ON THE FIRST STEPS OF THE MINIMAL MODEL PROGRAM FOR THE MODULI SPACE OF STABLE POINTED CURVES

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Abstract   The aim of this paper is to study all the natural first steps of the minimal model program for the moduli space of stable pointed curves. We prove that they admit a modular interpretation, and we study their geometric properties. As a particular case, we recover the first few Hassett–Keel log canonical models. As a by-product, we produce many birational morphisms from the moduli space of stable pointed curves to alternative modular projective compactifications of the moduli space of pointed curves.

Keywords and phrases: Minimal model program, moduli space of curves, modular compactifications

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Contents

1. Introduction

The motivation of this work comes from the following vague but inspiring thought:

Question: If we run a minimal model program of a moduli space, do all the steps admit a modular interpretation?

For example, this is true for the moduli spaces of vector bundles over many classes of surfaces (see [16, 54, 48, 43, 17, 19, 22] or surveys [21, 35, 44]).

In this paper, we look at this question for the coarse moduli space $\overline{M}_{g,n}$ of Deligne–Mumford stable n-pointed curves of genus g. The main result of the paper is that *all* the first natural steps of the minimal model program (MMP) for $\overline{M}_{g,n}$ admit a modular interpretation; more precisely, they are moduli spaces of suitable singular curves.

The MMP for $\overline{M}_{g,n}$ is closely related to the Hassett–Keel program (see [33, 34, 9, 8, 7]), which is interested in studying the modular interpretation of the log canonical models

$$\overline{M}_{g,n}(\alpha) := \mathrm{Proj} \bigoplus_{m \geq 0} H^0 \left(\overline{\mathcal{M}}_{g,n}, \left\lfloor m \left(K_{\overline{\mathcal{M}}_{g,n}} + \psi + \alpha(\delta - \psi) \right) \right\rfloor \right) \qquad (1.1)$$

of $\overline{M}_{g,n}$ with respect to $K_{\overline{\mathcal{M}}_{g,n}} + \psi + \alpha(\delta - \psi)$ as α decreases from 1 to 0. However, the point of view of the MMP is slightly different, since it is interested in contracting K-negative rays, or more generally faces, of the Mori cone $\overline{M}_{g,n}$ and then flipping them if the resulting contraction is small. It turns out that the first three steps of the Hassett–Keel program coincide with some of the steps of the MMP described in this paper, as we explain in detail toward the end of the introduction.

As a by-product of our investigation, we produce many morphisms (with connected fibres) from $\overline{M}_{g,n}$ to other normal projective varieties. The number of these morphisms grows exponentially in (g,n). This gives a partial answer to [29, Question, page 275]), which asks for a classification of all such morphisms. To the best of our knowledge, the only previously known birational morphisms from $\overline{M}_{g,n}$ (with $g > 5$) were the first two steps of the Hassett–Keel program and, for $n = 0$, the Torelli morphism from $\overline{M}_g$ to the Satake compactification of the moduli space of principally polarised abelian varieties (note that it is unknown whether the Satake compactification admits a modular interpretation as moduli space of curves).

The geometry of the morphisms that we construct in this paper will be further studied in our work. This paper is independent of its sequel [20], tu for the sake of completeness we have included here some results from that work.

As a further by-product, we produce many new weakly modular (and sometimes also modular) compactifications (in the sense of [26, Sec. 2.1]) of the moduli space $M_{g,n}$ of n-pointed smooth curves of genus g (see Remark 4.15). Moreover, our weakly modular

compactifications involve curves whose singularities are of the simplest kind, namely nodes, cusps and tacnodes – a problem that was explicitly discussed in [26, p. 21–22].

1.1. The first step

As a warm-up, let us describe the possible first steps of the MMP for $\overline{M}_{g,n}$, assuming for the moment that the characteristic of the base field k is 0.

A first natural K-negative[1] extremal ray of $\overline{NE}\left(\overline{M}_{g,n}\right)$ is generated by the *elliptic tail curve* C_{ell} – that is, the curve C_{ell} (well defined up to numerical equivalence) of $\overline{M}_{g,n}$ parametrising a moving 1-pointed elliptic curve (E,p) attached in p to a fixed $n+1$-pointed smooth irreducible curve of genus $g-1$. The contraction associated to the extremal ray $\mathbb{R}_{\geq 0} \cdot C_{\mathrm{ell}}$ has a modular meaning and can be identified with the modular contraction

$$\Upsilon : \overline{M}_{g,n} \to \overline{M}_{g,n}^{\mathrm{ps}}, \qquad (1.2)$$

where $\overline{M}_{g,n}^{\mathrm{ps}}$ is a projective normal $\mathbb{Q}$-factorial irreducible variety which is the coarse moduli space of the proper smooth Deligne–Mumford stack of *n-pointed pseudostable curves* of genus g – that is, n-pointed projective connected (reduced) curves of genus g with nodes and cusps as singularities, not having elliptic tails and with ample log canonical line bundle[2] – and Υ sends an n-pointed stable curve $C \in \overline{M}_{g,n}(k)$ into the n-pointed pseudostable curve $\Upsilon(C)$ of $\overline{M}_{g,n}^{\mathrm{ps}}(k)$ which is obtained by contracting the elliptic tails of C into cusps (see Propositions 3.11, 5.1 and 5.5).

The morphism Υ is a birational divisorial contraction of relative Picard number 1, and it is the unique such morphism if $g \geq 5$, by [29, Prop. 6.4]. Moreover, if the F-conjecture is true and $n \leq 2$, then a close inspection of formulae [29, Thm. 2.1] reveals that $\mathbb{R}_{\geq 0} \cdot C_{\mathrm{ell}}$ is the unique K-negative extremal ray of $\overline{NE}\left(\overline{M}_{g,n}\right)$. On the other hand, if the F-conjecture is true and $n \geq 3$, then there are other extremal rays of $\overline{NE}\left(\overline{M}_{g,n}\right)$ that are K-negative, but $\mathbb{R}_{\geq 0} \cdot C_{\mathrm{ell}}$ is the unique one which is also $K + \psi$-negative. In both the MMP and the Hassett–Keel program of $\overline{M}_{g,n}$, it seems that the divisor class $K + \psi$ is more natural than the divisor K; one reason is that, on the stack, it is stable under the clutching morphisms (see, for example, [15, Chap. XVII, Sec. 4]). The upshot of this discussion is that the morphism (1.2) is the 'natural' (and conjecturally unique for $n \leq 2$) first step of the MMP for $\overline{M}_{g,n}$.

1.2. The next steps

Let us now analyse the natural possible ways of continuing the MMP of $\overline{M}_{g,n}$ by looking for K-negative extremal rays of $\overline{M}_{g,n}^{\mathrm{ps}}$.

Given a hyperbolic pair (g,n) (that is, such that $2g-2+n > 0$), consider the set

$$T_{g,n} := (\{\mathrm{irr}\} \cup \{(\tau,I) : 0 \leq \tau \leq g, I \subseteq [n] := \{1,\ldots,n\}\} \setminus \{(0,\emptyset),(g,[n])\}) / \sim, \qquad (1.3)$$

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[1] In this introduction, we will be deliberately vague on the canonical class K; what we are going to say works both for the canonical class of the stack and for its coarse moduli space.

[2] We assume from now on that $(g,n) \neq (1,1),(2,0)$, because $\overline{\mathcal{M}}_{1,1}^{\mathrm{ps}}$ is empty, while $\overline{\mathcal{M}}_{2,0}^{\mathrm{ps}}$ is neither separated nor with finite inertia and $\overline{M}_{2,0}^{\mathrm{ps}}$ is only an adequate moduli space.

where $\sim$ is the equivalence relation such that irr is equivalent only to itself and $(\tau,I) \sim (\tau',I')$ if and only if $(\tau,I) = (\tau',I')$ or $(\tau',I') = (g-\tau,I^c)$, where $I^c = [n] \setminus I$. We will denote the class of (τ,I) in $T_{g,n}$ by $[\tau,I]$ and the class of irr in $T_{g,n}$ again by irr. Set $T_{g,n}^* = T_{g,n} \setminus \{\text{irr}\}$.

Definition 1.1 (Elliptic bridge curves, see Figure 1). Consider the following irreducible curves (well defined up to numerical equivalence) in $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ (or in $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$), which we call *elliptic bridge curves*:

1. If $g \geq 2$ and $(g,n) \neq (2,0)$, we denote by $C(\text{irr})$ the closure of the curve formed by a varying 2-pointed rational nodal elliptic curve (R,p,q) attached to a fixed n-pointed smooth irreducible curve D of genus $g-2$ in the two points p and q. If $(g,n) = (2,0)$, $C(\text{irr})$ is the closure of the curve formed by a varying rational curve with two nodes.

2. For every $\{[\tau,I],[\tau+1,I]\} = \{[\tau,I],[g-1-\tau,I^c]\} \subset T_{g,n} - \{(1,\emptyset),\text{irr}\}$, we denote by $C([\tau,I],[\tau+1,I])$ the curve formed by a varying 2-pointed rational nodal elliptic curve (R,p,q) attached in p to a fixed smooth irreducible curve D_1 of genus τ and with marked points $\{p_i\}_{i \in I}$ and attached in q to a fixed smooth irreducible curve D_2 of genus $g-1-\tau$ with marked points $\{p_i\}_{i \in I^c}$, with the convention that if $\tau = 0$ and $I = \{k\}$ for some $k \in [n]$, then instead of attaching the fixed curve D_1, we consider p as the kth marked points, and similarly for the case $(g-1-\tau,I^c) = (0,\{k\})$.

The *type* of an elliptic bridge curve is defined as follows: $C(\text{irr})$ has type $\{\text{irr}\} \subset T_{g,n}$, and $C([\tau,I],[\tau+1,I])$ has type equal to $\{[\tau,I],[\tau+1,I]\} \subset T_{g,n}$.

The elliptic bridge curves generate linearly independent extremal rays of $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ that are both K- and $K+\psi$-negative (see Proposition 5.9). For an arbitrary subset $T \subseteq T_{g,n}$, we denote by F_T the K-negative face of $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ spanned by the classes of the elliptic bridge curves whose type is contained in T (see Lemma 5.12 for the properties of F_T).

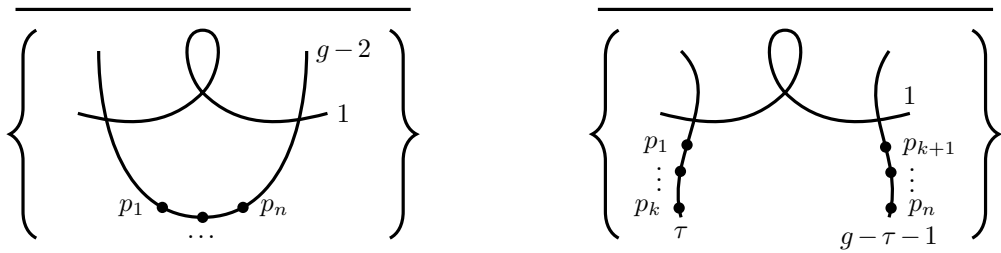If the F-conjecture (see [29, Conj. (0.2)]) holds, then the following are true:

Figure 1. The elliptic bridge curves $C(\text{irr})$ and $C([\tau,I],[\tau+1,I])$, where $I = \{1,\ldots,k\}$. The varying component is a 2-pointed rational nodal curve.

- The elliptic bridge curves are the unique 1-strata of $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ which are $K_{\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}} + \psi$-negative. In particular, they are the unique 1-strata of $\overline{\mathcal{M}}^{\mathrm{ps}}_g$ which are $K_{\overline{\mathcal{M}}^{\mathrm{ps}}_g}$-negative.
- The elliptic bridge curves are the unique $K_{\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}}$-negative curves of $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ which are the image of $K_{\overline{\mathcal{M}}_{g,n}}$-positive 1-strata of $\overline{\mathcal{M}}_{g,n}$.

Hence the natural prosecution of the MMP for $\overline{\mathrm{M}}_{g,n}$ is the contraction of one of these extremal rays, or, more generally, of a face F_T and its flip. The goal of this paper is to show that both the contractions of these K-negative faces and their flips have a modular description, and to describe explicitly their geometrical properties.

1.3. T-semistable and T^+-semistable curves

To give these modular descriptions, we need new stability notions. Given a tacnode p of an n-pointed projective curve of genus g with ample log canonical line bundle, we define the type of p as

- $\mathrm{type}(p) := \{\mathrm{irr}\} \subseteq T_{g,n}$ if the normalisation of C at p is connected;
- $\mathrm{type}(p) := \{[\tau, I], [\tau + 1, I]\} \subseteq T_{g,n}$ if the normalisation of C at p consists of two connected components, one of which has arithmetic genus τ and marked points $\{p_i\}_{i \in I}$ and the other of which has arithmetic genus $g - 1 - \tau$ and marked points $\{p_i\}_{i \in I^c}$.

In a similar fashion, we define the type of an A_1/A_1-attached elliptic chain (see Definition 3.2).

Definition 1.2 (see Definition 3.16). Set $T \subseteq T_{g,n}$.

(i) We denote by $\overline{\mathcal{M}}^T_{g,n}$ the stack of T-*semistable curves* – that is, n-pointed projective connected curves of genus g having singularities that are nodes, cusps or tacnodes of type contained in T, not having either A_1-attached elliptic tails nor A_3-attached elliptic tails and with ample log canonical line bundle.

(ii) We denote by $\overline{\mathcal{M}}^{T+}_{g,n}$ the stack of T^+-*semistable curves* – that is, T-semistable curves without any A_1/A_1-attached elliptic chain of type contained in T.

1.4. Main results

We can now state the three main results of this paper. We work over an algebraically closed field k. For some of our results, we will need to assume that the characteristic of k is big enough with respect to the pair (g, n), which we write as $\mathrm{char}(k) \gg (g, n)$ (see Definition 4.1); for some others we assume that the characteristic of k is 0.

The first main result describes the relation between the stacks of pseudostable curves, T-semistable curves and T^+-semistable curves and their good moduli spaces.

Theorem A (= Theorems 3.19 and 4.4). *Assume $(g,n) \neq (2,0)$ and $T \subset T_{g,n}$.*

(1) *The stack $\overline{\mathcal{M}}_{g,n}^T$ is algebraic, smooth, irreducible and of finite type over k, and we have open embeddings*

$$\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \overset{\iota_T}{\lhook\joinrel\longrightarrow} \overline{\mathcal{M}}_{g,n}^T \overset{\iota_T^+}{\longleftarrow\!\!\!\!\supset} \overline{\mathcal{M}}_{g,n}^{T,+}.$$

(2) *Assume that $\mathrm{char}(k) \gg (g,n)$. Then the algebraic stacks $\overline{\mathcal{M}}_{g,n}^T$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ admit good moduli spaces $\overline{\mathrm{M}}_{g,n}^T$ and $\overline{\mathrm{M}}_{g,n}^{T+}$, respectively, which are proper normal irreducible algebraic spaces over k. Moreover, there exists a commutative diagram*

$$
\begin{array}{ccccc}
\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} & \overset{\iota_T}{\lhook\joinrel\longrightarrow} & \overline{\mathcal{M}}_{g,n}^T & \overset{\iota_T^+}{\longleftarrow\!\!\!\supset} & \overline{\mathcal{M}}_{g,n}^{T+} \\
\downarrow{\phi^{\mathrm{ps}}} & & \downarrow{\phi^T} & & \downarrow{\phi^{T+}} \\
\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} & \overset{f_T}{\longrightarrow} & \overline{\mathrm{M}}_{g,n}^T & \overset{f_T^+}{\longleftarrow} & \overline{\mathrm{M}}_{g,n}^{T+}
\end{array}
$$

where the vertical maps are the natural morphisms to the good moduli spaces (indeed, ϕ^{ps} is also a good moduli space if $\mathrm{char}(k) \gg (g,n)$) and the bottom horizontal morphisms f_T and f_T^+ are proper (and birational, if $(g,n) \neq (1,2)$) morphisms.

Theorem A(1) (which coincides with Theorem 3.19) is proved in Section 3. In this section, we also investigate the properties of the stacks $\overline{\mathcal{M}}_{g,n}^T$ and $\overline{\mathcal{M}}_{g,n}^{T+}$: we describe the containment relation among all these different stacks in Proposition 3.22; we describe the closed points and the isotrivial specialisations of $\overline{\mathcal{M}}_{g,n}^T$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ in Propositions 3.24 and 3.27; and we describe the Picard groups of $\overline{\mathcal{M}}_{g,n}^T$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ in Corollary 3.29.

Theorem A(2) is proved in Section 4 (see Theorem 4.4). The strategy is the same as the one pioneered by Alper, Fedorchuk, Smyth and van der Wyck in [9] to perform the first steps of the Hassett–Keel program. The key property is the fact that the open embeddings of stacks in part (1) of the theorem arise from local variation of geometric invariant theory (VGIT) with respect to $\delta - \psi$ (in the sense of [9, Def. 3.14]). One little improvement of those methods is provided in Proposition 4.9, which generalises [8, Prop. 1.4] from characteristic 0 to arbitrary characteristic and allows us to construct the good moduli spaces, provided that the automorphism group schemes of the algebraic stacks are linearly reductive, which is true if the characteristic is big enough (see Lemma 4.2 and Remark 4.3).

After the completion of this work, Alper, Halpern-Leistner and Heinloth posted on arXiv a preprint [11] in which they provide a necessary and sufficient criterion for a stack to admit a good moduli space. Hence it should be possible to prove the existence of the good moduli spaces $\overline{\mathrm{M}}_{g,n}^T$ and $\overline{\mathrm{M}}_{g,n}^{T+}$ (and also Proposition 4.9) using their criterion; however, we have not checked the details.

Our second main result identifies, in characteristic 0, the morphism f_T with the contraction of the K-negative face F_T of the Mori cone of $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$.

Theorem B (= Theorem 6.1). *Assume* char$(k) = 0$, $(g,n) \neq (2,0)$ *and* $T \subseteq T_{g,n}$. *The good moduli space* $\overline{\mathrm{M}}_{g,n}^{T}$ *is projective and the morphism* $f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^{T}$ *coincides with the contraction of the face* F_T.

The proof of this theorem follows, using the rigidity lemma (Lemma 2.1), from the fact that f_T is a contraction with the property that a curve $C \subset \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is contracted by f_T if and only if its class $[C]$ lies in F_T (see Lemma 5.12 and Proposition 6.2). From Theorem B and standard corollaries of the cone theorem, we derive a description of the rational Picard group of $\overline{\mathrm{M}}_{g,n}^{T}$ and of its nef/ample cone (see Corollary 6.4).

In our sequel paper [20], we investigate the geometric properties of the moduli space $\overline{\mathrm{M}}_{g,n}^{T}$ and of the morphism f_T (see Proposition 6.7 for a recap of some of those results).

Our last main result is a description of the morphism $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ (which turns out to be a projective contraction; see Propositions 7.12 and 7.15) as the flip (in the sense of Definition 7.1) of f_T with respect to suitable $\mathbb{Q}$-line bundles.

Theorem C (= Theorem 7.4, Corollary 7.13, Corollary 7.20). *Assume* char$(k) \gg (g,n)$, $(g,n) \neq (2,0),(1,2)$, *and* $T \subseteq T_{g,n}$. *Let* $L \in \mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} = \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} = \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{T}\right)_{\mathbb{Q}}$. *The morphism* f_T^+ *is the* L-*flip of* f_T *if and only if* L *is* f_T-*antiample and the restriction of* L *to* $\overline{\mathcal{M}}_{g,n}^{T+}$ *descends to a* $\mathbb{Q}$-*line bundle on* $\overline{\mathrm{M}}_{g,n}^{T+}$. *In particular:*

(i) *The morphism* $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T,+} \to \overline{\mathrm{M}}_{g,n}^{T}$ *is the* $\left(K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi\right)$-*flip of* f_T.

(ii) *The morphism* $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T,+} \to \overline{\mathrm{M}}_{g,n}^{T}$ *is the* $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-*flip of* f_T *if and only if* $\overline{\mathrm{M}}_{g,n}^{T,+}$ *is* $\mathbb{Q}$-*Gorenstein – that is, if and only if* T *does not contain subsets of the form* $\{[0,\{j\}],[1,\{j\}],[2,\{j\}]\}$ *for some* $j \in [n]$ *or* $(g,n) = (3,1),(3,2),(2,2)$.

Therefore, $\overline{\mathrm{M}}_{g,n}^{T+}$ *is projective if* char$(k) = 0$.

In proving this result, we investigate the properties of the space $\overline{\mathrm{M}}_{g,n}^{T+}$ and of the morphism $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ in Section 7. We compute the rational Picard group of $\overline{\mathrm{M}}_{g,n}^{T+}$ in Proposition 7.7 (and in particular, we describe explicitly when a $\mathbb{Q}$-line bundle on $\overline{\mathcal{M}}_{g,n}^{T+}$ descends to a $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^{T+}$) and we describe when $\overline{\mathrm{M}}_{g,n}^{T}$ is $\mathbb{Q}$-factorial or $\mathbb{Q}$-Gorenstein in Corollary 7.9. Moreover, we describe the exceptional locus of f_T^+ in Proposition 7.15 and its relative Mori cone in Proposition 7.19.

Finally, we prove in Corollary 7.21 that whenever $f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^{T}$ is small and $\overline{\mathrm{M}}_{g,n}^{T,+}$ is $\mathbb{Q}$-factorial, for any $\mathbb{Q}$-line bundle L on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ which is f_T-antiample, the rational map $\left(f_T^+\right)^{-1} \circ f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \dashrightarrow \overline{\mathrm{M}}_{g,n}^{T,+}$ can be decomposed as a sequence of elementary L-flips.

A posteriori, we can recover our stacks of T-semistable and T^+-semistable curves as semistable loci for convenient line bundles, as explained in the following remark:

Remark 1.3. Let $\mathcal{U}_{g,n}^{lci}$ be the stack of n-pointed curves of arithmetic genus g with locally complete intersection singularities and with ample log canonical line bundle, as in Section 3.2. Recall that $\mathcal{U}_{g,n}^{lci}$ is a smooth and irreducible algebraic stack of finite type over

k. The stack $\overline{\mathcal{M}}_{g,n}^T$ of T-semistable curves is an open substack of $\mathcal{U}_{g,n}^{lci}$, and its complement contains a unique divisor, namely the divisor $\Delta_{1,\emptyset}$ parametrising curves with an elliptic tail.

Assume that $\mathrm{char}(k)=0$ and consider the projective good moduli space $\phi^T : \overline{\mathcal{M}}_{g,n}^T \to \overline{\mathrm{M}}_{g,n}^T$ (see Theorem B). Let M be an ample line bundle on $\overline{\mathrm{M}}_{g,n}^T$ and let $\mathcal{L}$ be a line bundle on $\mathcal{U}_{g,n}^{lci}$ whose restriction to $\overline{\mathcal{M}}_{g,n}^T$ coincides with $(\phi^T)^*(M)$ (note that such a line bundle $\mathcal{L}$ exists, since $\mathcal{U}_{g,n}^{lci}$ is regular). By combining [4, Thm. 11.5] and the proof of [4, Thm. 11.14(ii)], it follows that the stack $\mathcal{M}_{g,n}^T$ is exactly the semistable locus of $\mathcal{U}_{g,n}^{lci}$ with respect to $\mathcal{L}_N := \mathcal{L} \otimes \mathcal{O}_{\mathcal{U}_{g,n}^{lci}}(N\Delta_{1,\emptyset})$ for $N \gg 0$ (in the sense of [4, Def. 11.1]) and $\overline{\mathrm{M}}_{g,n}^T$ is the good moduli space provided by [4, Thm. 11.5]. A similar statement holds true for $\phi^{T+} : \overline{\mathcal{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^{T+}$.

1.5. Relation with the Hassett–Keel program

We can now describe in detail the connection between our work and the first steps of the Hassett–Keel program, as established in [33, 34, 9, 8, 7]. From [7, Thm. 1.1] and Proposition 5.5(i), it follows (assuming $\mathrm{char}(k)=0$) that

$$\overline{\mathrm{M}}_{g,n}(\alpha) = \begin{cases} \overline{\mathrm{M}}_{g,n} & \text{if } \frac{9}{11} < \alpha \leq 1, \\ \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} & \text{if } \frac{7}{10} < \alpha \leq \frac{9}{11}, \\ \overline{\mathrm{M}}_{g,n}^{T_{g,n}} & \text{if } \alpha = \frac{7}{10}, \\ \overline{\mathrm{M}}_{g,n}^{T_{g,n}+} & \text{if } \frac{2}{3} < \alpha < \frac{7}{10}. \end{cases} \tag{1.4}$$

Therefore, Theorems B and C imply that at the second critical value of the Hassett–Keel program, $7/10$, the variety $\overline{\mathrm{M}}_{g,n}(7/10)$ is obtained from $\overline{\mathrm{M}}_{g,n}(7/10+\epsilon) \cong \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ by contracting the entire elliptic bridge face of the Mori cone of $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ (whose dimension is computed in Remark 5.10), while the variety $\overline{\mathrm{M}}_{g,n}(7/10-\epsilon)$ is obtained by flipping this contraction with respect to $K+\psi$. As a by-product of our analysis, we obtain some results on the geometry of $\overline{\mathrm{M}}_{g,n}(7/10)$ and of $\overline{\mathrm{M}}_{g,n}(7/10-\epsilon)$: we compute their rational Picard groups (see Example 6.5 and Corollary 7.11) and determine when they are $\mathbb{Q}$-factorial or $\mathbb{Q}$-Gorenstein (see Proposition 6.7 and Remark 7.10).

1.6. Open questions

This work leaves out some interesting questions, which we hope to be able to address in the future:

(1) For any $\mathbb{Q}$-line bundle L on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ which is f_T-antiample, we can construct the L-flip of f_T at least if $\mathrm{char}(k)=0$ (see Lemma 7.3(ii)). Theorem C implies that the L-flip of f_T coincides with f_T^+, provided that the restriction of L to $\overline{\mathcal{M}}_{g,n}^{T+}$ is T^+-compatible. If this condition fails (which can only happen if $\overline{\mathrm{M}}_{g,n}^{T+}$ is not $\mathbb{Q}$-factorial), is there a modular description of the L-flip of f_T?

