$. Then (3) gives $a = \pm b = \pm c$. Consequently some pair among $f(k_i)$, $f(k_j)$, $f(k_s)$ is freely homotopic on F. This is not consistent with the fact that f is (n, F) essential, and so the proof of 2.4 is complete.

2.5 LEMMA. *Let M denote a compact, orientable, irreducible 3-manifold and F an incompressible surface in M with $c(F) = (g, b)$. If f is an (n, F) essential map with $n > \exp((g + b + 1)^2)$, then one of the following is true.*

i) *There is a bundle E embedded in M with fiber a nondisk, nonannular incompressible submanifold of F.*

ii) *There is a double twisted I bundle E in M such that the doubling surface of E is an incompressible submanifold of F.*

PROOF: Let $\mathbf{S} = \{(S_1, T_1), \ldots, (S_n, T_n)\}$ be the $(\bar{M}, \pm F)$ Seifert set. Let L denote a regular neighborhood in M of $F \cup \cup_{i=1}^{n} S_i$. By Theorem 1.1 we may take $f(S^1 \times I) \subset L$. Put $G_1, G_2, \ldots, G_p = (+F) \cap (\cup_i T_i)$ and $H_1, H_2, \ldots, H_q = (-F) \cap (\cup_i T_i)$.

2.6 CLAIM. *One of the following is true.*

a) *There is a bundle B embedded in M with fiber F.*

b) *There is a double twisted I bundle E embedded in M with doubling surface F.*

c) *There is some i for which $c(G_i) = (g_i, b_i)$, $c(G_i) < c(F)$, and $F: S^1 \times I \to L$ is an (s, G_i) essential map with $s \geq \exp((g_i + b_i + 1)^2)$.*

d) *There is some j for which $c(H_j) = (g_j, b_j)$, $c(H_j) < c(F)$, and $f: S^1 \times I \to L$ is an (s, H_j) essential map with $s \geq \exp((g_j + b_j + 1)^2)$.*

Consider first the case that for some i, j, both $i^*: \pi_1(G_i) \to \pi_1(F)$ and $i^* \pi_1(H_j) \to \pi_1(F)$ are epimorphisms. Let $G_i \subset T_i$ and $H_j \subset T_j$. Then each loop $f(k_t)$ lies either in G_i or in a regular neighborhood of a boundary component of F. It is a consequence of lemma 2.4 that no more than two such loops can lie in a regular neighborhood of a given component of $bd\,F$. Since F has at most $b + 1$ boundary components, G_i surely

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contains at least three of the loops $f(k_i)$. Once again by lemma 2.4 some pair among these has the property that no power of one is freely homotopic to any power of the other. In particular, G_i is not an annulus. More importantly, if S_i is a Seifert space, it must be either $S^1 \times S^1 \times I$ or the orientable I bundle over the Klein bottle. A similar argument applies to H_j. Thus in this case both S_i and S_j are I bundles.

If $S_i = S_j$, then the boundary of S_i less the frontier of S_i in N is disconnected, and so S_i is a product. The bundle B of a) is gotten then by gluing G_i to H_j in M. If on the other hand $S_i \neq S_j$, then neither S_i nor S_j can be a product. (Clearly no essential map of an annulus into a product can have both its boundary components in a single boundary component of the product.) Thus both S_i and S_j are twisted I bundles. Gluing H_i to H_j gives the double twisted I bundle in b).

Consider now the possibility that for each i, $i^*: \pi_1(G_i) \to \pi_1(F)$ is not epic or that $i_*: \pi_1(H_j) \to \pi_1(F)$ is nonepic for each j. We take $i_*: \pi_1(G_i) \to \pi_1(F)$ to be nonepic for each i. Put $c(G_i) = (g_i, b_i)$. Then $g_i \leq g$ and $b_i \leq b$ for each $i (1 \leq i \leq p)$. Furthermore if $g_i = g$ for any i, then $b_i < b$. Thus $c(G_i) < c(G)$ and $g_i + b_i < g + b$ for each $i (1 \leq i \leq p)$.