(2) Can we describe modularly all the small $\mathbb{Q}$-factorialisations of $\overline{M}_{g,n}^T$ – that is, all the $\mathbb{Q}$-factorial normal proper algebraic spaces endowed with a small contraction $X \to \overline{M}_{g,n}^T$? Even more, it would be interesting to determine the chamber decomposition

$$\mathrm{Cl}\left(\overline{M}_{g,n}^T\right)_{\mathbb{R}} / \mathrm{Pic}\left(\overline{M}_{g,n}^T\right)_{\mathbb{R}} = \coprod \mathrm{Nef}\left(X_i / \overline{M}_g^T\right),$$

where $X_i \to \overline{M}_{g,n}^T$ vary among all the small $\mathbb{Q}$-factorialisations of $\overline{M}_g^T$ (see [41, Exercise 116] and [45, Thm. 12.2.7]).

In this paper, we have described modularly some of the $\mathbb{Q}$-factorialisations of $\overline{M}_{g,n}^T$, namely $\overline{M}_{g,n}^{T^{\mathrm{div}}}$ (which coincides with $\overline{M}_{g,n}^{\mathrm{ps}}$ whenever f_T is small; see Proposition 6.7) and $\overline{M}_{g,n}^{S+}$ for all subsets $S \subseteq T$ that satisfy the conditions of Corollary 7.9(ii). However, when $\overline{M}_{g,n}^{T+}$ is not $\mathbb{Q}$-factorial, we know for sure there are other $\mathbb{Q}$-factorialisations, namely the $\mathbb{Q}$-factorial flips of the morphisms $f_S : \overline{M}_{g,n}^{\mathrm{ps}} \to \overline{M}_{g,n}^S$ where $S \subseteq T$ and $\overline{M}_{g,n}^{S+}$ is not $\mathbb{Q}$-factorial (see the previous question).

(3) Theorem B implies that the moduli space $\overline{M}_{g,n}^T$ (and hence also $\overline{M}_{g,n}^{T+}$) is projective if $\mathrm{char}(k) = 0$. Is this true in positive characteristics (big enough so that $\overline{M}_{g,n}^T$ exists)? For the special case $T = T_{g,n}$, this is achieved in Example 6.5 by building upon the geometric invariant theory (GIT) analysis of [34] for $n = 0$. In the general case, when no GIT construction seems plausible, we could try to use Kollár's approach [40], but the main difficulties are that the stack $\overline{\mathcal{M}}_{g,n}^T$ does not have finite stabilisers and it parametrises nonnodal curves.

(4) Can we find some (or all) $\mathbb{Q}$-line bundles L (perhaps of adjoint type) on $\overline{M}_{g,n}$ for which $\mathrm{Proj} \bigoplus_{m \geq 0} H^0\left(\overline{M}_{g,n}, \lfloor mL \rfloor\right)$ is isomorphic to $\overline{M}_{g,n}^T$ or $\overline{M}_{g,n}^{T+}$? A quite complete answer for $\overline{M}_{g,n}^T$ is contained in [20, Sec. 4].

2. Notation and background

We work over a fixed algebraically closed field k of arbitrary characteristic. Further restrictions on the characteristic of k will be specified when needed.

2.1. Notations for curves

An *n-pointed curve* $(C, \{p_i\}_{i=1}^n)$ is a connected, reduced, projective 1-dimensional scheme C over k with n distinct smooth points $p_i \in C$ (called *marked points*). If the number of marked points is clear from the context, we will denote an *n*-pointed curve simply by C. The (arithmetic) genus of a curve C will be denoted by $g(C)$. The *log canonical line bundle* of an *n*-pointed curve $(C, \{p_i\}_{i=1}^n)$ is $\omega_C^{\log} := \omega_C\left(\sum_{i=1}^n p_i\right)$.

A singular point $p \in C$ is called:

- a *node* (or singularity of type A_1) if the complete local ring $\widehat{\mathcal{O}_{C,p}}$ of C at p is isomorphic to $k[[x,y]]/(xy)$ (or to $k[[x,y]]/\left(y^2 - x^2\right)$ if $\mathrm{char}(k) \neq 2$);

- a *cusp* (or singularity of type A_2) if $\widehat{\mathcal{O}_{C,p}} \cong k[[x,y]]/\left(y^2 - x^3\right)$;
- a *tacnode* (or singularity of type A_3) if $\widehat{\mathcal{O}_{C,p}}$ is isomorphic to $k[[x,y]]/\left(y\left(y - x^2\right)\right)$ (or to $k[[x,y]]/\left(y^2 - x^4\right)$ if $\operatorname{char}(k) \neq 2$).

When dealing with the deformation theory of a tacnode, we will often assume $\operatorname{char}(k) \neq 2$ for simplicity (note that the semiuniversal deformation space of a tacnode has dimension 3 if $\operatorname{char}(k) \neq 2$ and 4 if $\operatorname{char}(k) = 2$).

We use the notation $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$, where R is a k-discrete valuation ring with residue field k and fraction field K; we set 0, η and $\bar{\eta}$ to be, respectively, the closed point, the generic point and a geometric generic point of Δ. Given a flat and proper family $\pi : \mathcal{C} \to \Delta$, we denote by $\mathcal{C}_0$ the special fibre, by $\mathcal{C}_\eta$ the generic fibre and by $\mathcal{C}_{\bar{\eta}}$ a geometric generic fibre.

An *isotrivial specialisation* is a flat and proper family $\pi : \mathcal{C} \to \Delta$ of curves such that the restriction $\mathcal{C} \times_\Delta \Delta^* \to \Delta^*$ is trivial – that is, $\mathcal{C} \times_\Delta \Delta^* \cong C \times_k \operatorname{Spec} K$ for some curve C defined over k. In this case, we say that C *isotrivially specialises* to $\mathcal{C}_0$, and we write $C \rightsquigarrow \mathcal{C}_0$. This isotrivial specialisation is called nontrivial if $\mathcal{C}_0 \not\cong C$, or equivalently (compare [50, Prop. 2.6.10]), if $\mathcal{C} \not\cong C \times_k \Delta$. Similar definitions can be given for pointed curves, by requiring that the family $\pi : \mathcal{C} \to \Delta$ admit sections.

2.2. Notations for Mori theory

A proper morphism $f : X \to Y$ between two reduced algebraic spaces of finite type over k is called a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Given a reduced proper k-algebraic space X, we denote by $N^1(X) \cong \mathbb{Z}^{\rho_X}$ the (numerical) *Néron–Severi* group, and we set $N^1(X)_\mathbb{R} = N^1(X) \otimes_\mathbb{Z} \mathbb{R}$ (the real Néron–Severi vector space). Via the intersection product, the dual of $N^1(X)$ is naturally identified with the group $N_1(X)$ of 1-cycles up to numerical equivalence, and we set $N_1(X)_\mathbb{R} = N_1(X) \otimes_\mathbb{Z} \mathbb{R}$. Inside $N_1(X)_\mathbb{R}$ are the *effective cone of curves* $\operatorname{NE}(X)$, which is the convex cone consisting of all effective 1-cycles on X, and its closure $\overline{\operatorname{NE}}(X)$, the *Mori cone*. Given a contraction $\pi : X \to Y$ between reduced proper k-algebraic spaces, the π-*relative effective cone of curves* is the convex subcone $\operatorname{NE}(\pi)$ of $\operatorname{NE}(X)$ spanned by the integral curves that are contracted by π (that is, the integral curves C of X such that $\pi(C)$ is a closed point of Y), and its closure $\overline{\operatorname{NE}}(\pi) := \overline{\operatorname{NE}(\pi)} \subseteq \overline{\operatorname{NE}}(X)$ is called the π-*relative Mori cone*. We will use the following facts:

- If Y is projective, then $\overline{\operatorname{NE}}(\pi)$ is a face of $\overline{\operatorname{NE}}(X)$, and hence $\operatorname{NE}(\pi)$ is a face of $\operatorname{NE}(X)$ (the proof of [24, Prop. 1.14(a)] for $\operatorname{NE}(\pi)$ works also for $\overline{\operatorname{NE}}(\pi)$). Moreover, the class of an integral curve $[C]$ belongs to $\operatorname{NE}(\pi)$ if and only if $\pi_*([C]) = 0$.
- If X and Y are projective (which implies that also π is projective), then π is uniquely determined by $\operatorname{NE}(\pi)$ up to isomorphism (see [24, Prop. 1.14(b)]).
- If π is projective, then the *relative Kleiman ampleness criterion* holds: a Cartier divisor D on X is π-ample if and only if D is positive on $\overline{\operatorname{NE}}(\pi) \setminus \{0\}$ (see [42, Thm. 1.44]).

Given a projective k-variety X and a face F of $\mathrm{NE}(X)$, if there exists a (projective) contraction $\pi : X \to Y$ into a projective k-variety Y such that $\mathrm{NE}(\pi) = F$, then $\pi : X \to Y$ (which is unique by what we have already said) is called the *contraction* of the face F and will be denoted by $\pi_F : X \to X_F$. Note that not all the faces F of $\mathrm{NE}(X)$ can have an associated contraction; a necessary condition for that to happen is that the closure of F be equal to a face of $\overline{\mathrm{NE}}(X)$. Contraction of faces of the effective cone of curves can also be characterised as follows:

Lemma 2.1. *Let X be a projective k-variety and let F be a face of $\mathrm{NE}(X)$ for which there exists a contraction $\pi_F : X \to X_F$. If $f : X \to Y$ is a contraction onto a reduced proper (not necessarily projective!) k-algebraic space Y such that an integral curve $C \subset X$ is contracted by f if and only if $[C] \in F$, then there exists an isomorphism $X_F \cong Y$ under which $f = \pi_F$.*

Proof. By the assumption on f and the definition of the contraction π_F of F, it follows that an integral curve $C \subset X$ is contracted by f if and only if it is contracted by π_F. Since X is assumed to be projective, the morphisms f and π_F are projective contractions, which implies that their closed fibres are connected projective k-varieties. Using suitable hyperplane sections, we can connect any two closed points of a closed fibre of f (resp., π_F) by a chain of integral curves contained in the given fibre of f (resp., π_F). Hence, from what we have said for curves, we conclude that a closed subscheme of X is a fibre of f if and only if it is a fibre of π_F.

We can now apply the rigidity lemma of [24, Lemma 1.15] to conclude that f factors through π_F and π_F factors through f. This implies that there exists an isomorphism $Y \cong X_F$ under which $f = \pi_F$. $\qquad\square$

In Lemma 2.1, the assumption that a curve $C \subset X$ is contracted by f if and only if $[C] \in F$ cannot be replaced by the weaker condition that $\mathrm{NE}(f) = F$, as the following example shows:

Example 2.2. Consider a projective smooth complex 3-fold X with a K_X-negative extremal ray R such that the contraction of R, $\pi_R : X \to Y$, contracts a divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ to a singular point in Y. In this case, by [46, Thm. 3.3], the normal bundle of E is $\mathcal{O}(-1, -1)$, and the two rulings of E are numerically equivalent on X. Such a 3-fold does exist, by [46, Section 10, Example 3.44.2].

By Nakano's theorem, E can also be contracted analytically along one of its rulings by a holomorphic map $f : X \to Z$. The end result Z is a proper complex smooth algebraic space (or equivalently, a proper Moishezon manifold) and $\mathrm{NE}(f) = R$. The complex manifold Z is therefore nonprojective, and it can be seen as a small resolution of Y.

3. The moduli stacks of T-semistable and T^+-semistable curves

The aim of this section is to define the relevant moduli stacks of n-pointed curves, with which we will work throughout the paper.

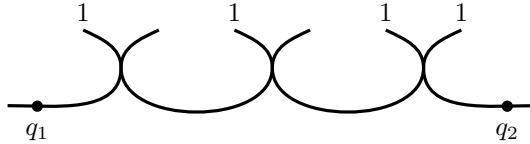*Figure 2.* An elliptic tail and an elliptic bridge.

Figure 3. An elliptic chain of length 4. The 1s indicate the genus of the irreducible components.

3.1. Special subcurves

In this subsection, we will introduce some special subcurves that will be used in the definition of our moduli stacks. The reader can safely skip this section at a first reading and come back to the relevant definitions when needed.

Definition 3.1 (Tails, bridges and chains [9, Def. 2.1 and 2.3, Lemma 2.13], see Figures 2 and 3)**.**

1. An *elliptic tail* is a 1-pointed irreducible curve (E,q) of arithmetic genus 1 (that is, E is either a smooth elliptic curve or a rational curve with one node or one cusp).

2. An *elliptic bridge* is a 2-pointed curve (E,q_1,q_2) of arithmetic genus 1 which either is irreducible or has two rational smooth components R_1 and R_2 that meet in either two nodes or one tacnode, and such that $q_i \in R_i$ for $i = 1,2$.

3. An *elliptic chain of length r* is a 2-pointed curve (E,q_1,q_2) which admits a finite, surjective morphism $\gamma : \bigcup_{i=1}^{r}(E_i,p_{2i-1},p_{2i}) \to (E,q_1,q_2)$ such that:
   (a) (E_i,p_{2i-1},p_{2i}) is an elliptic bridge for $i = 1,\ldots,r$;
   (b) γ induces an open embedding of $E_i \setminus \{p_{2i-1},p_{2i}\}$ into $E \setminus \{q_1,q_2\}$ for $i = 1,\ldots,r$;
   (c) $\gamma(p_{2i}) = \gamma(p_{2i+1})$ is a tacnode for $i = 1,\ldots,r-1$;
   (d) $\gamma(p_1) = q_1$ and $\gamma(p_{2r}) = q_2$.

Note that an elliptic chain of length r has arithmetic genus $2r-1$. An elliptic chain of length 1 is just an elliptic bridge.

Definition 3.2 (Attached elliptic tails and chains [9, Def. 2.4], see Figure 4)**.** Let $(C,\{p_i\}_{i=1}^{n})$ be an n-pointed curve of genus g. Let k,k_1,k_2 be equal to 1 or 3.

1. $(C,\{p_i\}_{i=1}^{n})$ has an A_k-*attached elliptic tail* if there exists a finite morphism $\gamma : (E,q) \to (C,\{p_i\}_{i=1}^{n})$ (called a gluing morphism) such that:
   (a) (E,q) is an elliptic tail;
   (b) γ induces an open embedding of $E - \{q\}$ into $C - \cup_{i=1}^{n}\{p_i\}$;
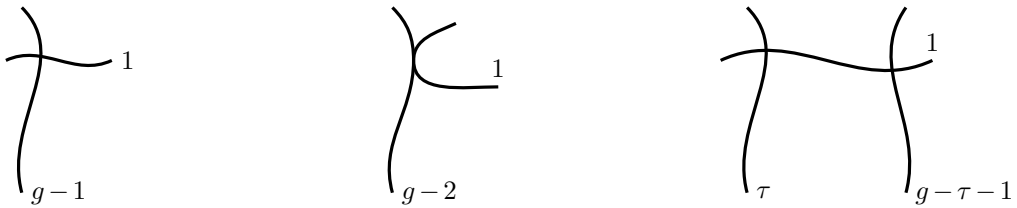   (c) $\gamma(q)$ is an A_k-singularity.

Figure 4. Three curves with, respectively, an A_1-attached elliptic tail, an A_3-attached elliptic tail and an A_1/A_1-attached elliptic bridge.

2. $(C,\{p_i\}_{i=1}^n)$ has an A_{k_1}/A_{k_2}-*attached elliptic chain* (of length r) if there exists a finite morphism $\gamma : (E,q_1,q_2) \to (C,\{p_i\}_{i=1}^n)$ (called a gluing morphism) such that:

(a) (E,q_1,q_2) is an elliptic chain (of length r);

(b) γ induces an open embedding of $E - \{q_1,q_2\}$ into $C - \cup_{i=1}^n \{p_i\}$;

(c) $\gamma(q_i)$ is an A_{k_i}-singularity or $k_i = 1$ and $\gamma(q_i)$ is a marked point (for $i = 1,2$).

An A_{k_1}/A_{k_2}-attached elliptic chain of length 1 is also called an A_{k_1}/A_{k_2}-*attached elliptic bridge*. An A_k/A_k-attached elliptic chain $\gamma : (E,q_1,q_2) \to (C,\{p_i\}_{i=1}^n)$ of length r such that $\gamma(q_1) = \gamma(q_2)$ is called *closed*. In this case, γ is surjective and $(g,n) = \left(2r - 1 + \frac{k+1}{2}, 0\right)$.

In analysing the automorphism group of the curves we will be dealing with, a central role is played by rosaries, as introduced in [34] (see also [9, Sec. 2.5]).

Definition 3.3 (Open and closed rosaries [34, Def. 6.1, 6.3], [9, Def. 2.26], see Figure 5)**.**

1. An *open rosary* of length r, or simply a *rosary* of length r, is a 2-pointed curve (R,q_1,q_2) which admits a finite, surjective morphism $\gamma : \bigcup_{i=1}^r (L_i,p_{2i-1},p_{2i}) \to (R,q_1,q_2)$ with:

(a) (L_i,p_{2i-1},p_{2i}) is 2-pointed smooth rational curve for $i = 1,\ldots,r$;

(b) γ induces an open embedding of $L_i \setminus \{p_{2i-1},p_{2i}\}$ into $R \setminus \{q_1,q_2\}$ for $i = 1,\ldots,r$;

(c) $a_i := \gamma(p_{2i}) = \gamma(p_{2i+1})$ is a tacnode for $i = 1,\ldots,r-1$;

(d) $\gamma(p_1) = q_1$ and $\gamma(p_{2r}) = q_2$.

2. A *closed rosary* of length r is a (0-pointed) curve R which admits a finite, surjective morphism $\gamma : \bigcup_{i=1}^r (L_i,p_{2i-1},p_{2i}) \to R$ such that:

(a) (L_i,p_{2i-1},p_{2i}) is 2-pointed smooth rational curve for $i = 1,\ldots,r$;

(b) γ induces an open embedding of $L_i \setminus \{p_{2i-1},p_{2i}\}$ into R for $i = 1,\ldots,r$;

(c) $a_i := \gamma(p_{2i}) = \gamma(p_{2i+1})$ for $i = 1,\ldots,r-1$, and $a_r := \gamma(p_1) = \gamma(p_{2r})$ are tacnodes.

Note that an open rosary (R,q_1,q_2) of length r has arithmetic genus $g(R) = r - 1$, while a closed rosary R of length r has arithmetic genus $g(R) = r + 1$.

An open rosary (R,q_1,q_2) of length r is such that $\omega_R(q_1 + q_2)$ is ample if (and only if) $r \geq 2$ (this is the reason why open rosaries of length 1 will not play any role in the sequel). An open rosary of length 2 is an elliptic bridge and is the unique elliptic bridge
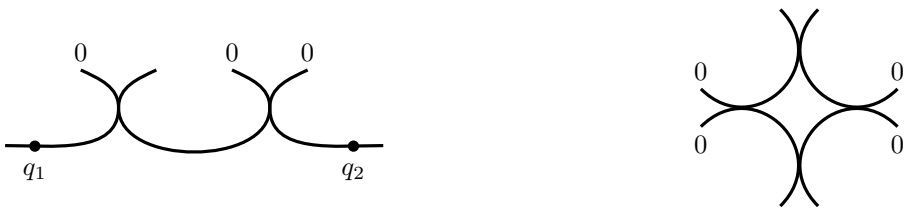
Figure 5. A rosary of length 3 and a closed rosary of length 4.

containing a tacnode; for this reason, we will also call it the *tacnodal elliptic bridge*. More generally, any open rosary of even length r can be regarded as an elliptic chain of length $r/2$ in which all the elliptic bridges are tacnodal.

Remark 3.4. Assume $\mathrm{char}(k) \neq 2$. Open rosaries and closed rosaries of even length share similar properties and can be described as follows, following [34, Prop. 6.5] (see also [9, Def. 2.20(2)] for open rosaries of length 2 that coincide with 7/10-atoms):[3]

(i) An open rosary (R, q_1, q_2) of length $r \geq 1$ can be obtained by gluing the disjoint union of r projective lines $\{L_i\}_{i=1}^r$ with homogeneous coordinate $[s_i, t_i]$ and the $r-1$ affine tacnodal curves $\mathrm{Spec}\, k[x_i, y_i]/\left(y_i^2 - x_i^4\right)$ via the gluing relations

$$x_i = \left(\frac{t_i}{s_i}, \frac{s_{i+1}}{t_{i+1}}\right) \in k\left[\frac{t_i}{s_i}\right] \times k\left[\frac{s_{i+1}}{t_{i+1}}\right], \tag{3.1}$$

$$y_i = \left(\left(\frac{t_i}{s_i}\right)^2, -\left(\frac{s_{i+1}}{t_{i+1}}\right)^2\right) \in k\left[\frac{t_i}{s_i}\right] \times k\left[\frac{s_{i+1}}{t_{i+1}}\right]. \tag{3.2}$$

Note that the marked points are equal to $q_1 = [0,1] \in L_1$ and $q_2 = [1,0] \in L_r$, while the tacnodes have coordinates (for every $1 \leq i \leq r-1$)

$$a_i = [1,0] \text{ on } L_i \quad \text{and} \quad a_i = [0,1] \text{ on } L_{i+1}.$$

The connected component of the automorphism group of (R, q_1, q_2) is equal to the multiplicative group $\mathbb{G}_\mathrm{m}$ which acts, in the coordinates already given, by

$$\lambda \cdot [s_i, t_i] = \left[\lambda^{(-1)^{i+1}} s_i, t_i\right], \qquad \lambda \cdot x_i = \lambda^{(-1)^i} x_i, \qquad \lambda \cdot y_i = \lambda^{2(-1)^i} y_i.$$

Note that the weights of the $\mathbb{G}_\mathrm{m}$-action on the tangent spaces at the marked points are

$$\mathrm{wt}_{\mathbb{G}_\mathrm{m}}\left(T_{q_1}(R)\right) = 1 \quad \text{and} \quad \mathrm{wt}_{\mathbb{G}_\mathrm{m}}\left(T_{q_2}(R)\right) = (-1)^r.$$

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[3]Closed rosaries of odd length have different properties: they depend on one modulus and they do not admit an infinite group of automorphisms. Since we will not need them, we refrain from giving an explicit description, and direct the interested reader to [34, Prop. 6.5].

(ii) A closed rosary R of even length $r \geq 1$ can be obtained by gluing the disjoint union of r projective lines $\{L_i\}_{i=1}^r$ with homogeneous coordinate $[s_i, t_i]$ and the r affine tacnodal curves $\operatorname{Spec} k[x_i, y_i]/\left(y_i^2 - x_i^4\right)$ via the gluing relations

$$x_i = \left(\frac{t_i}{s_i}, \frac{s_{i+1}}{t_{i+1}}\right) \in k\left[\frac{t_i}{s_i}\right] \times k\left[\frac{s_{i+1}}{t_{i+1}}\right], \tag{3.3}$$

$$y_i = \left(\left(\frac{t_i}{s_i}\right)^2, -\left(\frac{s_{i+1}}{t_{i+1}}\right)^2\right) \in k\left[\frac{t_i}{s_i}\right] \times k\left[\frac{s_{i+1}}{t_{i+1}}\right], \tag{3.4}$$

where we adopt the cyclic convention $L_{r+1} := L_1$, $x_{r+1} := x_1$ and $y_{r+1} := y_1$. Note that the tacnodes have coordinates (for every $1 \leq i \leq r$)

$$a_i = [1,0] \text{ on } L_i \quad \text{and} \quad a_i = [0,1] \text{ on } L_{i+1}.$$

The connected component of the automorphism group of R is equal to the multiplicative group $\mathbb{G}_m$ which acts, in the coordinates given, by

$$\lambda \cdot [s_i, t_i] = \left[\lambda^{(-1)^{i+1}} s_i, t_i\right], \qquad \lambda \cdot x_i = \lambda^{(-1)^i} x_i, \qquad \lambda \cdot y_i = \lambda^{2(-1)^i} y_i.$$

Note that this is well defined, since $(-1)^{r+1} = (-1)^1$ because r is even.

Similar to elliptic chains, open rosaries also can be attached in different ways inside a pointed curve. However, we will need to consider only nodal attachments, as we now define:

Definition 3.5 (Attached rosaries [34, Def. 6.3] and [9, Def. 2.26]). Let $(C, \{p_i\}_{i=1}^n)$ be an n-pointed curve. We say that $(C, \{p_i\}_{i=1}^n)$ has an A_1/A_1-*attached rosary* (of length r), or simply an *attached rosary*, if there exists a finite morphism $\gamma : (R, q_1, q_2) \to (C, \{p_i\}_{i=1}^n)$ (called a gluing morphism) such that:

(a) (R, q_1, q_2) is a rosary (of length r);

(b) γ induces an open embedding of $R - \{q_1, q_2\}$ into $C - \cup_{i=1}^n \{p_i\}$;

(c) $\gamma(r_i)$ is a node or a marked point (for any $i = 1, 2$).

Note that we could have an A_1/A_1-attached rosary $\gamma : (R, q_1, q_2) \to (C, \{p_i\}_{i=1}^n)$ of length r such that $\gamma(q_1) = \gamma(q_2)$: in this case we have $C = R$ and $(g, n) = (r, 0)$.

Next we want to define the type of a tacnode, of an A_{k_1}/A_{k_2}-attached elliptic chain, of an attached rosary and of a closed rosary, which will be a subset of the set $T_{g,n}$ (see definition (1.3)).

Definition 3.6 (Types of tacnodes, attached elliptic chains, attached and closed rosaries, see Figures 6). Let $(C, \{p_i\}_{i=1}^n)$ be a n-pointed curve such that C is Gorenstein and $\omega_C\left(\sum_{i=1}^n p_i\right)$ is ample.

(1) Let $p \in C$ be a tacnode. We say that p is of type:
   - $\operatorname{type}(p) := \{\text{irr}\} \subseteq T_{g,n}$ if the normalisation of C at p is connected;
   - $\operatorname{type}(p) := \{[\tau, I], [\tau + 1, I]\} \subseteq T_{g,n}$ if the normalisation of C at p has two connected components, one of which has arithmetic genus τ and marked points $\{p_i\}_{i \in I}$.