Let $G_1, G_2, \ldots, G_u$ be annuli and $G_{u+1}, G_{u+2}, \ldots, G_p$ nonannuli. Appealing once more to lemma 2.4, at most $2u$ elements of K can lie in $\cup_{i=1}^{u} G_i$. By lemma 2.2, there can exist a collection of at most $\exp(g + b)$ mutually exclusive annuli in F with the property that boundary components of distinct annuli are nonparallel in F. Thus at most $2 \exp(g + b) \leq \exp(g + b + 1)$ loops $f(k_j)$ can lie in $\cup_{i=1}^{u} G_i$. A final application of lemma 2.2 to the boundary components of $\{G_i | u + 1 \leq i \leq p\}$ gives $p - u < \exp(g + b + 1)$.

Suppose now that c) is wrong. That is that the number of loops $f(k_j)$ that lie in G_i is smaller than $\exp((g_i + b_i + 1)^2)$ for each i $(u + 1 \leq i \leq p)$.

$$
\begin{aligned}
n &< 2 + \exp(g + n + 1) + \sum_{i=u+1}^{p} \exp((g_i + b_i + 1)^2) \\
&\leq \exp(g + b + 1 + (g + b)^2) + \sum_{i=u+1}^{p} \exp((g + b)^2) \\
&\leq \exp(g + b + 1 + (g + b)^2) + \exp(g + b + 1) \exp((g + b)^2) \\
&\leq 2 \exp(g + b + 1 + (g + b)^2) \\
&\leq \exp((g + b) + 2 + (g + b)^2) \leq \exp((g + b + 1)^2).
\end{aligned}
$$

This is not consistent with the hypothesis of the lemma, and so claim 2.6 is established.

The aim now is to use 2.6 to give an inductive proof for 2.5. If $c(F) = (0, 2)$ then F is a planar 2-manifold with 3 boundary components. Since the hypothesis of the lemma cannot be satisfied if the surface involved is an annulus, and since any submanifold G of F with $c(G) < (0, 2)$ is a disk or annulus, c) or d) of claim 2.6 cannot be right. Thus in case $c(F) = (0, 2)$ we get either i) or ii) of lemma 2.5.

Suppose now that $c(F) > (0, 2)$, and that the lemma is known to be true for surfaces of smaller complexity. If either a) or b) of 2.6 is true, there is nothing left to prove. If on the other hand c) is true, then the inductive hypothesis applies to $f: S^1 \times I \to L$ and the surface G_i. The manifold E of i) or ii) is then embedded in L and hence in M. This completes the proof of lemma 2.5.

2.7 THEOREM. *Let F denote an incompressible surface in a compact, orientable, irreducible 3-manifold M with incompressible boundary. Let $f: S^1 \times I \to M$ be an (N, F) essential map. If $c(F) = (g, b)$, and if $n \geq \exp((g + b + 1)^2)$, then one of the following is true.*

i) *M contains an essential torus or annulus.*

ii) *M is a simple bundle with fiber F.*

iii) *M is a simple double twisted I bundle with doubling surface F.*

PROOF: Suppose M contains no essential torus or annulus. Apply 2.5 to obtain the submanifold E of M. Let $G \subset F$ be the fiber of E (the doubling surface of E if E is a double twisted I bundle).

If G is closed, then E is closed and so $E = M$. If on the other hand $bd\, G \neq \phi$, then $\pi_1(G)$ is free of rank at least 2. Now each component of $bd\, E$ is a torus. Since no such component is essential in M, each must be either compressible in M or be parallel to a component of $bd\, M$. Those components that can be are moved onto $bd\, M$. Let $T_1, T_2, \ldots, T_k$ be the compressible components of $bd\, E$. The irreducibility of M gives a solid torus R_i in M with $bd\, T_i = R_i$ $(1 \leq i \leq k)$. Now it is not possible that $E \subset R_i$ for any i. For then the inclusion of $\pi_1(G)$ into M would factor through the infinite cyclic group. This isn't possible since $\pi_1(G)$ is free of rank at least two. Let $E^* = E \cup \cup_i R_i$.

Since each boundary component of E^* is a boundary component of M, $E^* = M$. It follows that $i^*: \pi_1(E) \to \pi_1(M)$ is an epimorphism. Thus $\pi_1(G)$ is finitely generated normal subgroup of $\pi_1(M)$ that is free of rank at least two. Now $\pi_1(G)$ is a finitely generated normal subgroup of $\pi_1(F)$. It is an elementary property of free groups that $\pi_1(G)$ is of finite index in $\pi_1(F)$. Since $G \subset F$, it follows that $G = F$.

If $\pi_1(M)/\pi_1(F)$ is a finite group, then the Stallings-Swan theorem (7) assures us that $\pi_1(M)$ is also free. This is not consistent with the fact that $bd\, M$ is incompressible. $\pi_1(F)$ must then be of infinite index in $\pi_1(F)$. The desired conclusion is then given by (2), and the proof of 2.7 is complete.

Theorem 2.7 can be thought of as a "torus theorem" in which the ends of the singular torus don't match. The conclusion says that the ends can be made to match unless we are dealing with a simple bundle or a simple double twisted I bundle. There is a corresponding Theorem for annuli. It is a bit trickier to prove.

A graph J in the boundary of a 3-manifold is *injective* if $i^*: \pi_1(bdM - J) \to \pi_1(M)$ is monic. Proper paths α, β in M are *J-homotopic* if they are homotopic in M keeping endpoints in $bdM - J$. A cutting surface G for M is *J-good* if no arc in J is homotopic through bdM relative to its endpoints to a path in G. A map $f: I \times I \to M$ is (J, N, G) essential if

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i) $f^{-1}(G) = \{I \times x_1, I \times x_2, \ldots, I \times x_k\}$
$0 = x_1 < x_2 \ldots < x_k = 1$
ii) $f(\{0, 1\} \times I) \subset bdM - J$.
iii) For $x_i \neq x_j$, $I \times x_i$ is not $(J \cap bdG)$ homotopic in G to $I \times x_j$.
iv) $f(I \times 0)$ is not $J \cap G$-homotopic in G to a path in bdG.

2.8 THEOREM. *Let M be a compact, orientale, irreducible 3-manifold with incompressible boundary and J an injective graph in bd M. Let F be a J good cutting surface for M with $c(F) = (g, b)$ and $\#J \cap F = P$. Let $f: I \times I \to M$ be a (J, N, F) essential map. If $n \geq 2P(b + 1) + (2b + 1)^2 + \exp(2g + 4b + 2)^2)$, then one of the following is true.*
i) *M contains an essential torus or annulus.*
ii) *M is a simple bundle.*
iii) *M is a simple double twisted bundle.*

PROOF: Put $\ell_i = f(I \times x_i)$.

2.9 CLAIM. *ℓ_i is not homotopic in M to any path in bd M for any i.*

Suppose ℓ_1 is homotopic in M to a path in *bd M*. Then according to lemma 1.7 of [W1], ℓ_1 is homotopic in F to a path q_1 in *bd F*. Since ℓ_i is J homotopic to ℓ_j for each i, j, a second application of lemma 1.7 of (8) gives that ℓ_j is homotopic in F to a path q_j in *bd F* for each j $(1 \leq j \leq r)$. The incompressibility of *bd M* then allows a deformation of f into a component G of *bd M*. Since F is J good, lemma 1.7 of (8) gives $\#(q_i \cap J) = \#(q_j \cap J)$ for each i, j $(1 \leq i, j \leq r)$. In each component S of *bd F*, there are at most P components of S J. Thus up to $J \cap S$ homotopy in S, there are at most $2P$ distinct paths in S that meet J exactly $\#(\ell_1 \cap J)$ times. Since F has at most $b + 1$ boundary components, there are up to $J \cap$ *bd F* homotopy in *bd F* at most $2P(b + 1)$ paths in *bd F* that meet J exactly $\#(\ell_1 \cap J)$ times. Since $n > 2P(b + 1)$, this would give a $J \cap$ *bd F* homotopy of some ℓ_i to some ℓ_j in F. This proves claim 2.9.