(2) Let $\gamma : (E, q_1, q_2) \to (C, \{p_i\}_{i=1}^n)$ be an A_{k_1}/A_{k_2}-attached elliptic chain of length $r \geq 1$ and with $k_1, k_2 = 1$ or 3. Set

$$\epsilon(k_1, k_2) = \begin{cases} 0 & \text{if } k_1 = k_2 = 1, \\ 1 & \text{if } (k_1, k_2) = (1,3) \text{ or } (3,1), \\ 2 & \text{if } k_1 = k_2 = 3. \end{cases}$$

We say that (E, q_1, q_2) is of type:

- type$(E, q_1, q_2) := \{[0, \{p_i\}], [1, \{p_i\}], \ldots, [2r - 1 + \epsilon(k_1, k_2), \{p_i\}]\} \subseteq T_{g,n}$ if either $\gamma(q_1) = p_i$ or $\gamma(q_2) = p_i$;

    type$(E, q_1, q_2) := \{\mathrm{irr}\} \subseteq T_{g,n}$ if $\gamma(q_1)$ and $\gamma(q_2)$ are singular points (either nodes or tacnodes) of C and $\overline{C \setminus \gamma(E)}$ is connected (which includes also the case of a closed A_{k_1}/A_{k_2}-attached elliptic chain, in which case $\overline{C \setminus \gamma(E)} = \emptyset$);

- type$(E, q_1, q_2) := \{[\tau, I], [\tau + 1, I], \ldots, [\tau + 2r - 1 + \epsilon(k_1, k_2), I]\} \subseteq T_{g,n}$ if $\gamma(q_1)$ and $\gamma(q_2)$ are singular points (either nodes or tacnodes) of C and $\overline{C \setminus \gamma(E)}$ consists of two connected components, one of which has arithmetic genus τ with marked points $\{p_i\}_{i \in I}$.

(3) Let $\gamma : (R, q_1, q_2) \to (C, \{p_i\}_{i=1}^n)$ be an attached rosary of length r. We say that (R, q_1, q_2) is of type:

- type$(R, q_1, q_2) := \{[0, \{p_i\}], [1, \{p_i\}], \ldots, [r - 1, \{p_i\}]\} \subseteq T_{g,n}$ if either $\gamma(q_1) = p_i$ or $\gamma(q_2) = p_i$;

- type$(R, q_1, q_2) := \{\mathrm{irr}\} \subseteq T_{g,n}$ if $\gamma(q_1)$ and $\gamma(q_2)$ are nodes of C and $\overline{C \setminus \gamma(R)}$ is connected (which includes also the case where $\overline{C \setminus \gamma(R)} = \emptyset$, which can happen only if $(g, n) = (r, 0)$ and $\gamma(q_1) = \gamma(q_2)$));

- type$(R, q_1, q_2) := \{[\tau, I], [\tau + 1, I], \ldots, [\tau + r - 1, I]\} \subseteq T_{g,n}$ if $\gamma(q_1)$ and $\gamma(q_2)$ are nodes of C and $\overline{C \setminus \gamma(R)}$ consists of two connected components, one of which has arithmetic genus τ with marked points $\{p_i\}_{i \in I}$.

(4) The type of a closed rosary R is set to be type$(R) := \{\mathrm{irr}\}$.

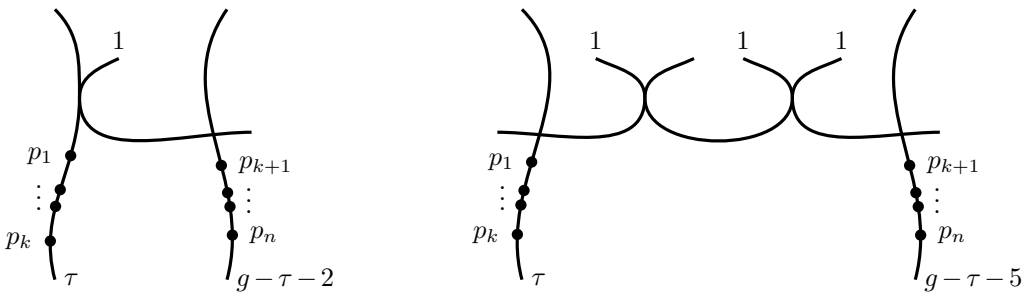One can check that these definitions are well posed.

Figure 6. A curve with an A_3/A_1-attached elliptic bridge of type $\{[\tau, I], [\tau + 1, I], [\tau + 2, I]\}$ and a curve with an A_1/A_1-attached elliptic chain of type $\{[\tau, I], [\tau + 1, I], \ldots, [\tau + 5, I]\}$, where $I = \{1, \ldots, k\}$.
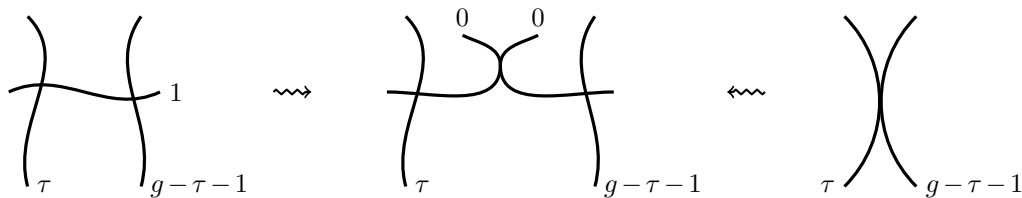
Figure 7. An A_1/A_1-attached elliptic bridge and a tacnode that isotrivially specialise to an A_1/A_1-attached rosary of length 2.

Remark 3.7. Note that the type $\gamma : (R, q_1, q_2) \to (C, \{p_i\}_{i=1}^n)$ of an attached rosary is the union of the types of all the tacnodes contained in $\gamma(R)$, and similarly for a closed rosary.

We conclude this subsection by describing some isotrivial specialisations that come from the $\mathbb{G}_m$-action on open rosaries and closed rosaries of even lengths (see Remark 3.4) and will play a crucial role in what follows. Given a (possibly n-pointed) curve C with a special subcurve R, we say that R specialises isotrivially to R' if there exists an isotrivial specialisation of C into a (possibly n-pointed) curve C' which is obtained by attaching R' to $\overline{C \setminus R}$.

Lemma 3.8 (see Figure 7 and 8)**.** *Assume that* $\mathrm{char}(k) \neq 2$*. We have the following isotrivial specialisations:*

(i) *an* A_1/A_1*-attached elliptic chain of length* $r \geq 1$ *isotrivially specialises to an attached rosary of length* $2r$*;*

(ii) *an* A_1/A_3*-attached elliptic chain of length* $r \geq 1$ *isotrivially specialises to an attached rosary of length* $2r + 1$*;*

(iii) *an* A_3/A_3*-attached elliptic chain of length* $r \geq 0$ *(which for* $r = 0$ *is a tacnode by convention) isotrivially specialises to an attached rosary of length* $2r + 2$*;*

(iv) *a closed* A_3/A_3*-attached elliptic chain of length* $r \geq 1$ *isotrivially specialises to a closed rosary of length* $2r$*.*

Moreover, each of these isotrivial specialisations preserves the type – that is, the type of the attached elliptic chain (or of the tacnode) is the same as the type of the closed or attached rosary to which it specialises.

Proof. See [34, Prop. 8.3, 8.6] □

3.2. The stacks of T-semistable and T^+-semistable curves

The aim of this subsection is to introduce the stacks of T-semistable and T^+-semistable n-pointed curves.

Let $\mathcal{U}_{g,n}$ (resp., $\mathcal{U}_{g,n}^{lci}$) be the algebraic stack of flat, proper families of n-pointed curves $(\pi : \mathcal{C} \to B, \{\sigma_i\}_{i=1}^n)$, where $\{\sigma_i\}_{i=1}^n$ are distinct sections that lie in the smooth locus of
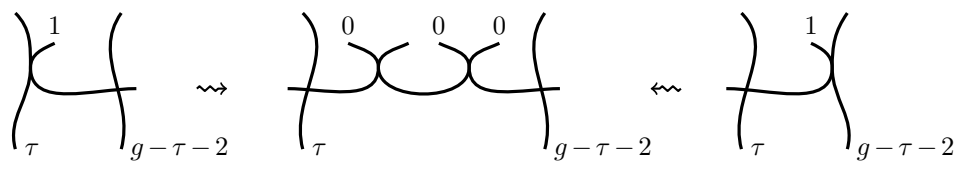
Figure 8. Two A_3/A_1-attached elliptic bridges that isotrivially specialise to an A_1/A_1-attached rosary of length 3.

π, such that the geometric fibres of π are Gorenstein (resp., locally complete intersection [lci]) curves of arithmetic genus g and the line bundle $\omega_{\mathcal{C}/B}\left(\sum \sigma_i\right)$ is relatively ample. Note that $\mathcal{U}_{g,n}$ is of finite type over k, since it parametrises log canonically polarised n-pointed curves, and $\mathcal{U}_{g,n}^{lci}$ is an open substack of $\mathcal{U}_{g,n}$ which is smooth and irreducible, since lci curves are unobstructed (see [50, Cor. 3.1.13(ii)] or [52, Tag 0DZX]) and smoothable (see [31, Ex. 29.0.1, Cor. 29.10]) and the condition of being lci is open (see [30, Def. 19.3.6, Cor. 19.3.8]). For any $1 \le k \le 3$, we denote by $\mathcal{U}_{g,n}(A_k) \subset \mathcal{U}_{g,n}^{lci}$ the open substack parametrising curves with at worst $A_1, \ldots, A_k$-singularities. Note that $\mathcal{U}_{g,n}(A_1) = \overline{\mathcal{M}}_{g,n}$.

Let us now recall the definition and the basic properties of the stack of pseudostable curves.

Definition 3.9.

(i) An n-pointed *pseudostable* curve of genus g is an n-pointed curve $(C, \{p_i\}_{i=1}^n)$ in $\mathcal{U}_{g,n}(A_2)$ that does not have A_1-attached elliptic tails.

(ii) The stack of pseudostable n-pointed curves of genus g is denoted by $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$.

The stack of pseudostable curves $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ coincides with the stack $\overline{\mathcal{M}}_{g,n}(9/11 - \varepsilon) = \overline{\mathcal{M}}_{g,n}(7/10 + \varepsilon)$ from [9, Def. 2.5 and Sec. 2.2]. We have decided to adopt this terminology because it is a natural extension of the case $n = 0$ originally considered by Schubert [49] (see also Hassett and Hyeon [33] and Hyeon and Morrison [37]).

Fact 3.10 ([9, Thm. 2.7])**.** *We have the following open embeddings:*

$$\overline{\mathcal{M}}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}(9/11) := \mathcal{U}_{g,n}(A_2) \hookleftarrow \overline{\mathcal{M}}_{g,n}(9/11 - \varepsilon) = \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}.$$

In particular, the stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is a smooth irreducible algebraic stack of finite type over k.

Note that for small values of (g,n), the stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is degenerate: if $g = 0$, then $\overline{\mathcal{M}}_{0,n}^{\mathrm{ps}} = \overline{\mathcal{M}}_{0,n}$, while for $(g,n) = (1,1)$ we have $\overline{\mathcal{M}}_{1,1}^{\mathrm{ps}} = \emptyset$.

The properties of the algebraic stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ and its relation with the stack $\overline{\mathcal{M}}_{g,n}$ of stable curves are collected in the following proposition:

Proposition 3.11. *Assume that $(g,n) \ne (1,1),(2,0)$.*

(i) *There is a surjective morphism $\widehat{\Upsilon} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ which, on geometric points, sends a stable n-pointed curve $(C, \{p_i\})$ into the pseudostable n-pointed curve $\widehat{\Upsilon}(C, \{p_i\})$ which is obtained by replacing every (A_1-attached) elliptic tail of $(C, \{p_i\})$ by a cusp.*

(ii) $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ *is a proper stack with finite inertia.*

(iii) *If* $\mathrm{char}(k) \neq 2$ *or* 3, *then* $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ *is a Deligne–Mumford (DM) stack.*

Proof. This is well known to experts, so we give only a sketch of the proof.

Part (i): the morphism of stacks $\widehat{\Upsilon}$ can be constructed (using the fact that $(g,n) \neq (2,0)$) as in [33, Thm. 1.1], which deals with $n = 0$ (note that the assumption $\mathrm{char}(k) = 0$ is not needed in that proof).

Part (ii): the properness of $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ can be deduced from the properness of $\overline{\mathcal{M}}_{g,n}$ and the existence of the surjective morphism $\widehat{\Upsilon} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, as in [26, Prop. 2.23].

In order to show that $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ has finite inertia and to prove part (iii), consider $(C, \{p_i\}) \in \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}(k)$, with $k = \overline{k}$, and denote by $\left(\widetilde{C}, \{q_j\}\right)$ the normalisation of C together with special points $\{q_j\}$ that are either the inverse images of the points $\{p_i\}$ or the inverse images of the singular points of C. It can be checked (using the fact that $(g,n) \neq (1,1)$) that

every component of $\widetilde{C}$ of genus 0 (resp., 1) has at least three (resp., one) special points. $\hspace{1em}$ (*)

Since the abstract automorphism group of $(C, \{p_i\})$ injects into the abstract automorphism group of $\left(\widetilde{C}, \{q_j\}\right)$, and this latter is finite by (*), we deduce that $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ has finite inertia.

Moreover, if $\mathrm{char}(k) \neq 2,3$, then the Lie algebra of the automorphism group scheme of $(C, \{p_i\})$, which is isomorphic to $H^0\left(C, T_C\left(-\sum p_i\right)\right)$, injects into $H^0\left(\widetilde{C}, T_{\widetilde{C}}\left(-\sum q_j\right)\right)$ by [51, Proposition 2.3], and this latter vector space is zero by (*), which shows part (iii). $\hspace{1em}\square$

Remark 3.12. If $\mathrm{char}(k)$ is equal to 2 or 3, [51, Example 1] shows that a high-genus cuspidal curve can have nonzero vector fields, hence $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is not a DM stack.

If $(g,n) = (2,0)$, then the stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ does not have finite inertia and is not separated (hence it is neither proper nor DM), as we now discuss.

Remark 3.13 (Pseudostable curves with $(g,n) = (2,0)$)**.** In the special case $(g,n) = (2,0)$, pseudostable curves are of these types: smooth curve $C_\emptyset$, integral curve C_n with one node and geometric genus 1, integral curve C_c with one cusp and geometric genus 1, rational curve with two nodes C_{nn}, rational curve C_{nc} with one node and one cusp, curve C_{nnn} made of two smooth rational curves meeting in three nodes, and rational curve C_{cc} with two cusps (see [20, Fig. 1] for a picture of all the strata of $\overline{\mathcal{M}}_2^{\mathrm{ps}}$). A pseudostable curve in $\overline{\mathcal{M}}_2^{\mathrm{ps}}$ is a closed point if and only if either it is nodal or it is the curve C_{cc} with two cusps. The pseudostable C_c and C_{nc} with only one cusp isotrivially specialise to C_{cc}, and hence they both contain C_{cc} in their closure (see [36, Thm. 1]). Moreover, the automorphism group of C_{cc} is equal to $\mathbb{G}_{\mathrm{m}}$.

Since $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is a proper (smooth and irreducible) stack with finite inertia, we can apply [38] to deduce the following result:

Corollary 3.14. *If* $(g,n) \neq (1,1),(2,0)$, *then there exist a proper normal irreducible algebraic space* $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *and a morphism* $\phi^{\mathrm{ps}} : \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *which is a coarse moduli space.*

Remark 3.15. If $(g,n) = (2,0)$, then it follows from [36, Thm. 1] that $\overline{\mathcal{M}}_2^{\mathrm{ps}}$ is the quotient stack of the GIT semistable locus in the Chow variety of tricanonical curves of genus 2. This implies that the associated GIT quotient, which we will denote by $\overline{\mathrm{M}}_2^{\mathrm{ps}}$, is a normal irreducible projective variety that comes equipped with a morphism $\phi^{\mathrm{ps}} : \overline{\mathcal{M}}_2^{\mathrm{ps}} \to \overline{\mathrm{M}}_2^{\mathrm{ps}}$ which is an adequate moduli space in the sense of Alper [5].

We now define the stack of T-semistable and T^+-semistable curves for $T \subseteq T_{g,n}$ (see definition (1.3)).

Definition 3.16. Fix a subset $T \subseteq T_{g,n}$.

(1) Let $\mathcal{U}_{g,n}(A_3(T))$ be the substack of $\mathcal{U}_{g,n}(A_3)$ parametrising n-pointed curves in $\mathcal{U}_{g,n}(A_3)$ such that all their tacnodes have type contained in T.

(2) In $\mathcal{U}_{g,n}$, define the following constructible loci:

$$\mathcal{B}^T := \{\text{curves containing an } A_1/A_1\text{-attached elliptic chain of type contained in } T\},$$

$$\mathcal{T}^{A_k} := \{\text{curves containing an } A_k\text{-attached elliptic tail}\}, \text{for } k = 1, 3.$$

(3) Consider the following substacks of $\mathcal{U}_{g,n}(A_3(T))$:

$$\overline{\mathcal{M}}_{g,n}^T := \mathcal{U}_{g,n}(A_3(T)) \setminus \left(\mathcal{T}^{A_1} \cup \mathcal{T}^{A_3} \right), \qquad \overline{\mathcal{M}}_{g,n}^{T,+} := \overline{\mathcal{M}}_{g,n}^T \setminus \mathcal{B}^T.$$

The n-pointed curves in $\overline{\mathcal{M}}_{g,n}^T$ are called T-*semistable*, while the n-pointed curves in $\overline{\mathcal{M}}_{g,n}^{T,+}$ are called T^+-*semistable*.

Remark 3.17. The two extreme cases of Definition 3.16 are easily described:

1. If $T = \emptyset$, then $\overline{\mathcal{M}}_{g,n}^T = \overline{\mathcal{M}}_{g,n}^{T,+} = \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$.

2. If $T = T_{g,n}$, then

$$\overline{\mathcal{M}}_{g,n}^T = \overline{\mathcal{M}}_{g,n}(7/10) \quad \text{and} \quad \overline{\mathcal{M}}_{g,n}^{T,+} = \overline{\mathcal{M}}_{g,n}(7/10 - \epsilon),$$

with the notation of [9, Def. 2.8].

We now want to prove that $\overline{\mathcal{M}}_{g,n}^T$ and $\overline{\mathcal{M}}_{g,n}^{T,+}$ are algebraic stacks of finite type over k. Let us first consider the stack $\mathcal{U}_{g,n}(A_3(T))$.

Lemma 3.18. *The locus* $\mathcal{U}_{g,n}(A_3(T))$ *is open in* $\mathcal{U}_{g,n}(A_3)$. *In particular,* $\mathcal{U}_{g,n}(A_3(T))$ *is an algebraic stack of finite type over* k.

Proof. We will show that $\mathcal{U}_{g,n}(A_3) \setminus \mathcal{U}_{g,n}(A_3(T))$ is closed. Since $\mathcal{U}_{g,n}(A_3(T))$ is clearly constructible in $\mathcal{U}_{g,n}(A_3)$, it suffices to show that $\mathcal{U}_{g,n}(A_3) \setminus \mathcal{U}_{g,n}(A_3(T))$ is closed under specialisations.

To this aim, consider a family $(\pi : \mathcal{C} \to \Delta, \{\sigma_i\}_{i=1}^n)$ of curves in $\mathcal{U}_{g,n}(A_3)$ (over the spectrum $\Delta = \mathrm{Spec}\, R$ of a discrete valuation ring) such that $\mathcal{C}_{\overline{\eta}}$ has a tacnode $p_{\overline{\eta}}$. It is enough to show that the central fibre $\mathcal{C}_0$ has a tacnode p_0 of the same type as $p_{\overline{\eta}}$. Up to passing to a finite base change of Δ, we can assume that there exists a section s of π such

that $s(\bar{\eta}) = p_{\bar{\eta}}$. We are now going to show that $p_0 := s(0)$ is a tacnode of $\mathcal{C}_0$ of the same type as $s(\bar{\eta})$.

Since the δ-invariant is upper semicontinuous and the tacnodes are the unique singular points of curves in $\mathcal{U}_{g,n}(A_3)$ that have δ-invariant equal to 2, we have that $s(0) \in C_0$ is also a tacnode. Hence the family $\pi : \mathcal{C} \to \Delta$ is equigeneric (even equisingular) along the section s; this implies that the partial normalisation of $\mathcal{C}$ along the section s produces a flat and proper family $\pi' : \mathcal{Y} \to \Delta$ of curves whose geometric fibres $\mathcal{Y}_0$ and $\mathcal{Y}_{\bar{\eta}}$ are the partial normalisations of, respectively, $\mathcal{C}_0$ and $\mathcal{C}_{\bar{\eta}}$ at, respectively, the points $s(0)$ and $s(\bar{\eta})$ (see [25, I.1.3.2] for $k = \mathbb{C}$ and [23, Thm. 4.1] for an arbitrary field $k = \bar{k}$; see also [9, Prop. 2.10] for an ad hoc proof in the case of outer A-singularities). In a flat and proper morphism with reduced geometric fibres, the number of connected components of the fibres stays constant and coincides with the number of connected components of the geometric fibres, so we see that there are two possibilities: either $\mathcal{Y}_0$ and $\mathcal{Y}_{\bar{\eta}}$ are both connected or they both have two connected components. In the first case, we have $\mathrm{type}(s(\bar{\eta})) = \mathrm{irr} = \mathrm{type}(s(0))$. In the second case, we have that $\mathcal{Y}$ is the disjoint union of two flat and proper families $\pi_1 : \mathcal{Y}_1 \to \Delta$ and $\pi_2 : \mathcal{Y}_2 \to \Delta$ with geometrically connected fibres of arithmetic genera equal to, respectively, $\tau \geq 0$ and $g - \tau - 1 \geq 0$. Moreover, since the sections σ_i of π do not meet the section s, they can be lifted uniquely to sections σ'_i of π' and hence there exists $I \subseteq [n]$ such that $\{\sigma'_i\}_{i \in I}$ are sections of π_1 and $\{\sigma'_i\}_{i \in I^c}$ are sections of π_2. This clearly implies that $\mathrm{type}(s(0)) = \{[\tau, I], [\tau + 1, I]\} = \mathrm{type}(s(\bar{\eta}))$. $\square$

This is the main result of this subsection.

Theorem 3.19. *Assume that* $(g,n) \neq (2,0)$ *and fix a subset* $T \subseteq T_{g,n}$. *The stack* $\overline{\mathcal{M}}_{g,n}^T$ *is algebraic, smooth, irreducible and of finite type over* k, *and we have open embeddings*

$$\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \overset{\iota_T}{\hookrightarrow} \overline{\mathcal{M}}_{g,n}^T \overset{\iota_T^+}{\hookleftarrow} \overline{\mathcal{M}}_{g,n}^{T,+} .$$

This result is false for $(g,n) = (2,0)$ (see [20, Rmk. 3.9]). If $T = T_{g,n}$, then using Remark 3.17, Theorem 3.19 reduces to [9, Thm. 2.7] for $\alpha_c = 7/10$ (but we have to assume that $(g,n) \neq (2,0)$).

Proof. Since the locus $\mathcal{T}^{A_1} \cup \mathcal{T}^{A_3}$ is closed in $\mathcal{U}_{g,n}(A_3)$ by [9, Prop. 2.15(1)], we have that $\overline{\mathcal{M}}_{g,n}^T$ is open in $\mathcal{U}_{g,n}(A_3(T))$, and hence it is open in $\mathcal{U}_{g,n}^{lci}$ by Lemma 3.18. Therefore, we conclude that $\overline{\mathcal{M}}_{g,n}^T$ is a smooth and irreducible algebraic stack of finite type over k because the same is true for $\mathcal{U}_{g,n}^{lci}$. Moreover, since $\mathcal{U}_{g,n}(A_2)$ is open in $\mathcal{U}_{g,n}(A_3(T))$, the inclusion

$$\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} = \mathcal{U}_{g,n}(A_2) \setminus \mathcal{T}^{A_1} = \mathcal{U}_{g,n}(A_2) \setminus (\mathcal{T}^{A_1} \cup \mathcal{T}^{A_3}) \subseteq \mathcal{U}_{g,n}(A_3(T)) \setminus (\mathcal{T}^{A_1} \cup \mathcal{T}^{A_3})$$

is an open embedding. It remains to prove that $\mathcal{B}^T$ is closed in $\overline{\mathcal{M}}_{g,n}^T$. Since $\mathcal{B}^T$ is constructible, it is enough to prove that $\mathcal{B}^T$ is closed under specialisation.

To this aim, consider a family $(\mathcal{C} \to \Delta, \{\sigma_i\})$ of curves in $\overline{\mathcal{M}}_{g,n}^T$ (over the spectrum $\Delta = \mathrm{Spec}\, R$ of a discrete valuation ring) such that $(\mathcal{C}_{\bar{\eta}}, \{\sigma_i(\bar{\eta})\})$ contains an A_1/A_1-attached elliptic chain (E, q_1, q_2) of length r (for some $r \geq 1$) and type contained in T. Since $(g,n) \neq (2,0)$, we have that q_1 is not attached to q_2. Therefore, following the proof of

[9, Prop. 2.15(2)][4] and using that $(\mathcal{C}_0, \{\sigma_i(0)\})$ is not contained in $\mathcal{T}^{A_1} \cup \mathcal{T}^{A_3}$, we get that $(\mathcal{C}_0, \{\sigma_i(0)\})$ contains an A_1/A_1-attached elliptic chain (E_0, t_1, t_2) of length $s \leq r$ which is contained in the limit of (E, q_1, q_2). From the explicit description of all such possible limits given in [9, Lemma 2.14], it follows that $\text{type}(E_0, t_1, t_2) \subseteq \text{type}(E, q_1, q_2)$, and hence that $\text{type}(E_0, t_1, t_2) \subseteq T$. Therefore the central fibre $(\mathcal{C}_0, \{\sigma_i(0)\})$ is contained in $\mathcal{B}^T$ and we are done. □

Remark 3.20. As observed after [9, Thm. 2.7], the stack $\mathcal{U}_{g,n}$ is the quotient stack of a locally closed smooth subscheme of an appropriate Hilbert scheme of some projective space $\mathbb{P}^N$ by PGL_{N+1}. Hence the same is true for all the stacks $\overline{\mathcal{M}}_{g,n}^T$ and $\overline{\mathcal{M}}_{g,n}^{T+}$, since they are open substacks of $\mathcal{U}_{g,n}$.

The containment relation among the different stacks $\overline{\mathcal{M}}_{g,n}^T$ is determined in Proposition 3.22, whose proof is given in [20]. Before that, we need the following definitions:

Definition 3.21.