2.10 CLAIM. *For $i \neq j$, ℓ_i is not homotopic in F to ℓ_j keeping endpoints in bd F.*

If 2.10 were wrong, then a map of an annulus into M would be obtained. The map must be essential since otherwise, ℓ_i deforms into *bd F* contrary to 2.9. Thus (9) gives the required annulus.

Let H_i denote the component of *bd* $M - J$ containing $f(i \times \{0, 1\})$ $(i = 0, 1)$. (The possibility that $H_0 = H_1$ is allowed.) There are several cases to be considered.

CASE 1. *Neither H_0 nor H_1 is a torus or an annulus.*

Let $2M$ denote M doubled along H_0 and H_1. A doubled copy $2F$ of F sits naturally as an incompressible surface in $2M$. A gross but satisfactory estimate gives $c(2F) \leq (2g + b, 3b + 1)$. By *doubling f*, a map g: $S^1 \times I \to 2M$ is obtained. The properties *a* thru *d* of F and claim 2.9 give immediately that g is an $(r, 2F)$ essential map into $2M$.

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Now apply Theorem 2.7. Since neither H_0 nor H_1 is a torus or annulus, M cannot be a bundle with fiber $2F$, nor a double twisted I bundle with doubling surface $2F$. Thus there is an essential torus or annulus T is $2M$.

Suppose that among all such tori and annuli in $2M$, T is chosen so that $T \cap (H_0 \cup H_1)$ is minimal. It follows immediately that no component of $T \cap (H_0 \cup H_1)$ is a contractible curve on T.

Suppose a is an innermost spanning arc on T that cobounds a disk E with an arc b in a component of $bd\,T$. Since F is boundary incompressible in M, lemma 1.7 of [8] gives that a cobounds a disk D on F with an arc c in $bd\,F$. Put $T' = (T - E) \cup D$. Then T' is essential in $2M$ and T' meets $H_0 \cup H_1$ fewer times than does T. This contradicts the minimality of T. Suppose a and b are innermost simple closed curves on T cobounding a subannulus T^* of T. If T^* is inessential in M, then T^* deforms in M to an annulus T' in $H_0 \cup H_1$. Then putting $T'' = (T - T^*) \cup T'$, an essential annulus (or torus) in $2M$ is obtained that meets $H_0 \cup H_1$ fewer times than does T. This is not consistent with the minimality of T and hence T^* would be he required essential annulus in M.

The proof proceeds assuming there is at most one simple closed curve in $T \cap (H_0 \cup H_1)$. Suppose c is the unique simple closed curve on $T \cap (H_0 \cup H_1)$. Then T cannot be a torus since $H_0 \cup H_1$ separates $2M$. Thus c separates T into two annuli T_1 and T_2. Suppose T_1 is inessential in M. (If T_1 is essential, there is nothing left to do.) It follows that T_1 deforms into $bd\,M$. An examination of this homotopy shows that c is parallel on F to a boundary component of F. It follows that T_2 must be essential in M. (A deformation of T_2 into $bd\,M$ would now give a deformation of T into $bd(2M)$.)

It remains to consider the case that each component of $T \cap (H_0 \cup H_1)$ is an arc that spans T. Since $H_0 \cup H_1$ separates $2M$, these components must occur in pairs. Let s and t be innermost such spanning arcs that cobound a disk E on T. Since $bd\,M$ is incompressible, $bd\,E$ bounds a disk D in $bd\,M$. Furthermore, the irreducibility of M gives a 3-cell bounded by $D \cup E$. Now if s and t lie separate components of $H_0 \cup H_1$, then D gives a deformation of s into $bd\,2M$ contradicting the essentiality of T. Thus both s and t lie in say H_0. Observe that D meets $bd\,H_0$ in 2 arcs u, v and that $s \cup t \cup u \cup v$ is the boundary of a disk $D' = D \cap H_0$. Finally observe that E can be isotoped into D. Once again an annulus T^* is obtained that meets $H_0 \cup H_1$ fewer times than does T. This completes Case 1.