(i) A subset $T \subseteq T_{g,n}$ is called *admissible* if $[1, \emptyset] \notin T$ and irr $\notin T$ if $g = 1$ and, for every $[\tau, I]$ in T, either $[\tau - 1, I]$ or $[\tau + 1, I]$ is in T.

(ii) Given a subset $T \subset T_{g,n}$, we obtain an admissible subset $T^{\text{adm}} \subseteq T$ as follows:
   - First we set $\widetilde{T} := T - \{[1, \emptyset]\}$ if $g \geq 2$ and $\widetilde{T} := T - \{[1, \emptyset], \text{irr}\}$ if $g \leq 1$.
   - Then we remove from $\widetilde{T}$ all the elements $[\tau, I] \in \widetilde{T}$ such that $[\tau - 1, I] \notin \widetilde{T}$ and $[\tau + 1, I] \notin \widetilde{T}$.

(iii) A subset $T \subset T_{g,n}$ is said to be *minimal* if $T = \{\text{irr}\}$ and $g \geq 2$ or $T = \{[\tau, I], [\tau + 1, I]\}$ (which then forces $g \geq 2$ or $g = 1$ and $n \geq 2$) for some element $[\tau, I] \neq [1, \emptyset]$ of $T_{g,n}$.

Observe that the empty set is admissible and is the unique admissible subset if $g = 0$ or $(g, n) = (1, 0)$. If $g \geq 2$ or $g = 1$ and $n \geq 2$, then the minimal subsets are exactly the minimal admissible nonempty subsets of $T_{g,n}$. Moreover, a subset $T \subset T_{g,n}$ is admissible if and only if it is the union of the minimal subsets contained in T.

Proposition 3.22. *[[20, Prop. 3.4]] Given two subsets $T, S \subseteq T_{g,n}$, we have*

$$\overline{\mathcal{M}}_{g,n}^T \subseteq \overline{\mathcal{M}}_{g,n}^S \subset \mathcal{U}_{g,n}(A_3) \iff T^{\text{adm}} \subseteq S^{\text{adm}}.$$

In particular, we have $\overline{\mathcal{M}}_{g,n}^T = \overline{\mathcal{M}}_{g,n}^S \iff T^{\text{adm}} = S^{\text{adm}}$, in which case we also have $\overline{\mathcal{M}}_{g,n}^{T,+} = \overline{\mathcal{M}}_{g,n}^{S,+}$.

On the other hand, it can be shown that if $S^{\text{adm}} \neq T^{\text{adm}}$, then $\overline{\mathcal{M}}_{g,n}^{T+}$ and $\overline{\mathcal{M}}_{g,n}^{S+}$ are incomparable.

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[4]The proof of this result is correct if we assume that q_1 is not attached to q_2 (which is always the case if $(g, n) \neq (2, 0)$), while it is not in general true if q_1 is attached to q_2 (which always happens for $(g, n) = (2, 0)$).

3.3. T-closed and T^+-closed curves

The aim of this subsection is to describe the closed points of the stacks of T-semistable and T^+-semistable curves.[5]

Let us start by describing the closed points of the stack of T-semistable curves.

Definition 3.23 (T-closed curves)**.** Assume that $(g,n) \neq (2,0)$. A curve $(C,\{p_i\})$ in $\overline{\mathcal{M}}_{g,n}^T(k)$ is *T-closed* if there is a decomposition $(C,\{p_i\}) = K \cup (E_1,q_1^1,q_2^1) \cup \cdots \cup (E_r,q_1^r,q_2^r)$, where the following are true:

(1) $(E_1,q_1^1,q_2^1), \ldots, (E_r,q_1^r,q_2^r)$ are attached rosaries of length 2, or equivalently A_1/A_1-attached tacnodal elliptic bridges, of type contained in T.

(2) K does not contain tacnodes nor A_1/A_1-attached elliptic bridges of type contained in T. In particular, every connected component of K is a pseudostable curve that does not contain any A_1/A_1-attached elliptic bridge of type contained in T.

Here K (which could be empty or disconnected) is regarded as a pointed curve with marked points given by the union of $\{p_i\}_{i=1}^n \cap K$ and $K \cap \left(\overline{C \setminus K}\right)$. We call K the *T-core* of $(C,\{p_i\}_{i=1}^n)$, and we call the decomposition $C = K \cup E_1 \cup \cdots \cup E_r$ the *T-canonical decomposition* of C.

Note that a $T_{g,n}$-closed curve is the same as a 7/10-closed curve as in [9, Def. 2.21].

Proposition 3.24. *Fix a subset $T \subset T_{g,n}$ and assume that $(g,n) \neq (2,0)$ and $\mathrm{char}(k) \neq 2$.*

(i) *A curve $(C,\{p_i\}) \in \overline{\mathcal{M}}_{g,n}^T(k)$ isotrivially specialises to the T-closed curve $(C,\{p_i\})^\star$, which is the stabilisation of the n-pointed curve obtained from $(C,\{p_i\})$ by replacing each tacnode (necessarily of type contained in T) by a rosary of length 2 and each A_1/A_1-attached elliptic bridge of type contained in T by a rosary of length 2.*

(ii) *A curve $(C,\{p_i\})$ is a closed point of $\overline{\mathcal{M}}_{g,n}^T$ if and only if $(C,\{p_i\})$ is T-closed.*

Note that if $T = T_{g,n}$, this proposition becomes [9, Thm. 2.22] for $\alpha_c = 7/10$ (or [34, Prop. 9.7] if, furthermore, $n = 0$).

This proposition is false for $(g,n) = (2,0)$ and $T = \{\mathrm{irr}\}$; see [20, Rmk. 3.8] for an explicit description of all the isotrivial specialisations and of the closed points of $\overline{\mathcal{M}}_2^{\mathrm{irr}}$.

Proof. Part (i) follows directly from Lemma 3.8.

We now prove part (ii). Part (i) implies that if $(C,\{p_i\}) \in \overline{\mathcal{M}}_{g,n}^T(k)$ is a closed point of $\overline{\mathcal{M}}_{g,n}^T$, then it must be T-closed. Conversely, let $(C,\{p_i\}) \in \overline{\mathcal{M}}_{g,n}^T(k)$ be T-closed and consider an isotrivial specialisation $(C,\{p_i\}) \rightsquigarrow (C',\{p_i'\})$ to a closed (and hence T-closed) point $(C',\{p_i'\})$ of $\overline{\mathcal{M}}_{g,n}^T$. Since the connected component $\mathrm{Aut}(C',\{p_i'\})^o$ of the automorphism group scheme $\mathrm{Aut}(C',\{p_i'\})$ is a torus (see [9, Prop. 2.6]), we can apply [10, Thm. 1.2] to deduce that $\overline{\mathcal{M}}_{g,n}^T$ is étale locally at $(C',\{p_i'\})$ isomorphic

---

[5]In analogy with GIT, we could call these closed points T-polystable (resp., T^+-polystable) curves. We decided not to use this terminology.

to $\left[W/\operatorname{Aut}(C', \{p_i'\})^o \right]$, for some affine variety W endowed with an action of the torus $\operatorname{Aut}(C', \{p_i'\})^o$. We can now apply [39, Thm. 1.4] to deduce that there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to \operatorname{Aut}(C', \{p_i'\})^o$ such that $\lim_{t \to 0} \lambda(t) \cdot [(C, \{p_i\})] = [(C', \{p_i'\})]$. In other words, $(C, \{p_i\})$ is in the basin of attraction of $(C', \{p_i'\})$ with respect to λ.

Now, mimicking the explicit analysis in [34, Prop. 9.7] of the basin of attraction of the one-parameter subgroups of $\operatorname{Aut}(C', \{p_i'\})^o$ (which come from the automorphism groups of the attached length 2 rosaries of $(C', \{p_i'\})$, as described in Remark 3.4), we deduce that $(C, \{p_i\}) \cong (C', \{p_i'\})$, and hence that $(C, \{p_i\})$ is a closed point of $\overline{\mathcal{M}}_{g,n}^T$. ☐

Remark 3.25. It is possible to give an alternative proof of Proposition 3.24(ii) (and also of Proposition 3.27(ii)) by proving directly, by arguing as in [9, Thm. 2.22], that any isotrivial specialisation of a T-closed (or T^+-closed) curve is actually trivial.

We now move to the description of the closed points of the stack of T^+-semistable curves.

Definition 3.26 (T^+-closed curves). We say that a curve $(C, \{p_i\})$ in $\overline{\mathcal{M}}_{g,n}^{T,+}$ is T^+-*closed* either if C is a closed rosary of even length r (which can happen only if $(g,n) = (r+1, 0)$ and $\operatorname{irr} \in T$) or if there is a decomposition $(C, \{p_i\}) = K \cup \left(R_1, q_1^1, q_2^1 \right) \cup \cdots \cup (R_r, q_1^r, q_2^r)$, where the following are true:

(1) $\left(R_1, q_1^1, q_2^1 \right), \ldots, (R_r, q_1^r, q_2^r)$ are attached rosaries of length 3 (automatically of type contained in T);

(2) K does not contain A_1/A_3-attached elliptic bridges of type contained in T nor closed A_3/A_3-attached elliptic chains of type contained in T.

Here K (which is allowed to be empty or disconnected) is regarded as a pointed curve with marked points given by the union of $\{p_i\}_{i=1}^n \cap K$ and of $K \cap (\overline{C \setminus K})$.

We call K the T^+-*core* of $(C, \{p_i\}_{i=1}^n)$, and we call the decomposition $C = K \cup R_1 \cup \cdots \cup R_r$ the T^+-*canonical decomposition* of C. Note that K does not contain any A_1/A_3-attached elliptic chain of type contained in T, because such a chain would necessarily contain an A_1/A_3-attached elliptic bridge of type contained in T, contradicting the assumptions on K.

Proposition 3.27. *Fix a subset $T \subset T_{g,n}$ and assume that $(g,n) \neq (2,0)$ and $\operatorname{char}(k) \neq 2$.*

(i) *A curve $(C, \{p_i\}) \in \overline{\mathcal{M}}_{g,n}^{T,+}(k)$ isotrivially specialises to the T^+-closed curve $(C, \{p_i\})^\dagger$, which is the stabilisation of the n-pointed curve obtained from $(C, \{p_i\})$ by replacing each A_1/A_3-attached elliptic bridge of type contained in T by a rosary of length 3 and each closed A_3/A_3-attached elliptic chain of length r and of type contained in T by a closed rosary of length $2r$.*

(ii) *A curve $(C, \{p_i\})$ is a closed point of $\overline{\mathcal{M}}_{g,n}^{T+}$ if and only if $(C, \{p_i\})$ is T^+-closed.*

Note that if $T = T_{g,n}$ and $n = 0$, then this proposition recovers [34, Prop. 9.9].

Proof. Part (i) follows directly from Lemma 3.8. Arguing as in the proof of Proposition 3.24(ii), part (ii) follows from (i) and the fact that a T^+-closed curve does not lie

on any basin of attraction of any other T^+-closed curve, a property that is checked as in [34, Prop. 9.9]. $\qquad\square$

3.4. Line bundles on the stacks $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, $\overline{\mathcal{M}}_{g,n}^{T}$ and $\overline{\mathcal{M}}_{g,n}^{T+}$

The aim of this section is to describe the Picard group of the three stacks $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, $\overline{\mathcal{M}}_{g,n}^{T}$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ that were introduced in Section 3.2.

From the deformation theory of lci singularities, it follows that the stack $\mathcal{U}_{g,n}^{lci}$ is smooth and the open substack $\overline{\mathcal{M}}_{g,n} = \mathcal{U}_{g,n}(A_1) \subset \mathcal{U}_{g,n}^{lci}$ has complement of codimension 2 (which can be proved as in [50, Prop. 3.1.5]). Hence, any line bundle on $\overline{\mathcal{M}}_{g,n}$ extends uniquely to a line bundle on $\mathcal{U}_{g,n}^{lci}$. In particular, we can define the Hodge line bundle λ, the canonical line bundle K, the cotangent line bundles ψ_i and the boundary line bundles δ_{irr} and $\delta_{i,I}$ (for every $[i,I] \in T_{g,n} - \{\mathrm{irr}\}$ such that $|I| \geq 2$ if $i = 0$) associated to the boundary divisors Δ_{irr} and $\Delta_{i,I}$ (for an explicit definition of the line bundles λ and K on the entire $\mathcal{U}_{g,n}$, see [6, Sec. 1.1].) Following [29], we will set $\delta_{0,\{i\}} = -\psi_i$ so that the divisors $\delta_{i,I}$ are defined for every $[i,I] \in T_{g,n}^*$. The total boundary line bundle, the total cotangent line bundle and the extended total boundary line bundle are defined as follows:

$$
\begin{cases}
\delta := \displaystyle\sum_{\substack{[i,I]\in T_{g,n}^*: \\ |I|\geq 2 \text{ if } i=0}} \delta_{i,I} + \delta_{\mathrm{irr}}, \\[1em]
\psi := \displaystyle\sum_{i=1}^{n} \psi_i, \\[1em]
\widehat{\delta} = \delta - \psi = \displaystyle\sum_{[i,I]\in T_{g,n}^*} \delta_{i,I} + \delta_{\mathrm{irr}}.
\end{cases}
$$

Fact 3.28.

(1) *The rational Picard group* $\mathrm{Pic}\left(\mathcal{U}_{g,n}^{lci}\right)_{\mathbb{Q}} = \mathrm{Pic}\left(\mathcal{U}_{g,n}^{lci}\right) \otimes \mathbb{Q}$ *of* $\mathcal{U}_{g,n}^{lci}$ *is generated by* λ, δ_{irr} *and* $\{\delta_{i,I}\}_{[i,I]\in T_{g,n}-\{\mathrm{irr}\}}$ *with no relations if* $g \geq 3$, *and with the following relations for* $g = 1,2$:
(i) *If* $g = 2$, *then*

$$10\lambda = \delta_{\mathrm{irr}} + 2\delta_1, \quad where \ \delta_1 := \sum_{[1,I]\in T_{2,n}^*} \delta_{1,I}.$$

(ii) *If* $g = 1$, *then*

$$
\begin{cases}
12\lambda = \delta_{\mathrm{irr}}, \\[0.5em]
\delta_{\mathrm{irr}} + 12 \displaystyle\sum_{\substack{[0,I]\in T_{1,n}^*: \\ p\in I}} \delta_{0,I} = 0 \quad for \ any \ 1 \leq p \leq n.
\end{cases}
$$

(2) *[Mumford's formula] The canonical line bundle K is equal to*

$$K = 13\lambda - 2\delta + \psi.$$

Indeed, the relations for $g = 0$ are also known, but we do not include them in this statement since we will not need them in this paper (see [15, Chap. XIX]).

Proof. Since $\mathcal{U}_{g,n}^{lci}$ is smooth, the Picard group of $\mathcal{U}_{g,n}^{lci}$ is equal to its divisor class group $\mathrm{Cl}(\mathcal{U}_{g,n}^{lci})$, and moreover, since $\overline{\mathcal{M}}_{g,n}$ is an open subset of $\mathcal{U}_{g,n}^{lci}$ whose complement has codimension 2, we have $\mathrm{Cl}(\mathcal{U}_{g,n}^{lci}) = \mathrm{Cl}(\overline{\mathcal{M}}_{g,n}) = \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$. Hence, both statements follow from the analogous statements for $\overline{\mathcal{M}}_{g,n}$: for (3.28), see [15, Chap. XIX] and the references therein if $\mathrm{char}(k) = 0$ and [47] if $\mathrm{char}(k) > 0$; for (3.28), see [15, Chap. XIII, Thm. 7.15] (whose proof works over an arbitrary field). $\qquad\square$

As a corollary of this fact, we can determine the rational Picard group of the stacks $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, $\overline{\mathcal{M}}_{g,n}^{T}$ and $\overline{\mathcal{M}}_{g,n}^{T+}$:

Corollary 3.29. *We have*

$$\begin{cases} \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} = \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{T}\right)_{\mathbb{Q}} = \dfrac{\mathrm{Pic}\left(\mathcal{U}_{g,n}^{lci}\right)_{\mathbb{Q}}}{(\delta_{1,\emptyset})}, \\[3mm] \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{T+}\right)_{\mathbb{Q}} = \dfrac{\mathrm{Pic}\left(\mathcal{U}_{g,n}^{lci}\right)_{\mathbb{Q}}}{(\delta_{1,\{i\}} : \{[0,\{i\}],[1,\{i\}] \subseteq T\})}. \end{cases}$$

Proof. Since $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is an open substack of the smooth stack $\mathcal{U}_{g,n}^{lci}$, its rational Picard group coincides with its rational divisor class group and it is a quotient of $\mathrm{Cl}(\mathcal{U}_{g,n}^{lci})_{\mathbb{Q}}$ by the classes of the irreducible divisors in $\mathcal{U}_{g,n}^{lci} \setminus \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, namely $\delta_{1,\emptyset}$. The argument for $\overline{\mathcal{M}}_{g,n}^{T}$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ is similar using the fact that the unique divisor in $\mathcal{U}_{g,n}^{lci} \setminus \overline{\mathcal{M}}_{g,n}^{T}$ is again $\Delta_{1,\emptyset}$, while the irreducible divisors in $\mathcal{U}_{g,n}^{lci} \setminus \overline{\mathcal{M}}_{g,n}^{T+}$ are $\Delta_{1,\emptyset}$ and $\{\Delta_{1,\{i\}} : \{[0,\{i\}],[1,\{i\}]\} \subseteq T\}$. $\qquad\square$

From now on, we will denote the restriction of a line bundle on $\mathcal{U}_{g,n}^{lci}$ to one of the open substacks $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, $\overline{\mathcal{M}}_{g,n}^{T}$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ with the same symbol.

Remark 3.30. Recently, Fringuelli and the third author proved in [27] that if $\mathrm{char}(k) \neq 2$, then $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ is generated by the tautological line bundles subject to the relations of Fact 3.28(3.28) (if $g \geq 1$). Hence Fact 3.28(3.28) and Corollary 3.29 hold true for the integral Picard group if $\mathrm{char}(k) \neq 2$.

4. Existence of good moduli spaces

In this section we want to prove that the moduli stacks of T-semistable and T^{+}-semistable curves admit a good moduli space in the sense of Alper [4].

From now on, we will assume that the characteristic is big enough, as specified in the following:

Definition 4.1 (Characteristic big enough with respect to T or (g,n))**.** Given $T \subseteq T_{g,n}$, we will say that the base field k has characteristic *big enough with respect to T*, and we will write $\mathrm{char}(k) \gg T$, either if $\mathrm{char}(k) = 0$ or if the characteristic is positive and does not divide the order of the étale group scheme of connected components of the automorphism

group schemes of every k-point of $\overline{\mathcal{M}}_{g,n}^T$. Given a hyperbolic pair (g,n), we will say that the base field k has characteristic *big enough with respect to* (g,n), and we will write $\mathrm{char}(k) \gg (g,n)$, if $\mathrm{char}(k) \gg T_{g',n'}$ for any hyperbolic pair (g',n') such that $g' \le g$ and $n' \le n + (g - g')$.

The relevance of the first condition, $\mathrm{char}(k) \gg T$ for the moduli stack $\overline{\mathcal{M}}_{g,n}^T$, is explained in the following lemma, and the definition of $\mathrm{char}(k) \gg (g,n)$ is dictated by the induction used in the proof of Theorem 4.4:

Lemma 4.2. *Given $T \subseteq T_{g,n}$, the automorphism group scheme of every k-point $\overline{\mathcal{M}}_{g,n}^T$ is linearly reductive if and only if $\mathrm{char}(k) \gg T$.*

Proof. The automorphism group scheme $\underline{\mathrm{Aut}}(C, \{p_i\})$ of every k-point $(C, \{p_i\})$ of $\overline{\mathcal{M}}_{g,n}^T$ is an extension of the étale group scheme $\pi_0\left(\underline{\mathrm{Aut}}(C, \{p_i\})\right)$ of its connected components by the connected component $\underline{\mathrm{Aut}}(C, \{p_i\})^o$ containing the identity, which is a torus by [9, Prop. 2.6]. Hence $\underline{\mathrm{Aut}}(C, \{p_i\})$ is linearly reductive if and only if $\mathrm{char}(k)$ does not divide the order of the étale group scheme $\pi_0\left(\underline{\mathrm{Aut}}(C, \{p_i\})\right)$ (see [1, §2]) – that is, if and only if $\mathrm{char}(k) \gg T$. $\qquad\square$

Remark 4.3. For any $T \subseteq T_{g,n}$, there exists a constant $C(T)$ such that if $\mathrm{char}(k) \ge C(T)$, then $\mathrm{char}(k) \gg T$. This follows from the fact that since $\overline{\mathcal{M}}_{g,n}^T$ is of finite type over k, the order of the finite group schemes of connected components of k-points of $\overline{\mathcal{M}}_{g,n}^T$ is bounded from above. Similarly, for any hyperbolic pair (g,n) there exists a constant $C(g,n)$ such that if $\mathrm{char}(k) \ge C(g,n)$, then $\mathrm{char}(k) \gg (g,n)$.

It would be interesting to find upper bounds for $C(T)$ and for $C(g,n)$ (for the analogous problem for $\overline{\mathcal{M}}_g$, see [53]).

Theorem 4.4. *Let $(g,n) \ne (2,0)$ and fix a subset $T \subseteq T_{g,n}$. Assume that $\mathrm{char}(k) \gg (g,n)$ as in Definition 4.1. The algebraic stacks $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, $\overline{\mathcal{M}}_{g,n}^T$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ admit good moduli spaces $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$, $\overline{\mathrm{M}}_{g,n}^T$ and $\overline{\mathrm{M}}_{g,n}^{T+}$, respectively, which are normal proper irreducible algebraic spaces over k. Moreover, there exists a commutative diagram*

$$
\begin{array}{ccccc}
\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} & \overset{\iota_T}{\hookrightarrow} & \overline{\mathcal{M}}_{g,n}^T & \overset{\iota_T^+}{\longleftarrow} & \overline{\mathcal{M}}_{g,n}^{T+} \\
\downarrow{\scriptstyle\phi^{\mathrm{ps}}} & & \downarrow{\scriptstyle\phi^T} & & \downarrow{\scriptstyle\phi^{T+}} \\
\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} & \overset{f_T}{\longrightarrow} & \overline{\mathrm{M}}_{g,n}^T & \overset{f_T^+}{\longleftarrow} & \overline{\mathrm{M}}_{g,n}^{T+}
\end{array}
\tag{4.1}
$$

where the vertical maps are the natural morphisms to the good moduli spaces and the bottom horizontal morphisms f_T and f_T^+ are proper morphisms.

By Remark 3.17, the two extremal cases of this theorem are either trivial or already known at least in characteristic 0: if $T^{\mathrm{adm}} = \emptyset$ (which is always the case for $g = 0$ or $(g,n) = (1,1)$), then the theorem is trivially true, since $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} = \overline{\mathcal{M}}_{g,n}^T = \overline{\mathcal{M}}_{g,n}^{T+}$; if $T^{\mathrm{adm}} =$

$T_{g,n}^{\mathrm{adm}}$ and $\mathrm{char}(k) = 0$, then the theorem reduces to [8, Thm. 1.1] for $\alpha_c = 7/10$ (but one has to exclude the case $(g,n) = (2,0)$).

Remark 4.5. Theorem 4.4 degenerates (but is still true) in the cases $(g,n) = (1,1)$ and $(g,n) = (1,2)$, while it is false for $(g,n) = (2,0)$ and $T^{\mathrm{adm}} \neq \emptyset$ (which implies that $T^{\mathrm{adm}} = \{\mathrm{irr}\}$), as we now discuss.

(1) If $(g,n) = (1,1)$, then $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} = \overline{\mathcal{M}}_{g,n}^T = \overline{\mathcal{M}}_{g,n}^{T+} = \emptyset$ for every T.

(2) If $(g,n) = (1,2)$ and $T^{\mathrm{adm}} \neq \emptyset$ (in which case it must be true that $T^{\mathrm{adm}} = \{[0,\{1\}],[1,\{1\}]\}$), then all the curves in $\overline{\mathcal{M}}_{1,2}^T$ isotrivially specialise to the tacnodal elliptic bridge, so that $\overline{\mathrm{M}}_{1,2}^T$ is equal to a point. On the other hand, the stack $\overline{\mathcal{M}}_{1,2}^{T+}$ (and hence also its good moduli space $\overline{\mathrm{M}}_{1,2}^{T+}$) is empty.

(3) If $(g,n) = (2,0)$ and $T^{\mathrm{adm}} = \{\mathrm{irr}\}$, then we do not know if the good moduli space for $\overline{\mathcal{M}}_2^T = \overline{\mathcal{M}}_2^{\mathrm{irr}}$ exists, but certainly if it exists it will not be separated [20]. On the other hand, the stack $\overline{\mathcal{M}}_2^{T+} = \overline{\mathcal{M}}_2^{\mathrm{irr}+}$ is not well defined, since it is not an open substack of $\overline{\mathcal{M}}_2^{\mathrm{irr}}$ (but only locally closed) [20, Rmk. 3.9].

Following the strategy of [8], there are two key ingredients in the proof. The first is the following:

Proposition 4.6. *Assume $(g,n) \neq (2,0)$ and $\mathrm{char}(k) \gg T$ and fix a subset $T \subseteq T_{g,n}$. Then the open embeddings*

$$\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \overset{\iota_T}{\hookrightarrow} \overline{\mathcal{M}}_{g,n}^T \overset{\iota_T^+}{\hookleftarrow} \overline{\mathcal{M}}_{g,n}^{T,+}$$

arise from local VGIT with respect to the line bundle $\delta - \psi$ on $\overline{\mathcal{M}}_{g,n}^T$.

We refer to [9, Def. 3.14] for the definition of when two open substacks of a given algebraic stack $\mathcal{X}$ arise from local VGIT at some (or any) closed point $x \in \mathcal{X}(k)$ with respect to a line bundle $\mathcal{L}$ on $\mathcal{X}$.

Proof. The proof of [9, Thm. 3.17] (or its expanded version in [13, Thm. 3.11]) carries through, as we now briefly indicate.

Let $(C,\{p_i\})$ be a closed point of $\overline{\mathcal{M}}_{g,n}^T$ – that is, $(C,\{p_i\})$ is a T-closed curve by Proposition 3.24(ii). As explained in Remark 3.4, for every A_1/A_1-attached tacnodal elliptic bridge (E_i,q_i^1,q_i^2) of $(C,\{p_i\})$, we have a one-parameter subgroup $\rho_i : \mathrm{Aut}\left((E_i,q_i^1,q_i^2)\right) \cong \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathrm{Aut}(C,\{p_i\})$, and these one-parameter subgroups freely generate the connected component $\mathrm{Aut}(C,\{p_i\})^o$ of $\mathrm{Aut}(C,\{p_i\})$ containing the identity. Arguing as in [9, Prop. 3.25], the character $\chi_{\delta-\psi}$ of $\mathrm{Aut}(C,\{p_i\})^o$ induced by the line bundle $\delta - \psi$ is equal to a positive power of the diagonal character

$$\chi_* : \mathrm{Aut}(C,\{p_i\})^o \cong \prod_{i=1}^r \mathrm{Aut}\left((E_i,q_i^1,q_i^2)\right) \longrightarrow \mathbb{G}_{\mathrm{m}},$$

$$(t_1,\ldots,t_r) \mapsto t_1 \cdots t_r.$$

We can choose coordinates on the first-order deformation space $T^1 := T^1((C, p_i))$ of $(C, \{p_i\})$ as in [13, Prop. 3.22], in such a way that the action of $\mathrm{Aut}(C, \{p_i\})^o$ is diagonal. Arguing as in [9, Prop. 3.26] (see also [13, Prop. 3.29] for more details), the VGIT ideals $I^+_{\chi_*}$ and $I^-_{\chi_*}$ of the action of $\mathrm{Aut}(C, \{p_i\})^o$ on $A := k[T^1]$ with respect to the character χ_* are such that $V(I^+_{\chi_*})$ is the locus of deformations of $(C, \{p_i\})$ that keep the tacnode of some (E_i, q_i^1, q_i^2) and smooth out the attaching nodes, while $V(I^-_{\chi_*})$ is the locus of deformations of $(C, \{p_i\})$ that smoothe out the tacnode of some (E_i, q_i^1, q_i^2) and keep the attaching nodes. Note that given an A_1/A_1-attached tacnodal elliptic bridge of type $S \subset T$, if we smooth out the attaching nodes we are left with a tacnode of type S, and if we smooth out the tacnode we are left with an elliptic bridge of type S. Therefore, the images $I^+_{\chi_*}\widehat{A}$ and $I^-_{\chi_*}\widehat{A}$ of the VGIT ideals in the completion $\widehat{A} := k[[T^1]]$ are equal to the ideals induced by, respectively, the reduced closed substacks $\overline{\mathcal{M}}^T_{g,n} \setminus \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ and $\overline{\mathcal{M}}^T_{g,n} \setminus \overline{\mathcal{M}}^{T,+}_{g,n}$ of $\overline{\mathcal{M}}^T_{g,n}$. By [9, Prop. 3.15], this is enough to conclude that the open embeddings

$$\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n} \overset{\iota_T}{\hookrightarrow} \overline{\mathcal{M}}^T_{g,n} \overset{\iota^+_T}{\hookleftarrow} \overline{\mathcal{M}}^{T,+}_{g,n}$$

arise from local VGIT with respect to the line bundle $\delta - \psi$ on $\overline{\mathcal{M}}^T_{g,n}$. $\qquad\square$

The second key point is the proof that the complements of $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ and of $\overline{\mathcal{M}}^{T,+}_{g,n}$ in $\overline{\mathcal{M}}^T_{g,n}$ admit good moduli spaces. Let us introduce a notation for these complements:

Definition 4.7. Consider the following closed substacks (with reduced structure) in $\overline{\mathcal{M}}^T_{g,n}$:

$$\mathcal{Z}^-_T = \overline{\mathcal{M}}^T_{g,n} \setminus \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}, \quad \text{and for } (g,n) \neq (2,0), \quad \mathcal{Z}^+_T = \overline{\mathcal{M}}^T_{g,n} \setminus \overline{\mathcal{M}}^{T,+}_{g,n}.$$

Observe that these loci have the following explicit description:

$$\mathcal{Z}^-_T = \left\{ \text{curves in } \overline{\mathcal{M}}^T_{g,n} \text{ with at least one tacnode (of type contained in } T) \right\},$$

$$\mathcal{Z}^+_T = \Big\{ \text{curves in } \overline{\mathcal{M}}^T_{g,n} \text{ with at least one } A_1/A_1\text{-attached elliptic chain} \qquad (4.2)$$

$$\text{of type contained in } T \Big\}. \qquad (4.3)$$

We first focus on the existence of a good moduli space for the stack $\mathcal{Z}^-_T$.

Proposition 4.8. *Fix $T \subseteq T_{g,n}$ and assume that $\mathrm{char}(k) \gg T$. If $\overline{\mathcal{M}}^{T'}_{g',n'}$ admits a proper good moduli space for all $T' \subseteq T_{g',n'}$ with either $g' < g$ and $1 \leq n' \leq n+1$ or $(g',n') = (g-2, n+2)$, then $\mathcal{Z}^-_T \subset \overline{\mathcal{M}}^T_{g,n}$ admits a proper good moduli space.*

Note that $\mathcal{Z}^-_T$ coincides with the stack $\overline{\mathcal{S}}_{g,n}(7/10)$ of [8, Section 4] in the case where $T^{\mathrm{adm}} = T^{\mathrm{adm}}_{g,n}$. Hence, this proposition generalises [8, Prop. 4.10] for $\alpha_c = 7/10$. At the other extreme, if $T^{\mathrm{adm}} = \emptyset$, then $\mathcal{Z}^-_T = \emptyset$ by Remark 3.17, and the result is trivial.

Moreover, if $(g,n) = (1,2)$ and $T^{\mathrm{adm}} \neq \emptyset$, then $\mathcal{Z}_T^- = \overline{\mathcal{S}}_{1,2}(7/10) \cong B\mathbb{G}_m$ because it consists of one point, namely the tacnodal elliptic bridge, which has automorphism group $\mathbb{G}_m$ (see [8, Lemma 4.3]) and the good moduli space is just a point.

The strategy for the proof of Proposition 4.8 is similar to the one of [8, Prop. 4.10] and consists in finding a finite cover of $\mathcal{Z}_T^-$ which is a stacky projective bundle over suitable stacks $\overline{\mathcal{M}}_{g',n'}^{T'}$ (as in the statement of Proposition 4.8) and then concluding by applying the criterion contained in the following proposition, which generalises [8, Prop. 1.4] from $\mathrm{char}(k) = 0$ to arbitrary characteristic:

Proposition 4.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks of finite type over an algebraically closed field k (of arbitrary characteristic). Suppose that the following are true:*

  (i) *the morphism $f : \mathcal{X} \to \mathcal{Y}$ is finite and surjective;*

 (ii) *there exists a good moduli space with $\mathcal{X} \to X$ with X separated;*

(iii) *the algebraic stack $\mathcal{Y}$ is a* global quotient stack – *that is, it is isomorphic to $[Z/G]$ for an algebraic space Z of finite type over k and a reductive algebraic k-group G, and it admits local quotient presentations (which implies that the stabilisers of its closed k-points are linearly reductive).*

Then there is a good moduli space $\mathcal{Y} \to Y$ with Y separated. Moreover, if X is proper, so is Y.

Proof. The proof of [8, Prop. 1.4] works verbatim, provided that we replace [8, Lemma 3.6] with Lemma 4.10. $\qquad\qquad\square$

Lemma 4.10 (Chevalley theorem for stacks). *Consider a commutative diagram*

$$\mathcal{X} \to \mathcal{Y} \to X$$

of algebraic stacks of finite type over an algebraically closed field k (of arbitrary characteristic), where X is an algebraic space. Suppose that the following are true:

  (i) *the morphism $\mathcal{X} \to \mathcal{Y}$ is finite and surjective;*

 (ii) *the morphism $\mathcal{X} \to X$ is cohomologically affine;*

(iii) *the algebraic stack $\mathcal{Y}$ is a global quotient stack such that the stabilisers of its closed k-points are linearly reductive.*

Then $\mathcal{Y} \to X$ is cohomologically affine.

Proof. The first part of the proof follows [8, Lemma 3.6]. Write $\mathcal{Y} = [V/G]$ for an algebraic space V of finite type over k and a reductive algebraic k-group G. Since $\mathcal{X} \to \mathcal{Y}$ is affine, $\mathcal{X}$ is the quotient stack $\mathcal{X} = [U/G]$, where $U = V \times_{\mathcal{Y}} \mathcal{X}$. Since $U \to \mathcal{X}$ is affine and $\mathcal{X} \to X$ is cohomologically affine, the morphism $U \to X$ is affine by Serre's criterion. The morphism $U \to V$ is finite and surjective, and therefore by Chevalley's theorem we can conclude that $p : V \to X$ is affine.

Since the affine morphism $p : V \to X$ is G-invariant and G is reductive, we can factor p as

$$p : V \to [V/G] \xrightarrow{\phi} V/G := \mathrm{Spec}_{\mathcal{O}_X} \, p_*(\mathcal{O}_V)^G \to X.$$

Since the morphism $V/G \to X$ is affine (and hence cohomologically affine), it is enough to show that ϕ is cohomologically affine (and indeed, we will show that it is a good moduli space).

Let v be a k-point of V with a closed G-orbit – that is, a closed k-point of $\mathcal{Y} = [V/G]$. Luna's slice theorem (in the generalised form [10, Thm. 2.1]) implies that we can find a G_v-invariant locally closed algebraic subspace $W_v \subset V$, containing v and affine over X, such that the morphism $f_v : W_v/G_v \to V/G$ is étale and the diagram

$$
\begin{array}{ccc}
[W_v/G_v] & \longrightarrow & [V/G] \\
\downarrow{\scriptstyle \phi_v} & & \downarrow{\scriptstyle \phi} \\
W_v/G_v & \xrightarrow{f_v} & V/G
\end{array}
$$

is Cartesian. Now, since G_v is linearly reductive, the morphism ϕ_v is a good moduli space by [4, Thm. 13.2]. Iterating this argument for all k-points of V with a closed G-orbit and using the quasi-compactness of V/G, we obtain an étale cover $f : Z \to V/G$ such that the pullback of ϕ via f is a good moduli space. This implies also that ϕ is a good moduli space by [4, Prop. 4.7(ii)], and we are done. □

Remark 4.11.

(i) Assumption (iii) in Proposition 4.9 is satisfied for quotient stacks of the form $[U/G]$, where U is a normal and separated scheme of finite type over k and G is a smooth linear algebraic k-group, having the property that the stabilisers of the closed k-points are linearly reductive (see [8, Prop. 2.3] and the references therein).

(ii) If $\mathrm{char}(k) = 0$, then the condition of the stabilisers in Lemma 4.10 can be removed (indeed, it follows from the first two assumptions on the lemma), as in [8, Lemma 3.6]. However, if $\mathrm{char}(k) = p > 0$, then the condition cannot be dropped, as the following example (suggested to us by Maksym Fedorchuk) shows:

$$\mathcal{X} = \mathrm{Spec}\, k \to \mathcal{Y} = [\mathrm{Spec}\, k/(\mathbb{Z}/p\mathbb{Z})] \to X = \mathrm{Spec}\, k.$$

Now, before entering into the proof of Proposition 4.8, we will need to review some constructions from [8, Sec. 4.2], adapted to our setting and notation.

The *sprouting* stack $\mathrm{Sprout}_{g,n}(A_3)$ is the algebraic stack (see [8, Def. 4.6]) consisting of flat and proper families of curves $(\mathcal{C} \to S, \{\sigma_i\}_{i=1}^{n+1})$ with $n+1$-sections σ_i such that

- the family $(\mathcal{C} \to S, \{\sigma_i\}_{i=1}^{n})$ is a S-point of $\mathcal{U}_{g,n}(A_3)$ and
- $\mathcal{C}$ has a tacnodal singularity along σ_{n+1}.

Note that the type of the tacnode remains constant along σ_{n+1} (see the proof of Lemma 3.18), so that $\mathrm{Sprout}_{g,n}(A_3)$ is the disjoint union of closed and open substacks where the type of σ_{n+1} is fixed. We will denote by $\mathrm{Sprout}_{g,n}(A_3)^{\mathrm{irr}}$

(resp., $\mathrm{Sprout}_{g,n}(A_3)^{0,\{j\}}$; resp., $\mathrm{Sprout}_{g,n}(A_3)^{h,M}$) the closed and open substack of $\mathrm{Sprout}_{g,n}(A_3)$ where the tacnodal section σ_{n+1} has type $\{\mathrm{irr}\}$ (resp., $\{[0,\{j\}],[1,\{j\}]\}$; resp., $\{[h,M],[h+1,M]\}$ with $[h,M]\neq[0,\{j\}]$ for any $j\in[n]$).

There is an obvious forgetful morphism

$$\mathcal{F}:\mathrm{Sprout}_{g,n}(A_3)\to\mathcal{U}_{g,n}(A_3)$$

given by forgetting the last section σ_{n+1}. The morphism $\mathcal{F}$ is finite (and representable) by [8, Prop. 4.7]. The restriction of $\mathcal{F}$ to $\mathrm{Sprout}_{g,n}(A_3)^{\mathrm{irr}}$ (resp., $\mathrm{Sprout}_{g,n}(A_3)^{0,\{j\}}$; resp., $\mathrm{Sprout}_{g,n}(A_3)^{h,M}$) will be denoted by $\mathcal{F}_{\mathrm{irr}}$ (resp., $\mathcal{F}_{0,\{j\}}$; resp., $\mathcal{F}_{h,M}$).

As explained in [8, Sec. 4.2], given a family $\left(\mathcal{C}\to S,\{\sigma_i\}_{i=1}^{n+1}\right)\in\mathrm{Sprout}_{g,n}(A_3)(S)$, we can normalise along the section σ_{n+1} and then stabilise in order to get a new family $\left(\mathcal{C}^s\to S,\{\sigma_i^s\}_{i=1}^{n+l}\right)$ (with $l=0$ or 2). The number of connected components of $\mathcal{C}^s\to S$, their genera and number of marked points and the number l is determined by the type of the tacnodal section σ_{n+1}. We can distinguish the following three cases:

- If the tacnodal section σ_{n+1} is of type $\{\mathrm{irr}\}$, then $\mathcal{C}^s\to S$ is connected, hence we get a morphism

$$\mathcal{N}_{\mathrm{irr}}:\mathrm{Sprout}_{g,n}(A_3)^{\mathrm{irr}}\longrightarrow\mathcal{U}_{g-2,n+2}(A_3),$$
$$\left(\mathcal{C}\to S,\{\sigma_i\}_{i=1}^{n+1}\right)\mapsto\left(\mathcal{C}^s\to S,\{\sigma_i^s\}_{i=1}^{n+2}\right),$$

  where the first n sections σ_i^s are the images of the first n sections σ_i, and the last sections $\left\{\sigma_{n+1}^s,\sigma_{n+l}^s\right\}$ are the two inverse images of σ_{n+1} under the normalisation along σ_{n+1}.

- If the tacnodal section has type equal to $\{[0,\{j\}],[1,\{j\}]\}$, then the normalisation of $\mathcal{C}\to S$ will have two connected components: one a family of genus $g-1$ curves with n marked points, and the other one a family of genus 0 curves with two marked points. When we stabilise, the second component gets contracted and hence we end up with a morphism

$$\mathcal{N}_{0,\{j\}}:\mathrm{Sprout}_{g,n}(A_3)^{0,\{j\}}\longrightarrow\mathcal{U}_{g-1,n}(A_3),$$
$$\left(\mathcal{C}\to S,\{\sigma_i\}_{i=1}^{n+1}\right)\mapsto\left(\mathcal{C}^s\to S,\{\sigma_i^s\}_{i=1}^{n}\right),$$

  where the first $n-1$ sections σ_i^s are the images of the sections $\{\sigma_i\}_{i\neq j,n+1}$, and the last section σ_n^s is one of the two inverse images of σ_{n+1} under the normalisation along σ_{n+1}.

- If the tacnodal section has type equal to $\{[h,M],[h+1,M]\}$ with $[h,M]\neq[0,\{j\}]$ for any $j\in[n]$, then the normalisation of $\mathcal{C}\to S$ will have two connected components, $\mathcal{C}_1\to S$ of genus h curves and with $|M|+1$ marked points and $\mathcal{C}_2\to S$ of genus $g-h-1$ and with $|M^c|+1$ marked points. Hence, after stabilising, we get a morphism

$$\mathcal{N}_{h,M}:\mathrm{Sprout}_{g,n}(A_3)^{h,M}\longrightarrow\mathcal{U}_{h,|M|+1}(A_3)\times\mathcal{U}_{g-h-1,|M^c|+1}(A_3),$$
$$\left(\mathcal{C}\to S,\{\sigma_i\}_{i=1}^{n+1}\right)\mapsto\left(\left(\mathcal{C}_1^s\to S,\{\sigma_i^s\}_{i\in M},\sigma_{n+1}^s\right),\left(\mathcal{C}_2^s\to S,\{\sigma_i^s\}_{i\in M^c},\sigma_{n+2}^s\right)\right),$$

  where the sections $\{\sigma_i^s\}_{i\in M\cup M^c}$ are the images of the first n sections σ_i, and the last sections $\left\{\sigma_{n+1}^s,\sigma_{n+l}^s\right\}$ are the images of the two inverse images of σ_{n+1} under the normalisation along σ_{n+1}.

By [8, Prop. 4.9], the maps $\mathcal{N}_{\mathrm{irr}}$, $\mathcal{N}_{0,\{j\}}$ and $\mathcal{N}_{h,M}$ are stacky projective bundles. For later use, observe that the codomains of these stacky projective bundles are always stacks of pointed curves with at least one marked point. This is clear for $\mathcal{N}_{\mathrm{irr}}$ and $\mathcal{N}_{h,M}$, and for $\mathcal{N}_{0,\{j\}}$ it follows from the fact that the morphism $\mathcal{N}_{0,\{j\}} : \mathrm{Sprout}_{g,n}(A_3)^{0,\{j\}} \to \mathcal{U}_{g-1,n}(A_3)$ is defined only if $\{[0,\{j\}],[1,\{j\}]\} \subset T_{g,n}$ which implies that $n \geq 1$.

We now study the compatibility of the maps $\mathcal{N}_{\mathrm{irr}}$, $\mathcal{N}_{0,\{j\}}$ and $\mathcal{N}_{h,M}$ and of $\mathcal{F}_{\mathrm{irr}}$, $\mathcal{F}_{0,\{j\}}$ and $\mathcal{F}_{h,M}$ with the open substacks of T-semistable curves.

Lemma 4.12. *Set $T \subseteq T_{g,n}$. Then the preimages of $\overline{\mathcal{M}}_{g,n}^T$ via the maps $\mathcal{F}_{\mathrm{irr}}$, $\mathcal{F}_{0,\{j\}}$ and $\mathcal{F}_{h,M}$ are computed as follows:*

(i) $\mathcal{F}_{\mathrm{irr}}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) = \begin{cases} \emptyset & \text{if } \mathrm{irr} \notin T, \\ \left(\mathcal{N}_{\mathrm{irr}}^{-1}\right)\left(\overline{\mathcal{M}}_{g-2,n+2}^{T_{g-2,n+2}}\right) & \text{if } \mathrm{irr} \in T. \end{cases}$

(ii) $\mathcal{F}_{0,\{j\}}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) = \begin{cases} \emptyset & \text{if } \{[0,\{j\}],[1,\{j\}]\} \not\subset T, \\ \left(\mathcal{N}_{0,\{j\}}^{-1}\right)\left(\overline{\mathcal{M}}_{g-1,n}^{\widehat{T}}\right) & \text{if } \{[0,\{j\}],[1,\{j\}]\} \subset T, \end{cases}$

where $\widehat{T}$ is the subset of $T_{g-1,n}$ defined by

$$\begin{cases} \mathrm{irr} \in \widehat{T} \Leftrightarrow \mathrm{irr} \in T, \\ [\tau,I] \in \widehat{T} \Leftrightarrow \begin{cases} [\tau,I] \in T & \text{if } n+1 \notin I, \\ [g-1-\tau,[n+1]-\{I\}] & \text{if } n+1 \in I. \end{cases} \end{cases}$$

(iii) $\mathcal{F}_{h,M}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) = \begin{cases} \emptyset & \text{if } \{[h,M],[h+1,M]\} \not\subset T, \\ \left(\mathcal{N}_{h,M}^{-1}\right)\left(\overline{\mathcal{M}}_{h,|M|+1}^{\widetilde{T}_{h,M}} \times \overline{\mathcal{M}}_{g-1-h,|M^c|+1}^{\widetilde{T}_{g-h-1,M^c}}\right) & \text{if } \{[h,M],[h+1,M]\} \subset T, \end{cases}$

where $\widetilde{T}_{h,M}$ is the subset of $T_{h,|M|+1}$ defined by

$$\begin{cases} \mathrm{irr} \in \widetilde{T}_{h,M} \Leftrightarrow \mathrm{irr} \in T, \\ [\tau,I] \in \widetilde{T}_{h,M} \Leftrightarrow \begin{cases} [\tau,I] \in T & \text{if } |M|+1 \notin I, \\ [h-\tau,[|M|+1]-\{I\}\} & \text{if } |M|+1 \in I, \end{cases} \end{cases}$$

with the convention that $[|M|] = [|M|+1] - \{|M|+1\}$ is identified with the subset $M \subset [n]$, and where $\widetilde{T}_{g-h-1,M^c} \subseteq T_{g-h-1,|M^c|+1}$ is defined similarly by replacing h with $g-h-1$ and M with M^c.

Proof. Recall that $\overline{\mathcal{M}}_{g,n}^T$ is the open substack whose k-points are n-pointed curves $(C,\{p_i\}) \in \mathcal{U}_{g,n}(A_3)$ that do not have A_1- or A_3-attached elliptic chains and whose tacnodes have type contained in T. Hence we can argue with families of curves over k – that is, with n-pointed curves.

Let us first prove (i). First of all, since for any $\left(C,\{p_i\}_{i=1}^{n+1}\right) \in \mathrm{Sprout}_{g,n}(A_3)^{\mathrm{irr}}(k)$ the n-pointed curve $\mathcal{F}_{\mathrm{irr}}\left(C,\{p_i\}_{i=1}^{n+1}\right) = (C,\{p_i\}_{i=1}^n) \in \mathcal{U}_{g,n}(A_3)(k)$ will have a tacnode of type $\{\mathrm{irr}\}$ in p_{n+1}, we deduce that $\mathcal{F}_{\mathrm{irr}}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) = \emptyset$ if $\mathrm{irr} \notin T$. We can therefore assume that $\mathrm{irr} \in T$. Note that $\mathcal{F}_{\mathrm{irr}}\left(C,\{p_i\}_{i=1}^{n+1}\right) = (C,\{p_i\}_{i=1}^n)$ will have an A_1- or A_3-attached elliptic

chain if and only if the same property holds for $\mathcal{N}_{\mathrm{irr}}\left(C,\{p_i\}_{i=1}^{n+1}\right) = \left(C^s, \{p_i^s\}_{i=1}^{n+2}\right)$. Hence the result follows because every tacnode of $\left(C^s, \{p_i^s\}_{i=1}^{n+2}\right)$ becomes a tacnode of type $\{\mathrm{irr}\}$ when seen in $(C, \{p_i\}_{i=1}^{n})$.

Let us now prove (ii). Since for any $\left(C, \{p_i\}_{i=1}^{n+1}\right) \in \mathrm{Sprout}_{g,n}(A_3)^{0,\{j\}}(k)$ the n-pointed curve $\mathcal{F}_{0,\{j\}}\left(C, \{p_i\}_{i=1}^{n+1}\right) = (C, \{p_i\}_{i=1}^{n}) \in \mathcal{U}_{g,n}(A_3)(k)$ will have a tacnode of type $\{[0,\{j\}],[1,\{j\}]\}$ in p_{n+1}, we deduce that $\mathcal{F}_{0,\{j\}}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) = \emptyset$ if $\{[0,\{j\}],[1,\{j\}]\} \not\subseteq T$. We can therefore assume that $\{[0,\{j\}],[1,\{j\}]\} \subseteq T$. Note that $\mathcal{F}_{0,\{j\}}\left(C, \{p_i\}_{i=1}^{n+1}\right) = (C, \{p_i\}_{i=1}^{n}) \in \mathcal{U}_{g,n}(A_3)(k)$ will have an A_1- or A_3-attached elliptic chain if and only if the same property holds for $\mathcal{N}_{0,\{j\}}\left(C,\{p_i\}_{i=1}^{n+1}\right) = \left(C^s, \{p_i^s\}_{i=1}^{n+1}\right)$. Hence the result follows because every tacnode of $\left(C^s, \{p_i^s\}_{i=1}^{n+1}\right)$ of type $\{\mathrm{irr}\}$ remains of type $\{\mathrm{irr}\}$ when seen in $(C, \{p_i\}_{i=1}^{n})$, while every tacnode of $\left(C^s, \{p_i^s\}_{i=1}^{n+1}\right)$ of type $\{[\tau,I],[\tau+1,I]\}$ becomes, when seen in $(C, \{p_i\}_{i=1}^{n})$, of type $\{[\tau,I],[\tau+1,I]\}$ if $n+1 \notin I$ and of type $\{[g-2-\tau,[n+1]-\{I\}],[g-1-\tau,[n+1]-\{I\}]\}$ if $n+1 \in I$.