CASE 2. *Both H_0 and H_1 are tori and $H_0 = H_1$.*

The idea here is to *straighten* $f|\{0, 1\} \times [0, 1]$ so that $f(0 \times [0, 1] = f(1 \times [0, 1])$. Then f can be traded for a map of an annulus into M. Consider H_0 as $S^1 \times [0, k]$ where k is identified with 0. This product structure is chosen so that the boundary components of F that meet H_0 occur at $S^1 \times t$ where t ranges over the integers $0, 1, 2, \ldots, k$.

2.11 CLAIM. *For each integer i, $f(1 \times [x_i, x_{i+1})]$ either has its endpoints in separate components of $bd\,F$, or both endpoints lie in $S^1 \times 0$ and $f(1 \times [x_i, x_{i+1}])$ meets both sides of a regular neighborhood of $S^1 \times 0$.*

Suppose for some integer i, $f(1 \times [x_i, x_{i+1}])$ meets only one side of a regular neighborhood of $S^1 \times t$ and that $f(1 \times [x_i, x_i + 1\}) \subset S^1 \times t$. Then using the goodness of F with respect to J, $f(1 \times [x_i, x_{i+1}])$ can be deformed through $bd\ M - J$ into $(S^1 \times t) - J$. Consider now $f|I \times [x_i, x_{i+1}]$ as a homotopy of the path $a = f(1 \times (x_i, x_{i+1}))$ to the path $b = f(0 \times [x_i, x_{i+1}])$. Since a $\cap J = \phi$, lemma 1.7 of [W1] gives a deformation of $f(0 \times [x_i, x_{i+1}])$ into $(S^1 \times t) - J$. Observe now that $f(bd\ (I \times [x_i, x_{i+1}]))$ lies in $F - J$. The incompressibility of F in M gives a contraction of $f(bd(I \times [x_i, x_{i+1}]))$ in F. But this is a $J \cap bd\ F$ homotopy of ℓ_i to ℓ_{i+1}. This inconsistency proves claim 2.11.

Choose a point y on S^1. The first step in *straightening* f is to deform $f|[x_i, x_{i+1}] \times 1$ so that it maps linearly into $y \times [t, t + 1]$ (assuming $f(x_i, 1) \subset S^1 \times t$). (The graph J will for the moment be ignored.) Suppose this has been accomplished for all $i < j$. Then $f(1, x_i) = (y, t)$. By 2.11 $f(1, x_{i+1}) \subset S^1 \times (t + 1)$. Since the fundamental group of $S^1 \times [t, t + 1]$ is generated by $S^1 \times (t + 1)$, it is possible to move $f(1, x_{i+1})$ around $S^1 \times (t + 1)$ so that as a result, $f|1 \times [x_i, x_{i+1}]$ is homotopic with endpoints fixed to $y \times [t, t + 1]$. Thus f can be inductively straightened so that $f|S^1 \times [0, 1]$ looks like the standard exponential map onto the circle $y \times [0, k]$. $f|0 \times [0, 1]$ is altered in a similar fashion.

Observe now that each path ℓ_i has been deformed so that it is a loop in F. Now ℓ_i is not homotopic in F to any loop in $bd\ F$, but ℓ_i may be freely homotopic in F to such a loop. We wish to correct this. Let Z be the boundary component of F containing the base point of ℓ_i. Let $m_i = [Z,[Z, \ell_i']]$. Let $G = \pi_1(F)$, and let $\gamma_n G$ denote the nth term of the lower central series of G. Suppose $\ell_i \in \gamma_n G$ but $\ell_i \notin \gamma_{n+1} G$. (Such an n exists [6]). Then $[Z, \ell_i] \in \gamma_{n+1} G$ so that $[Z,[Z, \ell_i]] \neq 1$. Also, $[Z,[Z, \ell_i]] \in \gamma_2 G$ and no boundary component of $bd\ F$ lies in $\gamma_2 G$. Thus m_i is not freely homotopic in F to any boundary component of F. By piecing together f with maps of $S^1 \times i$ to $S^1 \times (i + 1)$ through H_0, one can make a map g: $S^1 \times [0, 1] \to M$ such that $g(S^1 \times x_r) = m_r$ for each integer t, and $g^{-1}(G) = S^1 \times x_0 S^1 \times x_1 \dots S^1 \times x_r$.