Let us finally prove (iii). First of all, since for any $\left(C, \{p_i\}_{i=1}^{n+1}\right) \in \mathrm{Sprout}_{g,n}(A_3)^{h,M}(k)$ the n-pointed curve $\mathcal{F}_{h,M}\left(C, \{p_i\}_{i=1}^{n+1}\right) = (C, \{p_i\}_{i=1}^{n}) \in \mathcal{U}_{g,n}(A_3)(k)$ will have a tacnode of type $\{[h,M],[h+1,M]\}$ in p_{n+1}, we deduce that $\mathcal{F}_{h,M}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) = \emptyset$ if $\{[h,M],[h+1,M]\} \not\subset T$. We can therefore assume that $\{[h,M],[h+1,M]\} \subset T$. Note that $\mathcal{F}_{h,M}\left(C, \{p_i\}_{i=1}^{n+1}\right) = (C, \{p_i\}_{i=1}^{n}) \in \mathcal{U}_{g,n}(A_3)(k)$ will not have an A_1- or A_3-attached elliptic chain if and only if the same property holds for both $\left(C_1^s, \{p_i^s\}_{i\in M}, \{p_{n+1}\}\right) \in \mathcal{U}_{h,|M|+1}(A_3)(k)$ and $\left(C_2^s, \{p_i^s\}_{i\in M}, \{p_{n+2}\}\right) \in \mathcal{U}_{g-h-1,|M^c|+1}(A_3)(k)$. Hence it remains to determine the type of the tacnodes of $\left(C_1^s, \{p_i^s\}_{i\in M}, \{p_{n+1}\}\right)$ and $\left(C_2^s, \{p_i^s\}_{i\in M}, \{p_{n+2}\}\right)$ when considered in $(C, \{p_i\}_{i=1}^{n})$. We will only examine the tacnodes of $\left(C_1^s, \{p_i^s\}_{i\in M}, \{p_{n+1}\}\right)$, the other case being analogous. A tacnode of $\left(C_1^s, \{p_i^s\}_{i\in M}, \{p_{n+1}\}\right)$ of type $\{\mathrm{irr}\}$ remains of type $\{\mathrm{irr}\}$ when seen in $(C, \{p_i\}_{i=1}^{n})$, while a tacnode of $\left(C^s, \{p_i^s\}_{i=1}^{n+1}\right)$ of type $\{[\tau,I],[\tau+1,I]\}$ becomes, when seen in $(C, \{p_i\}_{i=1}^{n})$, of type $\{[\tau,I],[\tau+1,I]\}$ if $|M|+1 \notin I$ and of type $\{[h-\tau-1,[|M|+1]-\{I\}],[h-\tau,[|M|+1]-\{I\}]\}$ if $|M|+1 \in I$. This implies the result. $\qquad\square$

Using this lemma, we can prove the existence of the proper good moduli space for $\mathscr{Z}_T^-$.

Proof.[Proof of Proposition 4.8] Consider the open substack of $\mathrm{Sprout}_{g,n}(A_3)$:

$$E_T := \mathcal{F}_{\mathrm{irr}}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) \coprod_{[0,\{j\}]\in T_{g,n}} \mathcal{F}_{0,\{j\}}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right) \coprod_{\substack{[h,M]\in T_{g,n}: \\ 0\le h\le g-1, [h,M]\ne[0,\{j\}]}} \mathcal{F}_{h,M}^{-1}\left(\overline{\mathcal{M}}_{g,n}^T\right).$$

The morphism $\mathcal{F}$ restricted to E_T gives rise to a morphism

$$\mathcal{F}_T = \mathcal{F}_{|E_T} : E_T \to \overline{\mathcal{M}}_{g,n}^T,$$

which is finite by [8, Prop. 4.7]. By construction, the image of $\mathcal{F}_T$ is the locus of $\overline{\mathcal{M}}_{g,n}^T$ having at least one tacnode – that is, exactly $\mathscr{Z}_T^-$.

Observe that the algebraic stack $\mathcal{Z}_T^-$, being a closed substack of $\overline{\mathcal{M}}_{g,n}^T$, is a global quotient stack of a normal variety by Remark 3.20 and has linearly reductive stabilisers by Lemma 4.2 and our assumption on char(k). Moreover, Lemma 4.12 and [8, Prop. 4.9] imply that E_T is a stacky projective bundle over the disjoint unions of stacks of the form $\overline{\mathcal{M}}_{g',n'}^{T'}$ for suitable $T' \subseteq T_{g',n'}$ with either $g' < g$ and $1 \leq n' \leq n+1$ or $(g',n') = (g-2,n+2)$. Since all these stacks $\overline{\mathcal{M}}_{g',n'}^{T'}$ admit proper good moduli spaces by assumption, E_T also admits a proper good moduli space. We can now apply Proposition 4.9 and Remark 4.11(i) to infer that $\mathcal{Z}_T^-$ admits a proper good moduli space. $\qquad\square$

Now we turn to the existence of a good moduli space for the stack $\mathcal{Z}_T^+$.

Proposition 4.13. *Fix $T \subseteq T_{g,n}$ with $(g,n) \neq (2,0)$ and assume that* char(k) $\gg T$. *If $\overline{\mathcal{M}}_{g',n'}^{T'}$ admits a proper good moduli space for all $T' \subseteq T_{g',n'}$ with either $g' < g$ and $1 \leq n' \leq n+1$ or $(g',n') = (g-2,n+2)$, then $\mathcal{Z}_T^+ \subset \overline{\mathcal{M}}_{g,n}^T$ admits a proper good moduli space.*

Note that $\mathcal{Z}_T^+$ coincides with the stack $\overline{\mathcal{H}}_{g,n}(7/10)$ of [8, Sec. 4] in the case where $T^{\mathrm{adm}} = T_{g,n}^{\mathrm{adm}}$. Hence, this proposition generalises [8, Prop. 4.15] for $\alpha_c = 7/10$ (but one has to assume that $(g,n) \neq (2,0)$). At the other extreme, if $T^{\mathrm{adm}} = \emptyset$, then $\mathcal{Z}_T^+ = \emptyset$ by Remark 3.17 and the result is trivial. Moreover, if $(g,n) = (1,2)$ and $T^{\mathrm{adm}} \neq \emptyset$, then $\mathcal{Z}_T^+ = \overline{\mathcal{M}}_{1,2}^T$ admits a point as a good moduli space by Remark 4.5 (which follows also from the description $\mathcal{Z}_T^+ = \overline{\mathcal{H}}_{1,2}(7/10) \cong [\mathbb{A}^3/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^3$ with weights 2,3 and 4, see [8, Lemma 4.11]).

The strategy of the proof of Proposition 4.13 is similar to that of [8, Proposition 4.15], and consists in finding a finite cover of $\mathcal{Z}_T^+$ consisting of the disjoint union of the product of a stack admitting a good moduli space with suitable stacks $\overline{\mathcal{M}}_{g',n'}^{T'}$ (as in the statement of Proposition 4.13) and then concluding by applying Proposition 4.9. In order to use this strategy we will need to review some constructions from [8, Sec. 4.3], adapted to our setting and notation.

For any integer $r \geq 1$, let

$$\mathcal{EC}_r \subset \overline{\mathcal{M}}_{2r-1,2}(7/10) = \overline{\mathcal{M}}_{2r-1,2}^{T_{2r-1,2}}$$

be the closure (with reduced structure) of the locally closed substack of elliptic chains of length r. It is proved in [8, Lemma 4.12] that $\mathcal{EC}_r$ admits a proper good moduli space.

By gluing to an elliptic chain of length r suitable pointed curves, we can obtain n-pointed curves in $\mathcal{U}_{g,n}(A_3)$. More precisely, there are the following two types of gluing morphisms:

- For any $1 \leq r \leq g/2$, we consider the gluing morphism

$$\mathcal{G}_{\mathrm{irr}}^r : \mathcal{U}_{g-2r,n+2}(A_3) \times \mathcal{EC}_r \longrightarrow \mathcal{U}_{g,n}(A_3),$$
$$\left((C,\{p_i\}_{i=1}^{n+2}),(Z,q_1,q_2)\right) \mapsto (C \cup Z,\{p_i\}_{i=1}^n)/(p_{n+1} \sim q_1, p_{n+2} \sim q_2).$$

Note that we include in this case also the limit case $(g,n) = (2r,0)$, where $\mathcal{U}_{g-2r,n+2}(A_3) = \mathcal{U}_{0,2}(A_3) = \emptyset$ and in this construction we have to glue q_1 with q_2.

- For any $0 \le h \le g - 2r + 1$ and any $M \subseteq [n]$ with the restriction that $|M| \ge 1$ if either $h = 0$ or $h = g - 2r + 1$, we consider the gluing morphism

$$\mathcal{G}^r_{h,M} : \mathcal{U}_{h,|M|+1}(A_3) \times \mathcal{U}_{g-h-2r+1,|M^c|+1}(A_3) \times \mathcal{EC}_r \longrightarrow \mathcal{U}_{g,n}(A_3),$$
$$\left((C, \{p_i\}_{i \in M}, s_1), (C', \{p'_i\}_{i \in M^c}, s_2), (Z, q_1, q_2) \right) \mapsto (C \cup C' \cup Z, \{p_i\}_{i=1}^n) \, / $$
$$(s_1 \sim q_1, s_2 \sim q_2).$$

Note that we include in this case also three degenerate cases: $(h, M) = (0, \{j\})$, in which case $\mathcal{U}_{h,|M|+1}(A_3) = \mathcal{U}_{0,2}(A_3) = \emptyset$ and the point q_1 is regarded as the jth marked point; and $(g - h - 2r + 1, M^c) = (0, \{j\})$, in which case $\mathcal{U}_{g-h-2r+1,|M^c|+1}(A_3) = \mathcal{U}_{0,2}(A_3) = \emptyset$ and the point q_2 is regarded as the jth marked point; and the case where both occurrences happen, namely $(g, n) = (2r - 1, 2)$, when the gluing morphism is the embedding of $\mathcal{EC}_r$ into $\mathcal{U}_{2r-1,2}(A_3)$.

It follows from [8, Lemma 4.13 and 4.14] that the morphisms $\mathcal{G}^r_{\mathrm{irr}}$ and $\mathcal{G}^r_{h,M}$ are finite. For later use, observe that the stacks of the form $\mathcal{U}_{g',n'}(A_3)$ that appear in the domain of the morphisms $\mathcal{G}^r_{\mathrm{irr}}$ and $\mathcal{G}^r_{h,M}$ have the property that $n' \ge 1$ – that is, there is at least one marked point.

We now study the compatibility of the maps $\mathcal{G}^r_{\mathrm{irr}}$ and $\mathcal{G}^r_{h,M}$ with the open substacks of T-semistable curves.

Lemma 4.14. *Set* $T \subseteq T_{g,n}$.

(i) *If* $\mathrm{irr} \in T$, *then*

$$(\mathcal{G}^r_{\mathrm{irr}})^{-1} \left(\overline{\mathcal{M}}^T_{g,n} \right) = \overline{\mathcal{M}}^{T_{g-2r,n+2}}_{g-2r,n+2} \times \mathcal{EC}_r.$$

(ii) *If* $\{[h, M], \dots, [h + 2r - 1, M]\} \subseteq T$ *and* $(h, M), (g - h - 2r + 1, M^c) \ne (1, \emptyset)$, *then*

$$\left(\mathcal{G}^r_{h,M} \right)^{-1} \left(\overline{\mathcal{M}}^T_{g,n} \right) = \overline{\mathcal{M}}^{\widetilde{T}_{h,M}}_{h,|M|+1} \times \overline{\mathcal{M}}^{\widetilde{T}_{g-h-2r+1,M^c}}_{g-h-2r+1,|M^c|+1} \times \mathcal{EC}_r,$$

where $\widetilde{T}_{h,M}$ *is the subset of* $T_{h,|M|+1}$ *defined by*

$$\begin{cases} \mathrm{irr} \in \widetilde{T}_{h,M} \Leftrightarrow \mathrm{irr} \in T, \\ [\tau, I] \in \widetilde{T}_{h,M} \Leftrightarrow \begin{cases} [\tau, I] \in T & \text{if } |M| + 1 \notin I, \\ [h - \tau, [|M| + 1] - \{I\}] \in T & \text{if } |M| + 1 \in I, \end{cases} \end{cases}$$

with the convention that $[|M|] = [|M| + 1] - \{|M| + 1\}$ *is identified with the subset* $M \subset [n]$ *(which allows us to consider any subset of* $[|M|]$ *as a subset of* $[n]$*), and where* $\widehat{T}_{g-h-2r+1,M^c} \subseteq T_{g-h-2r+1,|M^c|+1}$ *is defined similarly by replacing* h *with* $g - h - 2r + 1$ *and* M *with* M^c.

Proof. Let us first prove (i). First of all, note that $\mathcal{G}^r_{\text{irr}}\left(\left(C,\{p_i\}_{i=1}^{n+2}\right),(Z,q_1,q_2)\right)$ does not have an A_1- or A_3-attached elliptic chain if and only if the same is true for $\left(C,\{p_i\}_{i=1}^{n+2}\right)$. Moreover, every tacnode of Z and of C becomes of type $\{\text{irr}\}$ in $(C\cup Z,\{p_i\}_{i=1}^n)/(p_{n+1}\sim q_1,p_{n+2}\sim q_2)$, from which the conclusion follows.

Let us now prove (ii). We will assume that we are not in one of the three degenerate cases discussed after the definition of $\mathcal{G}^r_{h,M}$, and leave those three limit cases (which are easier to deal with) to the reader. First of all, note that since $(h,M),(g-h-2r+1,M^c)\neq(1,\emptyset)$ by assumption, $\mathcal{G}^r_{h,M}\left(\left(C,\{p_i\}_{i\in M},s_1\right),\left(C',\{p_i'\}_{i\in M^c},s_2\right),(Z,q_1,q_2)\right)$ does not have an A_1- or A_3-attached elliptic chain if and only if the same is true for $\left(C,\{p_i\}_{i\in M},s_1\right)$ and $\left(C',\{p_i'\}_{i\in M^c},s_2\right)$. Next, every tacnode of Z, when considered in $(C\cup C'\cup Z,\{p_i\}_{i=1}^n)/(s_1\sim q_1,s_2\sim q_2)$, is of type contained in $\{[h,M],\dots,[h+2r-1,M]\}$, and hence in T by our assumption. On the other hand, a tacnode of $(C,\{p_i\}_{i\in M},s_1)$ of type $\{\text{irr}\}$ remains of type $\{\text{irr}\}$ when seen in $(C\cup C'\cup Z,\{p_i\}_{i=1}^n)/(s_1\sim q_1,s_2\sim q_2)$, while if it has type $\{[\tau,I],[\tau+1,I]\}$, it remains of the same type if $|M|+1\notin I$ and becomes of type $\{[h-\tau-1,[|M|+1]-\{I\}],[h-\tau,[|M|+1]-\{I\}]\}$ if $|M|+1\in I$. A similar analysis can be done for C', and this concludes the proof. $\square$

Using this lemma, we can prove the existence of the good moduli space for $\mathcal{Z}^+_T$.

Proof. [Proof of Proposition 4.13] First of all, note that $\overline{\mathcal{M}}^T_{g,n}=\overline{\mathcal{M}}^{T\setminus[1,\emptyset]}_{g,n}$, because a tacnode of type $[1,\emptyset]$ corresponds to a tacnodal elliptic tail, and we have already removed such a tail from $\overline{\mathcal{M}}^T_{g,n}$ in Definition 3.16. We can thus assume that $[1,\emptyset]\notin T$.

Consider the stack

$$
H_T := \begin{cases}
\displaystyle\coprod_{\{[h,M],\dots,[h+2r-1,M]\}\subseteq T}(\mathcal{G}^r_{h,M})^{-1}\left(\overline{\mathcal{M}}^T_{g,n}\right) & \text{if irr}\notin T, \\
\displaystyle\coprod_{\{[h,M],\dots,[h+2r-1,M]\}\subseteq T}(\mathcal{G}^r_{h,M})^{-1}\left(\overline{\mathcal{M}}^T_{g,n}\right)\coprod_{1\leq r\leq g/2}(\mathcal{G}^r_{\text{irr}})^{-1}\left(\overline{\mathcal{M}}^T_{g,n}\right) & \text{if irr}\in T.
\end{cases}
$$

The finite morphisms $\mathcal{G}^r_{\text{irr}}$ and $\mathcal{G}^r_{h,M}$ give rise to a finite morphism

$$
\mathcal{G}_T : H_T \to \overline{\mathcal{M}}^T_{g,n},
$$

whose image, by construction, is the locus of $\overline{\mathcal{M}}^T_{g,n}$ having at least one A_1/A_1-attached elliptic chain of type contained in T – that is, exactly $\mathcal{Z}^+_T$.

Observe that the algebraic stack $\mathcal{Z}^+_T$, being a closed substack of $\overline{\mathcal{M}}^T_{g,n}$, is a global quotient stack of a normal variety by Remark 3.20 and has linearly reductive stabilisers by Lemma 4.2 and our assumption on char(k). Moreover, Lemma 4.14 implies that the stack H_T is a (finite) disjoint union of products of the stacks $\mathcal{EC}_r$, which admit a proper good moduli space by [8, Lemma 4.12], and of the stacks $\overline{\mathcal{M}}^{T'}_{g',n'}$ for suitable $T'\subseteq T_{g',n'}$ with either $g'<g$ and $1\leq n'\leq n+1$ or $(g',n')=(g-2,n+2)$, which admit a proper good moduli space by assumption. Therefore, H_T also admits a proper good moduli space. We can now apply Proposition 4.9 and Remark 4.11(i) to infer that $\mathcal{Z}^+_T$ admits a proper good moduli space. $\square$

We can now prove the main result of this section.

Proof.[Proof of Theorem 4.4] First of all, Proposition 4.6 implies that the two open embeddings

$$\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \hookrightarrow \overline{\mathcal{M}}_{g,n}^{T} \hookleftarrow \overline{\mathcal{M}}_{g,n}^{T+}$$

arise from local VGIT with respect to the line bundle $\delta - \psi$ on $\overline{\mathcal{M}}_{g,n}^{T}$.

Next, the stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ admits a coarse proper moduli space $\phi^{\mathrm{ps}} : \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ (see Proposition 3.11). Since the stabiliser of any k-point of $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is linearly reductive by our assumption on the characteristic (see Lemma 4.2 and recall that $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \subseteq \overline{\mathcal{M}}_{g,n}^{T}$), we infer that ϕ^{ps} is also a good moduli space by [1, Thm. 3.2].

Therefore, thanks to [8, Theorem 1.3], the existence of proper good moduli spaces fitting into commutative diagram (4.1) will follow if we show that the stacks $\mathcal{Z}_T^- = \overline{\mathcal{M}}_{g,n}^{T} \setminus \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ and $\mathcal{Z}_T^+ = \overline{\mathcal{M}}_{g,n}^{T} \setminus \overline{\mathcal{M}}_{g,n}^{T+}$ admit good moduli spaces. This follows from Propositions 4.8 and 4.13 using induction on g: the base of the induction is the case $g = 0$ when $\overline{\mathcal{M}}_{0,n}^{T} = \overline{\mathcal{M}}_{0,n}$ is a variety (hence it is its own good moduli space) and the assumption on the characteristic of the base field k allows us to apply induction. Observe that the nonexistence of a proper moduli space for $\overline{\mathcal{M}}_{2,0}^{\mathrm{irr}}$ (see Remark 4.5) does not interfere with this inductive proof, since all the stacks $\overline{\mathcal{M}}_{g',n'}^{T'}$ appearing in the inductive hypothesis of Propositions 4.8 and 4.13 are such that $n' \geq 1$.

Finally, observe that the morphisms f_T and f_T^+ are proper (being morphisms between proper algebraic spaces) and all the good moduli spaces are normal and irreducible, since the corresponding algebraic stacks are smooth and irreducible by Theorem 3.19 (see [4, Theorem 4.16(viii)]). □

Remark 4.15. Since the stacks $\overline{\mathcal{M}}_{g,n}^{T}$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ contain the stack $\mathcal{M}_{g,n}$ of n-pointed smooth curves of genus g as an open substack, the spaces $\overline{\mathrm{M}}_{g,n}^{T}$ and $\overline{\mathrm{M}}_{g,n}^{T+}$ are weakly modular compactifications of $M_{g,n}$ in the sense of [26, Def. 2.6]. Moreover, they are modular compactifications of $M_{g,n}$ in the sense of [26, Def. 2.1] whenever the spaces $\overline{\mathrm{M}}_{g,n}^{T}$ and $\overline{\mathrm{M}}_{g,n}^{T+}$ are coarse moduli spaces, or equivalently whenever the stacks $\overline{\mathcal{M}}_{g,n}^{T}$ and $\overline{\mathcal{M}}_{g,n}^{T+}$ are DM, and this happens under the following conditions:

- $\overline{\mathcal{M}}_{g,n}^{T}$ is a DM stack if and only if $\mathrm{char}(k) \gg T$ and $\overline{\mathcal{M}}_{g,n}^{T} = \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ – that is, if and only if $T^{\mathrm{adm}} = \emptyset$.
- Assume that $\mathrm{char}(k) \gg T$. Then $\overline{\mathcal{M}}_{g,n}^{T+}$ is a DM stack if and only if T does not contain subsets of the form $\{[\tau, I], [\tau+1, I], [\tau+2, I]\}$ with $[\tau, I], [\tau+2, I] \neq [1, \emptyset]$.

5. The moduli space of pseudostable curves and the elliptic bridge face

The aim of this section is to study the geometric properties of the moduli space $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ of pseudostable curves and to describe a face of its Mori cone, which we call the elliptic bridge face, that will play a special role in the sequel.

We start by studying the singularities, the Picard group and the canonical class of $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$.

Proposition 5.1. *Assume* $(g,n) \neq (2,0)$ *and* $\mathrm{char}(k) \neq 2,3$. *Consider the stack* $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ *of pseudostable curves of genus* g *with* n *marked points and let* $\phi^{\mathrm{ps}} : \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *be the morphism into its coarse moduli space.*

(i) *The space* $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *is normal with finite quotient singularities, hence it is* $\mathbb{Q}$-*factorial. If* $\mathrm{char}(k) = 0$, *then* $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *is klt.*

(ii) *The pullback via the morphism* ϕ^{ps} *induces an isomorphism*

$$(\phi^{\mathrm{ps}})^* : \mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} \xrightarrow{\cong} \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}}.$$

(iii) *If* $(g,n) \neq (1,2),(2,1),(3,0)$, *then the canonical line bundle of* $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *is such that*

$$(\phi^{\mathrm{ps}})^* \left(K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}\right) = K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}}.$$

In particular, using (ii) *and Mumford's formula for* $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}}$ *(see Fact 3.28(3.28)), we get*

$$K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} = 13\lambda - 2\delta + \psi.$$

From now on, we identify (in $\mathrm{char}(k) \neq 2,3$) $\mathbb{Q}$-line bundles on $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ with $\mathbb{Q}$-line bundles on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ via the isomorphism $(\phi^{\mathrm{ps}})^*$ of (ii), similar to what is usually done for $\mathbb{Q}$-line bundles on $\overline{\mathcal{M}}_{g,n}$ and on $\overline{\mathrm{M}}_{g,n}$.

Proof. Part (i): since $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is a smooth and separated DM stack by Fact 3.10 and Proposition 3.11, its coarse moduli space $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ is normal with finite quotient singularities by [2, Lemma 2.2.3]. We conclude since finite quotient singularities are $\mathbb{Q}$-factorial and, if $\mathrm{char}(k) = 0$, klt by [42, Prop. 5.15].

Part (iii): it is enough to show that the morphism ϕ^{ps} is an isomorphism in codimension 1. First of all, the assumptions on (g,n) guarantee that the locus of n-pointed smooth curves with nontrivial automorphisms has codimension at least 2 (see [15, Chap. XII, Prop. 2.15]); hence the morphism ϕ^{ps} is an isomorphism in codimension 1 when restricted to $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$. Secondl a generic point of a boundary divisor of $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ does not have nontrivial automorphisms, and hence ϕ^{ps} is also an isomorphism in codimension 1 at the boundary of $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$.

Part (ii): consider the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} & \longrightarrow & \mathrm{Cl}\left(\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} \\
{\scriptstyle(\phi^{\mathrm{ps}})^*}\uparrow & & \downarrow{\scriptstyle(\phi^{\mathrm{ps}})_*} \\
\mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} & \longrightarrow & \mathrm{Cl}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}}.
\end{array}
$$

The upper horizontal morphism is an isomorphism because $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is a smooth stack; the lower horizontal arrow is an isomorphism because $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ is normal and $\mathbb{Q}$-factorial

by (i); and the right vertical arrow is an isomorphism because ϕ^{ps} is an isomorphism in codimension 1, as observed in the proof of (iii). Hence, by the commutativity of the diagram, we infer that $(\phi^{\mathrm{ps}})^*$ is also an isomorphism. □

Remark 5.2. We do not know whether $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ has finite quotient singularities or is simply $\mathbb{Q}$-factorial when $\mathrm{char}(k) = 2,3$ (see also [3, Rmk. 3.6]). If $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ is also $\mathbb{Q}$-factorial when $\mathrm{char}(k) = 2,3$, then all the results of this section extend to $\mathrm{char}(k) = 2,3$.

Remark 5.3. The first two points of Proposition 5.1 remain true for $(g,n) = (2,0)$.

Indeed, part (i) follows from the fact that $\overline{\mathrm{M}}_2^{\mathrm{ps}}$ is isomorphic to the GIT quotient of binary sextics (see [36, Thm. 2]), which is isomorphic to the weighted projective space $\mathbb{P}(1,2,3,5)$ (see [32, Prop. 2.2(1)] and the references therein), and hence it has finite quotient singularities.

On the other hand, part (ii) follows from the fact that $\overline{\mathcal{M}}_2^{\mathrm{ps}}$ is smooth, $\overline{\mathrm{M}}_2^{\mathrm{ps}}$ has $\mathbb{Q}$-factorial singularities and the morphism $\phi^{\mathrm{ps}} : \overline{\mathcal{M}}_2^{\mathrm{ps}} \to \overline{\mathrm{M}}_2^{\mathrm{ps}}$ is finite in codimension 1 by Remark 3.13.

Remark 5.4. In the exceptional cases excluded by Proposition 5.1(iii) (and also for $(g,n) = (2,0)$), we can apply Hurwitz formula to the morphism $\phi^{\mathrm{ps}} : \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ to get

$$K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} = (\phi^{\mathrm{ps}})^* \left(K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} \right) + R = K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} + R,$$

where R is (the class of) the effective ramification divisor. Using Mumford's formula for $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}}$, we have

$$K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} = 13\lambda - 2\delta + \psi - R.$$

Moreover, from the proof of Proposition 5.1(iii), it follows that R is an effective divisor not contained in the boundary of $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$.

We now focus on the relation of the coarse moduli space $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ of pseudostable curves with the coarse moduli space $\overline{\mathrm{M}}_{g,n}$ of stable curves. Note that for $(g,n) \neq (1,1),(2,0)$, the morphism of stacks $\widehat{\Upsilon} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ of Proposition 3.11(i) induces a proper morphism between their coarse moduli spaces

$$\Upsilon : \overline{\mathrm{M}}_{g,n} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}. \tag{5.1}$$

Proposition 5.5. *Assume* $(g,n) \neq (1,1),(2,0)$ *and* $g \geq 1$.

(i) *The space* $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *is isomorphic to the following log canonical model of* $\overline{\mathrm{M}}_{g,n}$:

$$\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \cong \overline{\mathrm{M}}_{g,n}\left(\frac{9}{11}\right) := \mathrm{Proj} \bigoplus_{m \geq 0} H^0 \left(\overline{\mathcal{M}}_{g,n}, \left\lfloor m \left(K_{\overline{\mathcal{M}}_{g,n}} + \psi + \frac{9}{11}(\delta - \psi) \right) \right\rfloor \right).$$

In particular, $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ *is a normal projective variety.*

(ii) *The morphism* Υ *is the contraction of the extremal ray* $\mathbb{R}_{\geq 0} \cdot [C_{\mathrm{ell}}]$ *of the Mori cone* $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}\right)$, *which negatively intersects* $K_{\overline{\mathcal{M}}_{g,n}}$, $K_{\overline{\mathcal{M}}_{g,n}} + \psi$, $K_{\overline{\mathrm{M}}_{g,n}}$ *and* $K_{\overline{\mathrm{M}}_{g,n}} + \psi$. *Moreover,* Υ *is a divisorial contraction and the exceptional locus is the divisor* $\Delta_{1,\emptyset}$.

(iii) *Assume that* $\mathrm{char}(k) \neq 2, 3$. *The pullback map* $\Upsilon^* : \mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} \to \mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}\right)_{\mathbb{Q}}$ *is determined by the following relations:*

$$\begin{cases} \Upsilon^*(\lambda) = \lambda + \delta_{1,\emptyset}, \\ \Upsilon^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}} + 12\delta_{1,\emptyset}, \\ \Upsilon^*(\delta_{i,I}) = \delta_{i,I} \quad \text{for any } [i, I] \neq [1, \emptyset]. \end{cases}$$

Proof. Some parts of this theorem are proved for $n = 0$ in [33] and [37], and some other parts are proved in [7] under the assumption that $\mathrm{char}(k) = 0$. Let us convince the reader that those proofs work for any n and over an arbitrary algebraically closed field k. Consider the $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}$

$$L_{g,n} := K_{\overline{\mathcal{M}}_{g,n}} + \psi + \frac{9}{11}(\delta - \psi) = K_{\overline{\mathcal{M}}_{g,n}} + \frac{9}{11}\delta + \frac{2}{11}\psi.$$

By [7, Introduction], the line bundle $L_{g,n}$ is nef and has degree 0 precisely on the curves that are numerically equivalent to C_{ell}. Moreover, we claim that $L_{g,n}$ is semiample on $\overline{\mathrm{M}}_{g,n}$. Indeed, when $n = 0$, $L_{g,0}$ is the pullback via Υ of the natural polarisation coming from the identification of $\overline{\mathrm{M}}_g^{\mathrm{ps}}$ with the GIT quotient of the Chow variety of 4-canonical curves (see [37, Thm. 7] and [34, Thm. 3.1]). When $n > 0$, $L_{g,n}$ is the pullback of $L_{g+n,0}$ via the regular morphism $\overline{\mathrm{M}}_{g,n} \to \overline{\mathrm{M}}_{g+n}$ that attaches a fixed smooth elliptic curve to each of the marked points of an n-pointed stable curve of genus g (see [15, Lemma (4.38)]).

These facts imply that a sufficiently high multiple of $L_{g,n}$ induces a regular morphism

$$\pi : \overline{\mathrm{M}}_{g,n} \to \mathrm{Proj} \bigoplus_{m \geq 0} H^0\left(\overline{\mathrm{M}}_{g,n}, \left\lfloor m\left(K_{\overline{\mathcal{M}}_{g,n}} + \frac{9}{11}\delta + \frac{2}{11}\psi\right)\right\rfloor\right)$$

which is the contraction of the extremal ray $\mathbb{R}_{\geq 0} \cdot C_{\mathrm{ell}}$ of $\mathrm{NE}\left(\overline{\mathrm{M}}_{g,n}\right)$. The codomain coincides with $\overline{\mathrm{M}}_{g,n}\left(\frac{9}{11}\right)$ because

$$H^0\left(\overline{\mathrm{M}}_{g,n}, \left\lfloor m\left(K_{\overline{\mathcal{M}}_{g,n}} + \frac{9}{11}\delta + \frac{2}{11}\psi\right)\right\rfloor\right) = H^0\left(\overline{\mathcal{M}}_{g,n}, \left\lfloor m\left(K_{\overline{\mathcal{M}}_{g,n}} + \frac{9}{11}\delta + \frac{2}{11}\psi\right)\right\rfloor\right)$$

for all m divisible by the cardinality of all inertia groups of $\overline{\mathcal{M}}_{g,n}$ (see also [33, Prop. A.13]).

Now observe that by the modular description of Υ, an integral curve of $\overline{\mathrm{M}}_{g,n}$ lies on a closed fibre of Υ if and only if its class lies in $\mathbb{R}_{\geq 0} \cdot C_{\mathrm{ell}}$. Moreover, Υ is a contraction by the Zariski main theorem, since it is a proper morphism between irreducible normal algebraic spaces, which is moreover birational (being an isomorphism when restricted to the dense open subset of smooth curves). Therefore, using the rigidity lemma (Lemma 2.1), we get an isomorphism $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \cong \overline{\mathrm{M}}_{g,n}\left(\frac{9}{11}\right)$ under which Υ gets identified to π.

Using Mumford's formula $K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda - 2\delta + \psi$ and the formulae from [29, Thm. 2.1], we compute

$$C_{\mathrm{ell}} \cdot K_{\overline{\mathcal{M}}_{g,n}} = C_{\mathrm{ell}} \cdot \left(K_{\overline{\mathcal{M}}_{g,n}} + \psi\right) = -9.$$

If $(g,n) \neq (1,2),(2,1),(3,0)$, then we have $K_{\overline{M}_{g,n}} = K_{\overline{\mathcal{M}}_{g,n}} - \delta_{1,\emptyset}$ by [15, Chap. XII, Cor. 7.16], and then, again using the formulae from [29, Thm. 2.1], we compute

$$C_{\mathrm{ell}} \cdot K_{\overline{M}_{g,n}} = C_{\mathrm{ell}} \cdot \left(K_{\overline{M}_{g,n}} + \psi \right) = -8.$$

In these exceptional cases, we have $K_{\overline{M}_{g,n}} = K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} - \delta_{1,\emptyset} - R$, with R the ramification divisor of the morphism $\phi : \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ not entirely contained in the boundary of $\overline{M}_{g,n}$. We can choose the curve C_{ell} (in its numerical equivalence class) in such a way that the automorphism group of its generic point is generated by the elliptic involution along the elliptic tail, which implies that C_{ell} is not contained in R. This ensures that C_{ell} intersects R nonnegatively and hence negatively intersects $K_{\overline{M}_{g,n}^{\mathrm{ps}}}$ and $K_{\overline{M}_{g,n}^{\mathrm{ps}}} + \psi$.

Finally, the exceptional locus of Υ contains $\Delta_{1,\emptyset}$, since the curves numerically equivalent to C_{ell} cover $\Delta_{1,\emptyset}$. On the other hand, since $\delta_{1,\emptyset} \cdot C_{\mathrm{ell}} = -1 < 0$ by [29, Thm. 2.1], any curve numerically equivalent to C_{ell} is contained in $\Delta_{1,\emptyset}$. Therefore the exceptional locus of Υ is equal to $\Delta_{1,\emptyset}$, and hence Υ is a divisorial contraction. This concludes the proof of (i) and (ii).

In order to prove part (iii), observe that since the exceptional locus of Υ is equal to $\Delta_{1,\emptyset}$, the pullback of a $\mathbb{Q}$-line bundle L on $\overline{M}_{g,n}^{\mathrm{ps}}$ is equal to $L + \alpha(L)\delta_{1,\emptyset}$ for some $\alpha(L) \in \mathbb{Q}$. The rational number $\alpha(L)$ is uniquely determined by imposing $C_{\mathrm{ell}} \cdot \Upsilon^*(L) = \Upsilon_*(C_{\mathrm{ell}}) \cdot L = 0$ (because C_{ell} is contracted by Υ), and can be computed using [29, Thm. 2.1]. $\qquad\square$

Remark 5.6. Some parts of this proposition are true also for $(g,n) = (2,0)$. More specifically, Hyeon and Lee construct in [36, Sec. 4] (see also [32, Prop. 4.2]) a contraction $\Upsilon : \overline{M}_2 \to \overline{M}_2^{\mathrm{ps}}$ which contracts $\Delta_{1,\emptyset}$ (even though Υ does not come from a morphism between the corresponding stacks). Moreover, we have the identification $\overline{M}_2^{\mathrm{ps}} \cong \overline{M}_2 \left(\dfrac{9}{11} \right)$, as it follows by combining [32, Thm. 4.10] and [36, Thm. 4.2]. Finally, the proof of (iii) extends verbatim to the case where $(g,n) = (2,0)$.

Remark 5.7. In characteristic 0, the morphism Υ admits another description.

Indeed, from the two open embeddings of Fact 3.10, passing to their good moduli spaces (in $\mathrm{char}(k) = 0$), we get the following proper birational morphisms between normal proper algebraic spaces (see [8, Thm. 1.1] for $\alpha_c = 9/11$):

$$\overline{M}_{g,n} \xrightarrow{\ j_1^+\ } \overline{M}_{g,n}(9/11) \xleftarrow{\ j_1^-\ } \overline{M}_{g,n}^{\mathrm{ps}} = \overline{M}_{g,n}(9/11 - \epsilon) \ .$$

By [9, Thm. 2.2], the morphism j_1^+ (resp., j_1^-) is defined on geometric points by sending a stable (resp., pseudostable) curve into the curve which is obtained by replacing each elliptic tail (resp., cusp) by a cuspidal elliptic tail. Since cusps do not have local moduli, the map j_1^- is bijective on geometric points, and hence, being proper and birational between normal algebraic spaces, it is an isomorphism by Zariski's main theorem. Comparing these descriptions of j_1^+ and j_1^- on geometric points and the description of Υ contained in Proposition 3.11(i), we deduce that

$$\Upsilon = \left(j_1^- \right)^{-1} \circ j_1^+.$$

We now study the elliptic bridge curves in $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ introduced in Definition 1.1. Let us first determine their intersections with the $\mathbb{Q}$-line bundles on $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ (or on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$).

Lemma 5.8. *Assume that* $\mathrm{char}(k) \neq 2,3$. *Given a $\mathbb{Q}$-line bundle* $L = a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{[i,I] \in T_{g,n}^* - \{[1,\emptyset]\}} b_{i,I}\delta_{i,I}$ *in* $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, *we have the intersection formulas*

$$\begin{cases} C(\mathrm{irr}) \cdot L = a + 10b_{\mathrm{irr}}, \\ C([\tau,I],[\tau+1,I]) \cdot L = a + 12b_{\mathrm{irr}} - b_{\tau,I} - b_{\tau+1,I}. \end{cases}$$

Proof. We can compute the intersection on the moduli space $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$. The curves $C(\mathrm{irr})$ and $C([\tau,I],[\tau+1,I])$ in $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ are push-forwards via Υ of irreducible curves $\widetilde{C}(\mathrm{irr})$ and $\widetilde{C}([\tau,I],[\tau+1,I])$ in $\overline{\mathrm{M}}_{g,n}$ that are defined in the same way. Therefore, by the projection formula, we have

$$\begin{cases} C(\mathrm{irr}) \cdot L = \widetilde{C}(\mathrm{irr}) \cdot \Upsilon^*(L), \\ C([\tau,I],[\tau+1,I]) \cdot L = \widetilde{C}([\tau,I],[\tau+1,I]) \cdot \Upsilon^*(L). \end{cases} \tag{5.2}$$

Now, Proposition 5.5(iii) gives

$$\Upsilon^*(L) = a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + (a + 12b_{\mathrm{irr}})\delta_{1,\emptyset} + \sum_{[i,I] \in T_{g,n}^* - \{[1,\emptyset]\}} b_{i,I}\delta_{i,I}. \tag{5.3}$$

Finally, observe that the curve $\widetilde{C}(\mathrm{irr})$ coincides with the curve of [29, Thm. 2.2(4)] for $(i,I) = (0,\emptyset)$, while the curve $\widetilde{C}([\tau,I],[\tau+1,I])$ coincides with the curve of [29, Thm. 2.2(5)] for $(i,I) = (\tau,I)$ and $(j,J) = (g-1-\tau,I^c)$. Hence, using [29, Thm. 2.1], we have

$$\begin{cases} \widetilde{C}(\mathrm{irr}) \cdot \left(a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \displaystyle\sum_{[i,I] \in T_{g,n} - \{\mathrm{irr}\}} b_{i,I}\delta_{i,I} \right) = -2b_{\mathrm{irr}} + b_{1,\emptyset}, \\ \widetilde{C}([\tau,I],[\tau+1,I]) \cdot \left(a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \displaystyle\sum_{[i,I] \in T_{g,n} - \{\mathrm{irr}\}} b_{i,I}\delta_{i,I} \right) = -b_{\tau,I} - b_{\tau+1,I} + b_{1,\emptyset}. \end{cases} \tag{5.4}$$

We conclude by putting together equations (5.2), (5.3) and (5.4). $\square$

Now we look at the subcone of the Mori cone $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ spanned by the elliptic bridge curves.

Proposition 5.9. *Assume that* $\mathrm{char}(k) \neq 2,3$.

(i) *The elliptic bridge curves are linearly independent in* $N_1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ *and they intersect* $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}}$, $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi$, $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$ *and* $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} + \psi$ *negatively.*

(ii) *The convex cone spanned by elliptic bridge curves is a face of the Mori cone* $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ *(which we call the* elliptic bridge face*). In particular, each elliptic bridge curve generates an extremal ray of the Mori cone of* $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$.

(iii) *If $(g,n) \neq (1,2),(2,0)$, then a curve $B \subset \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is such that its class in $N_1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ lies in the elliptic bridge face if and only if the only nonisotrivial components of the corresponding family of pseudostable curves $\mathcal{C} \to B$ are A_1/A_1-attached elliptic bridges.*

Note that part (i) implies that the elliptic bridge face is polyhedral and simplicial. Observe also that part (iii) is false for $(g,n) = (1,2)$ (resp., $(2,0)$): in these two cases, $\dim N_1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} = 1$ and the elliptic bridge face, which is spanned by $C([0,\{1\}],[0,\{2\}])$ (resp., $C(\mathrm{irr})$), coincides with the entire Mori cone $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ and it is therefore a half-line. Hence, the class of any effective curve on $\overline{\mathrm{M}}_{g,n}$ lies in the elliptic bridge face, and there are plenty of effective curves in the projective varieties $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$.

Proof. Part (i): the fact that the elliptic bridge curves are linearly independent in $N_1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ follows by a close inspection of the intersection formulas in Lemma 5.8 using the relations among the generators of $\mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}}$ (see Fact 3.28(3.28), Corollary 3.29 and Proposition 5.1(ii)).

The fact that the elliptic bridge curves negatively intersect $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}}$ and $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi$ follows again from Lemma 5.8 and Mumford's formula $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} = 13\lambda - 2\delta + \psi$ (see Fact 3.28(3.28)). This implies the analogous result for $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$ and $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} + \psi$ if $(g,n) \neq (1,2),(2,0),(2,1),(3,0)$, by Proposition 5.1(iii). In the four exceptional cases mentioned, we have $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} = K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} - R$, with R being the ramification divisor of the morphism $\phi^{\mathrm{ps}} : \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ by Remark 5.4. We can choose the elliptic bridge curves (in their numerical equivalence class) in such a way that their generic point does not have nontrivial automorphisms, which implies that they are not contained in R. This ensures that the elliptic bridge curves intersect R nonnegatively, and hence they negatively intersect $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$ and $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} + \psi$.

Let us now prove parts (ii) and (iii). If $(g,n) = (1,2)$ or $(2,0)$, then $\dim N_1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right) = 1$ and part (ii) is obvious (while part (iii) is clearly false!).

Otherwise, consider the $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$

$$N_{g,n} := K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \frac{7}{10}\delta + \frac{3}{10}\psi = \frac{13}{10}(10\lambda - \delta + \psi).$$

By [7, Thm. 1.2(a)] (whose proof works in arbitrary characteristics and can be applied, since $(g,n) \neq (1,2),(2,0)$),[6] the line bundle $N_{g,n}$ is nef and as degree 0 precisely on the curves of $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ described in part (iii). Note that such curves are numerically equivalent to a nonnegative linear combination of elliptic bridge curves in $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ (since $\overline{\mathrm{M}}_{1,2}^{\mathrm{ps}}$ has

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[6]Note that in that theorem, not only $(g,n) = (2,0)$ but also $(g,n) = (1,2)$ must be excluded. The reason is that these are the only two cases where the line bundle $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \frac{7}{10}\delta + \frac{3}{10}\psi$, which is proportional to $10\lambda - \delta + \psi = 10\lambda - \widehat{\delta}$, is zero on $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$.

Picard number 1 by Corollary 3.29 and Proposition 5.1(ii)) and every elliptic bridge curve intersects $N_{g,n}$ in 0 by Lemma 5.8.

Moreover, we claim that $N_{g,n}$ is semiample on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$. Indeed, when $n = 0$, $N_{g,0}$ is the pullback of the natural polarisation on the GIT quotient $\overline{\mathrm{M}}_g^c$ of the Chow variety of bicanonical curves of genus g via a regular morphism $\Psi : \overline{\mathrm{M}}_g^{\mathrm{ps}} \to \overline{\mathrm{M}}_g^c$ (see [34, Thm. 2.13] and [34, Thm. 3.1], whose proofs work in arbitrary characteristic). When $n > 0$, fixing an integer $h \geq 2$ we have that $N_{g,n}$ is the pullback of $N_{g+nh,0}$ via the regular morphism $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g+nh}^{\mathrm{ps}}$ that attaches a fixed smooth irreducible curve of genus h to each of the marked points of an n-pointed stable curve of genus g (see [15, Lemma (4.38)] and [7, Sec. 5.4]).

These facts imply that if we denote by η the fibration induced by a sufficiently high power of $N_{g,n}$, the convex cone spanned by the elliptic bridge curves coincides with the η-relative effective cone $\mathrm{NE}(\eta)$ of curves and is therefore a face of the effective cone $\mathrm{NE}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ of curves (see subsection 2.2). Moreover, property (iii) holds.

It remains to show that the convex cone spanned by the elliptic bridge curves is also a face of the Mori cone $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$. However, this convex cone, which coincides with $\mathrm{NE}(\eta)$, is polyhedral (because it is generated by a finite number of curves) and hence closed. Since the closure of $\mathrm{NE}(\eta)$ is equal to the π-relative Mori cone $\overline{\mathrm{NE}}(\eta)$ (see subsection 2.2), we deduce that the convex cone spanned by the elliptic bridge curves is equal to $\overline{\mathrm{NE}}(\eta)$ and hence is a face of $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$. □

Remark 5.10. Assume that $g \geq 1$ (to avoid trivialities, since for $g = 0$ there are no elliptic bridge curves).

The dimension of the elliptic bridge face, which is equal to the number of elliptic bridge curves, is equal to

$$\dim(\text{elliptic bridge face}) = \begin{cases} 1 & \text{if } (g,n) = (2,0), \\ \frac{g-1}{2} & \text{if } n = 0 \text{ and } g \geq 3 \text{ is odd}, \\ \frac{g}{2} - 1 & \text{if } n = 0 \text{ and } g \geq 4 \text{ is even}, \\ g2^{n-1} - 1 & \text{if } g \geq 1 \text{ and } n \geq 1. \end{cases}$$

Comparing it with the Picard number of $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$, which can be obtained from Fact 3.28(3.28), Corollary 3.29 and Proposition 5.1(ii), we get

$$\mathrm{codim}(\text{elliptic bridge face}) = \begin{cases} 0 & \text{if } (g,n) = (2,0), \\ 1 & \text{if } n = 0 \text{ and } g \geq 3 \text{ is odd}, \\ 2 & \text{if } n = 0 \text{ and } g \geq 4 \text{ is even}, \\ 2^{n-1} + 1 - \delta_{2,g} - (n+1)\delta_{1,g} & \text{if } g \geq 1 \text{ and } n \geq 1, \end{cases}$$

where $\delta_{2,g}$ and $\delta_{1,g}$ are the Kronecker symbols.

The subfaces of the elliptic bridge face can be described as follows:

Definition 5.11 (T-faces). For any $T \subseteq T_{g,n}$, we denote by F_T the cone in $N_1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ generated by the classes of elliptic bridge curves of type contained in T. We will call F_T the *T-face* of the Mori cone.

The poset of T-faces is described by the following result, where we use the terminology of Definition 3.21:

Lemma 5.12. *Assume that* $\mathrm{char}(k) \neq 2, 3$.

(i) *For any* $T \subseteq T_{g,n}$, *the cone* F_T *is a simplicial polyhedral face of the Mori cone* $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ *whose dimension is equal to the number of minimal subsets of* $T_{g,n}$ *contained in* T. *In particular, the extremal rays of the elliptic bridge face are given by* $\{F_T : T \text{ is minimal}\}$.

(ii) *If* $(g,n) \neq (1,2), (2,0)$, *then a curve* $B \subset \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ *is such that its class in* $N_1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ *lies in* F_T *if and only if the only nonisotrivial components of the corresponding family of pseudostable curves* $\mathcal{C} \to B$ *are* A_1/A_1-*attached elliptic bridges of type contained in* T.

(iii) *We have* $F_T \subseteq F_S$ *if and only if* $T^{\mathrm{adm}} \subseteq S^{\mathrm{adm}}$. *In particular, we have* $F_T = F_S$ *if and only if* $T^{\mathrm{adm}} = S^{\mathrm{adm}}$.

Proof. Part (i): the cone F_T is a face of the elliptic bridge face of $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$, which is a simplicial polyhedral face of the Mori cone $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ whose extremal rays are generated by the elliptic bridge curves (by Proposition 5.9). Hence F_T is a simplicial polyhedral face of the Mori cone $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ whose extremal rays are generated by the elliptic bridge curves of type contained in T. We conclude because the elliptic bridge curves correspond to the minimal subsets of $T_{g,n}$. Part (ii) follows from Proposition 5.9(iii) and the fact that F_T is a face of the elliptic bridge face. Part (iii): by part (i), we have $F_T \subseteq F_S$ if and only if every minimal subset of $T_{g,n}$ contained in T is also contained in S, and this is equivalent to the inclusion $T^{\mathrm{adm}} \subseteq S^{\mathrm{adm}}$. $\square$

6. The moduli space of T-semistable curves

The aim of this section is to study the geometric properties of the moduli space $\overline{\mathrm{M}}_{g,n}^T$ of T-semistable curves and of the morphism $f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^T$. Throughout this section, we assume that $\mathrm{char}(k) \gg (g,n)$ (see Definition 4.1), which is needed for the existence of the good moduli space $\overline{\mathrm{M}}_{g,n}^T$. The main result of this section says that, in characteristic 0, the morphism f_T is the contraction of the T-face F_T (see Definition 5.11) of the Mori cone $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_g^{\mathrm{ps}}\right)$.

Theorem 6.1. *Set* $T \subseteq T_{g,n}$ *with* $(g,n) \neq (2,0)$. *Assume* $\mathrm{char}(k) = 0$. *The good moduli space* $\overline{\mathrm{M}}_{g,n}^T$ *is projective, and the morphism* $f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^T$ *is the contraction of the face* F_T. *Moreover,* f_T *is a* $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-*negative contraction.*

The theorem is trivially true in the following cases:

- If $T^{\mathrm{adm}} = \emptyset$ (which is always the case for $g = 0$ or $(g,n) = (1,1)$), then f_T is the identity by Remark 3.17. On the other hand, $F_T = (0)$, and hence γ_T is also the identity.
- If $(g,n) = (1,2)$ and $T^{\mathrm{adm}} \neq \emptyset$ (in which case it must be true that $T^{\mathrm{adm}} = \{[0,\{1\}],[1,\{1\}]\}$), then $f_T : \overline{\mathrm{M}}_{1,2}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{1,2}^{T} = \operatorname{Spec} k$ by Remark 4.5. On the other hand, $F_T = \overline{\mathrm{NE}}\left(\overline{\mathcal{M}}_{1,2}^{\mathrm{ps}}\right)$ (see the discussion following Proposition 5.9), so that the contraction γ_T of F_T is the map to $\operatorname{Spec} k$.

Before proving this theorem, we will need a description of the fibres of f_T.

Proposition 6.2. *Set $T \subseteq T_{g,n}$ with $(g,n) \neq (2,0)$ and $\operatorname{char}(k) \gg (g,n)$.*

(i) *The projective morphism f_T is a contraction – that is, $(f_T)_* \left(\mathcal{O}_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}} \right) = \mathcal{O}_{\overline{\mathrm{M}}_{g,n}^{T}}$.*

(ii) *Let B an integral curve inside $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ with associated family of pseudostable curves $\mathcal{C} \to B$, and let C be the image of B inside $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$. Then C is contracted by f_T if and only if the only nonisotrivial components of the family $\mathcal{C}$ are A_1/A_1-attached elliptic bridges of type contained in T.*

(iii) *The exceptional locus of f_T is the union of the irreducible closed subsets*

$$\mathrm{Ell}([\tau,I],[\tau+1,I]) := \overline{\{(C,\{p_i\}) \in \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \text{ having an elliptic bridge of type } \{[\tau,I],[\tau+1,I]\}\}}$$

for every $\{[\tau,I],[\tau+1,I]\} \subseteq T - \{[1,\emptyset]\}$ and

$$\mathrm{Ell}(\mathrm{irr}) := \overline{\{(C,\{p_i\}) \in \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \text{ having an elliptic bridge of type } \{\mathrm{irr}\}\}} \quad \text{if } \mathrm{irr} \in T \text{ and } g \geq 2.$$

Moreover, if $(g,n) \neq (1,2)$, then all these closed subsets have codimension 2 except $\mathrm{Ell}([0,\{i\}],[1,\{i\}])$, which coincides with the divisors $\Delta_{1,\{i\}}$ (for any $1 \leq i \leq n$). In particular, f_T is always birational, and it is small if and only if T does not contain any subset of the form $\{[0,\{i\}],[1,\{i\}]\}$ for some $1 \leq i \leq n$.