Suppose that for some $p \neq q$, m_p is freely homotopic in F to m_q. Then $g|S^1 \times [x_p, x_q]$ can be thought of as a map of a torus into M. Observe that $g(0 \times [x_p, x_q])$ is a nonzero power of $y \times [0, k]$, and that m_p does not contract in M. Furthermore no nonzero power of $y \times [0, k]$ is homotopic to a loop in F. It follows that $g|S^1 \times [x_p, x_q]$ induces a monomorphism of the fundamental group of a torus into M. Finally m_p does not deform through F into $bd\ F$, and so $g|S^1 \times [x_p, x_q]$ cannot deform into $bd\ M$. It follows from [9] that M contains an essential torus or annulus.

If on the other hand m_p is not freely homotopic in F to m_q for any $p \neq q$, then theorem 2.7 gives the result.

CASE 3. *Both H_0 and H_1 are tori, but $H_0 \neq H_1$.*

The idea here is to loop the paths around H_0 and back up to H_1 to transfer the problem back to Case 2. Let Z_1 be the boundary component of F containing the initial point of ℓ_i and Z_2 the boundary component of F containing the terminal point of ℓ_i. It is easy to build a map $g: I \times [0, 1] \to M$ such that $g(I \times x_i) = \ell_i Z_2 \ell_i^{-1} = n_i$ and $g^{-1}(F) = \cup_{i=0}^{r} I \times x_i$. Now the loop n_i is freely homotopic in F to Z_2 and so n_i cannot deform into Z_1.

If for each $i \neq j$, n_i is not $J \cap bd\, F$ homotopic in F to n_j, then Case 2 gives the conclusion. Suppose on the other hand that for some $p \neq q$, n_i is $J\, bd\, F$ homotopic to n_j. Then $g|I \times [x_p, x_q]$ can be thought of as a map of an annulus into M. Since n_p does not deform into Z_1, g is essential. The required annulus is now given by [9].

CASE 4. *H_0 is a torus, but H_1 is not a torus.*

This case can be handled through a combination of the case 1 through 3. Only an outline of the procedure is given. First double M along H_1 to obtain $2M$ as in Case 1. Double f to obtain a map $g : I \times [0, 1] \to 2M$. Observed that $g(0 \times [0, 1])$ and $g(1 \times [0, 1])$ both lie in torus components of $bd\, 2M$. Then Case 3 gives an essential torus or annulus T in $2M$. Then T can be used to find an essential torus or annulus in M just as in Case 1.

2.9 COROLLARY. *Let M be a compact, orientable, irreducible 3-manifold and F an incompressible, boundary incompressible surface in M. If $\pi_1(F)$ contains any non-trivial subgroup that is normal in $\pi_1(M)$ then one of the following is true.*

i) *M contains an essential torus or annulus.*

ii) *M is a bundle over S^1 with fiber F.*

iii) *M is a double twisted I bundle with doubling surface F.*

iv) *M is an I bundle over F.*

PROOF: If $\pi_1(F)$ is of finite index in $\pi_1(M)$ we get iv) immediately. The proof proceeds assuming $\pi_1(F)$ is of infinite index in $\pi_1(M)$. Since no normal subgroup of a free product can live entirely in one factor, the boundary of M must be incompressible. If x is in the normal subgroup and $t \in \pi_1(F)$, then a long annulus that meets F in the loops $x, txt^{-1}, \ldots, t^N xt^{-N}$ is easily constructed. Thorem 2.7 applies to yield the result.

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