Note that the closed subsets $\mathrm{Ell}([\tau,I],[\tau+1,I])$ (resp., $\mathrm{Ell}(\mathrm{irr})$) are covered by the elliptic bridge curves $C([\tau,I],[\tau+1,I])$ (resp., $C(\mathrm{irr})$). Hence part (iii) is a necessary condition to f_T being the contraction of the face F_T. When $(g,n) = (1,2)$ and $T^{\mathrm{adm}} = \{[0,\{1\}],[1,\{1\}]\}$, the morphism f_T is the map to a point and its exceptional locus is equal to $\mathrm{Ell}([0,\{1\}],[1,\{1\}]) = \overline{\mathrm{M}}_{1,2}^{\mathrm{ps}}$.

Proof. Part (i) follows from the Zariski main theorem using the fact that f_T is a proper morphism between irreducible normal algebraic spaces (see Theorem 4.4), which is moreover birational because it is an isomorphism when restricted to the dense open subset of smooth curves.

Let us now prove parts (ii) and (iii). By Proposition 3.24(i), the morphism f_T sends a pseudostable curve $(C,\{p_i\})$ into the T-closed curve $f_T((C,\{p_i\}))$ which is obtained from $(C,\{p_i\})$ by replacing each A_1/A_1-attached elliptic bridge of type contained in T by an attached rosary of length 2. The type of any A_1/A_1-attached elliptic bridge of $(C,\{p_i\})$ can be equal to $\{\mathrm{irr}\}$ if $\mathrm{irr} \in T$ and $g \geq 2$, or $\{[\tau,I],[\tau+1,I]\}$ if $\{[\tau,I],[\tau+1,I]\} \subseteq$

$T - \{[1,\emptyset]\}$ (because $(C, \{p_i\})$ does not have elliptic tails). This implies part (ii) and that the exceptional locus of f_T is equal to

$$\mathbb{E}_T := \bigcup_{\{[\tau,I],[\tau+1,I]\} \subseteq T - \{[1,\emptyset]\}} \mathrm{Ell}([\tau,I],[\tau+1,I]) \bigcup_{\substack{\mathrm{irr} \in T \\ g \geq 2}} \mathrm{Ell}(\mathrm{irr}).$$

We conclude by observing that the closed subsets $\mathrm{Ell}([\tau,I],[\tau+1,I])$ and $\mathrm{Ell}(\mathrm{irr})$ are irreducible of the stated codimension. $\qquad\square$

Proof.[Proof of Theorem 6.1] As observed after the statement of the theorem, we can assume that $(g,n) \neq (1,2)$, for otherwise the theorem is trivially true.

Since F_T is a $K_{\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}}$-negative face of $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}^{\mathrm{ps}}_g\right)$ and $\overline{\mathrm{M}}^{\mathrm{ps}}_g$ has klt singularities by Proposition 5.1(i), the cone theorem [42, Thm. 3.7(3)] implies that there is a $K_{\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}}$-negative contraction of F_T

$$\gamma_T : \overline{\mathrm{M}}^{\mathrm{ps}}_{g,n} \to \left(\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}\right)_{F_T}.$$

Therefore, the theorem will follow from Lemma 2.1 if we show that an integral curve $C \subset \overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$ is contracted by f_T if and only if its class $[C]$ belongs to F_T.

In order to prove this, fix an integral curve $C \subset \overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$ and observe that since $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ has finite inertia by Proposition 3.11, the curve C admits a finite cover that lifts to $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$. Hence we can find an integral curve $B \subset \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$, with associated family of pseudostable curves $\mathcal{C} \to B$, whose image in $\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$ is the curve C. Now, Proposition 6.2(ii) says that C is contracted by f_T if and only if the only nonisotrivial components of the family $\mathcal{C}$ are A_1/A_1-attached elliptic bridges of type contained in T, which is equivalent to the fact that $[C]$ belongs to F_T by Lemma 5.12(ii). $\qquad\square$

As a corollary of Theorem 6.1 and some facts that are implicit in the proof of the cone theorem, we can describe the Néron–Severi group of $\overline{\mathrm{M}}^T_{g,n}$ and its nef/ample cone. We will need the following definition, where we freely identify the rational Picard groups of $\overline{\mathcal{M}}^T_{g,n}$, $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ and $\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$, using Corollary 3.29 and Proposition 5.1(ii):

Definition 6.3. A $\mathbb{Q}$-line bundle L on $\overline{\mathcal{M}}^T_{g,n}$ (or equivalently on $\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$ or on $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$) is said to be *T-compatible* if L intersects to zero all the elliptic bridge curves of type contained in T.

Explicitly, using Lemma 5.8, a $\mathbb{Q}$-line bundle

$$L = a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{[i,I] \in T_{g,n} - \{[1,\emptyset], \mathrm{irr}\}} b_{i,I}\delta_{i,I} \in \mathrm{Pic}\left(\overline{\mathcal{M}}^T_{g,n}\right)_{\mathbb{Q}}$$

is T-compatible if and only if

$$\begin{cases} a + 10b_{\mathrm{irr}} = 0 \text{ if } \mathrm{irr} \in T, \\ a + 12b_{\mathrm{irr}} - b_{\tau,I} - b_{\tau+1,I} = 0 \text{ for any } \{[\tau,I],[\tau+1,I]\} \subset T. \end{cases} \qquad (6.1)$$

Corollary 6.4. *Set $T \subseteq T_{g,n}$ with $(g,n) \neq (2,0)$. Assume that $\mathrm{char}(k) = 0$. Then the following are true:*

(i) *The real Néron–Severi vector space $N^1\left(\overline{\mathcal{M}}_{g,n}^T\right)_{\mathbb{R}}$ can be identified, via pullback along f_T, with the annihilator subspace $F_T^{\perp} \subset N^1\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{R}}$. This implies that a $\mathbb{Q}$-line bundle L on $\overline{\mathcal{M}}_{g,n}^T$ descends to a (necessarily unique) $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^T$ (which we will denote by L^T) if and only if L is T-compatible.*

(ii) *The nef (resp., ample) cone of $\overline{\mathrm{M}}_{g,n}^T$ can be identified, via pullback along f_T, with the dual face $F_T^{\vee} := F_T^{\perp} \cap \mathrm{Nef}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$ of F_T (resp., the interior of $F_T^{\vee}$).*

In particular, F_T and $F_T^{\vee}$ are perfect dual faces – that is, $\mathrm{codim}\, F_T = \dim F_T^{\vee}$ – and hence they are exposed faces (that is, they admit supporting hyperplanes). Moreover, every $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ whose class lies in the interior of $F_T^{\vee}$ defines a supporting hyperplane for F_T and is semiample with associated contraction equal to f_T.

In [20, Prop. 3.13], we will prove that the second assertion of part (i) holds true if $\mathrm{char}(k) \gg (g,n)$, arguing similarly to Proposition 7.7.

Proof. Since f_T is the contraction of the $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-negative face F_T by Theorem 6.1, it follows from [42, Thm. 3.7(4)] that $F_T^{\perp}$ is the pullback via f_T of $N^1\left(\overline{\mathrm{M}}_{g,n}^T\right)_{\mathbb{R}}$, which proves the first statement in (i). The second statement follows from the first one, the left part of commutative diagram (4.1) and Proposition 5.1(ii).

Next, since F_T is a $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-negative face of $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)$, it follows from step 6 of the proof of [42, Thm. 3.15] that F_T is an exposed face. Hence any $\mathbb{Q}$-line bundle L which is in the relative interior of $F_T^{\vee}$ is a supporting hyperplane for F_T, and conversely. Moreover, it follows from the base-point-free theorem (see step 7 of the proof of [42, Thm. 3.15]) that any $\mathbb{Q}$-line bundle L which is a supporting hyperplane for F_T is semiample, and the morphism associated to $|mL|$ (for $m \gg 0$) is f_T. In particular, it follows that the relative interior of $F_T^{\vee}$ is the pullback via f_T of the ample cone of $\overline{\mathrm{M}}_{g,n}^T$, and by taking the closures, we get that $F_T^{\vee}$ is the pullback via f_T of the nef cone of $\overline{\mathrm{M}}_{g,n}^T$, which proves (ii).

Finally, the last part of Corollary 6.4 follows from what has already been proven and the equalities

$$\mathrm{codim}\, F_T = \dim N^1\left(\overline{\mathrm{M}}_{g,n}^T\right)_{\mathbb{R}} = \dim F_T^{\vee},$$

where we have used [24, Rmk. 7.40] for the first equality and the fact that the nef cone is a full-dimensional cone in the real Néron–Severi vector space for the second one. $\square$

Note that the characteristic 0 assumption is used in the proof of Theorem 6.1 only to establish the projectivity of $\overline{\mathrm{M}}_{g,n}^T$. There is a special case, however, where we can prove the projectivity in arbitrary characteristic (provided that it is large enough so that $\overline{\mathrm{M}}_{g,n}^T$ exists).

Example 6.5. If $T = T_{g,n}$ (and $(g,n) \neq (2,0)$), then Theorem 6.1 is true for char$(k) \gg (g,n)$ and can be proved as it follows. From the proof of Proposition 5.9, it follows that the $\mathbb{Q}$-line bundle on $\overline{M}_{g,n}^{ps}$

$$N_{g,n} := K_{\overline{\mathcal{M}}_{g,n}^{ps}} + \frac{7}{10}\delta + \frac{3}{10}\psi = \frac{13}{10}(10\lambda - \delta + \psi)$$

is semiample and its dual face in NE $\left(\overline{\mathcal{M}}_{g,n}^{ps}\right)$ is the elliptic bridge face (note that this is true also for $(g,n) = (1,2)$, in which case $N_{1,2} = 0$ and the elliptic bridge face coincides with the entire effective cone of curves of $\overline{M}_{1,2}^{ps}$). Hence a sufficiently high multiple of $N_{g,n}$ induces a morphism

$$\psi : \overline{M}_{g,n}^{ps} \to \text{Proj} \bigoplus_{m \geq 0} H^0\left(\overline{M}_{g,n}^{ps}, \lfloor mN_{g,n} \rfloor\right),$$

which is the contraction of the elliptic bridge face and whose codomain coincides with $\overline{M}_{g,n}\left(\frac{7}{10}\right)$ by [7, Prop. 7.2]. Since the $f_{T_{g,n}}$-relative effective cone NE $\left(f_{T_{g,n}}\right)$ of curves is equal to the elliptic bridge face (see Proposition 6.2(ii)), Lemma 2.1 implies that we have an isomorphism

$$\overline{M}_{g,n}^{T_{g,n}} \cong \overline{M}_{g,n}(7/10), \tag{6.2}$$

under which $f_{T_{g,n}}$ gets identified with ψ. Note that formula (6.2) is a special case (if char$(k) = 0$) of [7, Thm. 1.1], and it was previously proved by Hassett and Hyeon [34] for $n = 0$.

From the foregoing discussion and Remark 5.10, we can compute the Picard number of $\overline{M}_{g,n}^{T_{g,n}} \cong \overline{M}_{g,n}(\frac{7}{10})$ and the relative Picard number of $f_{T_{g,n}}$ (assuming that $g \geq 1$, for otherwise we have that $\overline{M}_{0,n}(\frac{7}{10}) = \overline{M}_{0,n}$):

(1) The Picard number of $\overline{M}_{g,n}(\frac{7}{10})$ is equal to

$$\dim_{\mathbb{Q}} \text{Pic}\left(\overline{M}_{g,n}\left(\frac{7}{10}\right)\right)_{\mathbb{Q}} = \begin{cases} 1 & \text{if } n = 0 \text{ and } g \geq 3 \text{ is odd,} \\ 2 & \text{if } n = 0 \text{ and } g \geq 4 \text{ is even,} \\ 2^{n-1} + 1 - \delta_{2,g} - (n+1)\delta_{1,g} & \text{if } g \geq 1 \text{ and } n \geq 1. \end{cases}$$

(2) The relative Picard number of $f_{T_{g,n}}$ is equal to

$$\rho\left(f_{T_{g,n}}\right) = \begin{cases} \frac{g-1}{2} & \text{if } n = 0 \text{ and } g \geq 3 \text{ is odd,} \\ \frac{g}{2} - 1 & \text{if } n = 0 \text{ and } g \geq 4 \text{ is even,} \\ g2^{n-1} - 1 & \text{if } g \geq 1 \text{ and } n \geq 1. \end{cases}$$

In [20], we study several geometric properties of the space $\overline{M}_{g,n}^{T}$ and the morphism f_T. For completeness, we mention those results here. We will need the following definition:

Definition 6.6. Given a subset $T \subseteq T_{g,n}$, we define the *divisorial part* of T as the (possibly empty) subset $T^{\mathrm{div}} \subset T$ defined by

$$T^{\mathrm{div}} := \begin{cases} \emptyset & \text{if } (g,n) = (1,1) \text{ or } (2,1), \\ \{\{[0,\{i\}],[1,\{i\}]\} : \{[0,\{i\}],[1,\{i\}]\} \subset T\} & \text{otherwise.} \end{cases}$$

It is easily checked that T^{div} is admissible in the sense of Definition 3.21.

Proposition 6.7. *[[20, Prop. 3.16, 3.17]] Assume $(g,n) \neq (2,0)$, $\mathrm{char}(k) \gg (g,n)$ and $T \subseteq T_{g,n}$.*

(1) *The following conditions are equivalent:*
   (a) $\overline{\mathrm{M}}_{g,n}^T$ *is $\mathbb{Q}$-factorial.*
   (b) $\overline{\mathrm{M}}_{g,n}^T$ *is $\mathbb{Q}$-Gorenstein.*
   (c) $T^{\mathrm{adm}} = T^{\mathrm{div}}$.

(2) *The morphism $f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^T$ can be factorised as*

$$f_T \colon \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \xrightarrow{f_{T^{\mathrm{div}}}} \overline{\mathrm{M}}_{g,n}^{T^{\mathrm{div}}} \xrightarrow{\sigma_T} \overline{\mathrm{M}}_{g,n}^T \tag{6.3}$$

   in such a way that the following are true:
   (a) *The morphism $f_{T^{\mathrm{div}}}$ is a composition of $\frac{1}{2}\left|T^{\mathrm{div}}\right|$ divisorial contractions, each having the relative Mori cone generated by a K-negative extremal ray.*
   (b) *The algebraic space $\overline{\mathrm{M}}_{g,n}^{T^{\mathrm{div}}}$ is $\mathbb{Q}$-factorial and, if $\mathrm{char}(k) = 0$, klt.*
   (c) *The morphism σ_T is a small contraction.*
   (d) *The relative Mori cone of σ_T is a $K_{\overline{\mathrm{M}}_{g,n}^{T^{\mathrm{div}}}}$-negative face if and only if T does not contain subsets of the form $\{[0,\{j\}],[1,\{j\}],[2,\{j\}]\}$ for some $j \in [n]$ or $(g,n) = (3,1),(3,2),(2,2)$.*

Note that if $\mathrm{char}(k) = 0$, then all the spaces appearing in formula (6.3) are projective varieties, and hence $f_{T^{\mathrm{div}}}$ is the composition of divisorial contractions of K-negative rays, while σ_T is a small contraction of a K-negative face if and only if the condition on T appearing in (d) is satisfied.

7. The moduli space of T^+-semistable curves

The aim of this section (throughout which we assume that $\mathrm{char}(k) \gg (g,n)$; see Definition 4.1) is to describe the map $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T^+} \to \overline{\mathrm{M}}_{g,n}^T$ in terms of the minimal model program. In particular, we will describe f_T^+ as the flip of f_T with respect to suitable $\mathbb{Q}$-line bundles.

7.1. Preliminary definitions and results about flips

Definition 7.1. Let $f : X \to Y$ be a proper morphism between normal algebraic spaces of finite type over k and let D be an f-antiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. A *D-flip of*

f is a proper morphism $f_D^+ : X_D^+ \to Y$ of algebraic spaces fitting into the commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow^{\ \eta\ } & X_D^+ \\
& f \searrow \quad \swarrow f_D^+ & \\
& Y, &
\end{array}
\tag{7.1}
$$

where η is a rational map and such that:

(i) the algebraic space X_D^+ (which is automatically of finite type over k) is normal;

(ii) the morphism f_D^+ is a small contraction – that is, it is a contraction whose exceptional locus $\mathrm{Exc}\left(f_D^+\right)$ has codimension at least 2;

(iii) the $\mathbb{Q}$-divisor $D^+ := \eta_*(D)$ is $\mathbb{Q}$-Cartier and f_D^+-ample.

A D-flip is called *elementary* if f has relative Picard number 1.

The difference between Definition 7.1 and the classical definition of a flip is that we do not require the map f to be small.

Remark 7.2. Assume that f is birational. Then, since f_D^+ is small, we have that η^{-1} does not contract any divisor – that is, in the terminology of [18, Page 424] it is a birational contraction. Moreover, the map η is D-nonpositive in the sense of [18, Def. 3.6.1] and so η is the ample model of D over Y (see [18, Def. 3.6.5]).

In [14, Definition 11], a diagram analogous to diagram 7.1 is called an MMP-step.

We discuss the existence and uniqueness of flips in the following result. The proof is standard; we include it for completeness.

Lemma 7.3. *Let $f : X \to Y$ be a proper morphism of normal algebraic spaces of finite type over k and let D be an f-antiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(i) *If the D-flip of f exists, then it is given by*

$$
f_D^+ : X_D^+ = \mathrm{Proj} \bigoplus_{m \geq 0} \mathcal{O}_Y(\lfloor m f_*(D) \rfloor) \to Y.
\tag{7.2}
$$

In particular, the D-flip of f is unique.

Moreover, the D-flip depends only on the $\mathbb{Q}$-line bundle $L = \mathcal{O}_X(D)$ associated to D, and hence it will be denoted by $f_L : X_L^+ \to Y$ and called the L-flip of f.

(ii) *If $\mathrm{char}(k) = 0$, X is klt and K_X is f-antiample, then the coherent sheaf $\bigoplus_{m \geq 0} \mathcal{O}_Y(\lfloor m f_*(D) \rfloor)$ of $\mathcal{O}_Y$-algebras is finitely generated, and hence the D-flip of f exists.*

Proof. Part (i): suppose that the D-flip $f_D^+ : X_D^+ \to Y$ exists. Since D^+ is $\mathbb{Q}$-Cartier and f_D^+-ample, we have

$$
X_D^+ = \mathrm{Proj}_Y \bigoplus_{m \geq 0} \left(f_D^+\right)_* \left(\lfloor m D^+ \rfloor\right).
$$

Since X_D^+ is normal and the morphism f_D^+ is a small contraction, arguing as in the proof of [42, Lemma 6.2] and using the fact that $\left(f_D^+\right)_*(D^+) = \left(f_D^+\right)_*(\nu_*(D)) = f_*(D)$ because of the commutativity of diagram (7.1), we have the equality of $\mathcal{O}_Y$-algebras:

$$\bigoplus_{m \geq 0} \left(f_D^+\right)_* \left(\lfloor mD^+ \rfloor\right) = \bigoplus_{m \geq 0} \mathcal{O}_Y \left(\lfloor m\left(f_D^+\right)_*(D^+) \rfloor\right) = \bigoplus_{m \geq 0} \mathcal{O}_Y \left(\lfloor mf_*(D) \rfloor\right).$$

This concludes the proof of the first part of part (i). The second part follows from the fact that the push-forward of divisors respects the linear equivalence of divisors.

Part (ii): by [28, Corollary 4.5] there exists an effective $\mathbb{Q}$-divisor Δ on Y such that (Y, Δ) is klt. Hence we conclude by applying [41, Thm. 92], which is a consequence of [18] and says that the coherent sheaf $\bigoplus_{m \geq 0} \mathcal{O}_Y(\lfloor mf_*(D) \rfloor)$ of $\mathcal{O}_Y$-algebras is finitely generated. □

7.2. Main results about f_T^+ and $\overline{\mathcal{M}}_{g,n}^{T+}$

The following theorem, which is the main result of this section, describes the morphism f_T^+ as the flip of f_T with respect to suitable $\mathbb{Q}$-line bundles:

Theorem 7.4. *Assume* $(g,n) \neq (2,0), (1,2)$, $\mathrm{char}(k) \gg (g,n)$ *and* $T \subseteq T_{g,n}$. *Let* $L \in \mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}} = \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}\right)_{\mathbb{Q}}$. *Then* f_T^+ *is the* L-flip of f_T *if and only if* L *is* f_T-antiample *and the restriction of* L *to* $\overline{\mathcal{M}}_{g,n}^{T+}$ *is* T^+-compatible (see Definition 7.5).

The special cases $(g,n) = (1,2)$ and $(2,0)$ are discussed in Remark 4.5.

The proof of this theorem will be the outcome of several propositions that are interesting in their own. We first describe the rational Picard group of $\overline{\mathrm{M}}_{g,n}^{T+}$. Recall the description of the rational Picard group of $\overline{\mathcal{M}}_{g,n}^{T+}$ given in Corollary 3.29.

Definition 7.5. A $\mathbb{Q}$-line bundle on $\overline{\mathcal{M}}_{g,n}^{T+}$

$$L = a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{[i,I] \in T_{g,n} - \left\{[1,\emptyset], \bigcup_j [1,\{j\}], \mathrm{irr}\right\}} b_{i,I}\delta_{i,I} \tag{7.3}$$

is said to be T^+-*compatible* if $b_{\tau,I} = b_{\tau+2,I}$ for any pair $\{[\tau,I], [\tau+2,I]\} \subset T_{g,n}$ such that

$$\{[\tau,I], [\tau+1,I], [\tau+2,I]\} \subset T \text{ and } [\tau,I], [\tau+2,I] \notin \left\{[1,\emptyset], \bigcup_j [1,\{j\}]\right\}. \tag{7.4}$$

Remark 7.6. If a $\mathbb{Q}$-line bundle on $\overline{\mathcal{M}}_{g,n}^{T}$ is T-compatible (see Definition 6.3), then its restriction to $\overline{\mathcal{M}}_{g,n}^{T+}$ is T^+-compatible. This can be proven by direct inspection. Alternatively, it follows from the fact that T-compatible $\mathbb{Q}$-line bundles are exactly $\mathbb{Q}$-line bundles on $\overline{\mathrm{M}}_{g,n}^{T}$ by Corollary 6.4(i), while T^+ compatible $\mathbb{Q}$-line bundles are exactly the $\mathbb{Q}$-line bundles on $\overline{\mathrm{M}}_{g,n}^{T+}$ by Proposition 7.7, and we can pull back line bundles via the map $f_T^+ \colon \overline{\mathrm{M}}_{g,n}^{T} \to \overline{\mathrm{M}}_{g,n}^{T+}$.

Proposition 7.7. *Assume $(g,n) \neq (2,0),(1,2)$ and $\mathrm{char}(k) \gg (g,n)$. A $\mathbb{Q}$-line bundle L on $\overline{\mathcal{M}}_{g,n}^{T+}$ descends to a (necessarily unique) $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^{T+}$ (which we will denote by L^{T+}) if and only if L is T^+-compatible.*

Proof. Up to passisng to a multiple, it is enough to prove the statement for a line bundle on $\overline{\mathcal{M}}_{g,n}^{T+}$. Given such a line bundle L on $\overline{\mathcal{M}}_{g,n}^{T,+}$ and any one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{Aut}(C,\{p_i\})$ for some k-point $(C,\{p_i\}) \in \overline{\mathcal{M}}_{g,n}^{T+}(k)$, the group $\mathbb{G}_m$ will act via ρ onto the fibre $L_{(C,\{p_i\})}$ of the line bundle over $(C,\{p_i\})$ and we will denote by $\langle L, \rho \rangle \in \mathbb{Z}$ the weight of this action. According to [4, Theorem 10.3] applied to the good moduli space $\phi^{T,+} : \overline{\mathcal{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^{T+}$, the line bundle L descends to a $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^{T+}$ if and only if $\langle L, \rho \rangle = 0$ for any one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{Aut}(C,\{p_i\})$ of any closed k-point $(C,\{p_i\}) \in \overline{\mathcal{M}}_{g,n}^{T+}(k)$. We will now show that this is the case if and only if L is T^+-compatible.

To prove the 'if' implication, assume that L is T^+-compatible and fix a closed k-point $(C,\{p_i\})$ of $\overline{\mathcal{M}}_{g,n}^{T+}(k)$. By Proposition 3.27, either $(C,\{p_i\})$ is a closed rosary, and in this case the result follows from Lemma 7.8(ii), or it admits a T^+-canonical decomposition $C = K \cup \left(R_1, q_1^1, q_2^1\right) \cup \cdots \cup (R_r, q_1^r, q_2^r)$, where R_i is a rosary of length 3. In the second case, the connected component of the identity of $\mathrm{Aut}(C,\{p_i\})$ is isomorphic to $\Pi_{i=1}^r \mathrm{Aut}\left(R_i, q_1^i, q_2^i\right) \cong \mathbb{G}_m^{\times r}$, and hence it is enough to show that $\langle L, \rho_i \rangle = 0$ for $i = 1, \ldots, r$, where ρ_i is an isomorphism between $\mathbb{G}_m$ and $\mathrm{Aut}\left(R_i, q_1^i, q_2^i\right)$. The result now follows from Lemma 7.8(i).

To prove the converse direction, note that for each triple as in formula (7.4), there exists a T^+-closed curve with an attached rosary of length 3 and type $\{[\tau,I],[\tau+1,I],[\tau+2,I]\}$; denote by ρ the one-parameter subgroup associated to this rosary. The necessary condition $\langle L, \rho \rangle = 0$ implies, because of Lemma 7.8(i), that $b_{\tau,I} = b_{\tau+2,I}$. □

Lemma 7.8. *Assume that $\mathrm{char}(k) \neq 2$. Consider a line bundle L on $\overline{\mathcal{M}}_{g,n}^{T+}$ written as in equation (7.3).*

(i) *Let $(C,\{p_i\})$ be a k-point of $\overline{\mathcal{M}}_{g,n}^{T+}(k)$ that has an attached rosary (R,q_1,q_2) of length 3 and consider the one-parameter subgroup $\rho_R : \mathbb{G}_m \xrightarrow{\cong} \mathrm{Aut}((R,q_1,q_2))^o \subset \mathrm{Aut}((C,\{p_i\}))$ normalised so that $\mathrm{wt}_{\rho_R}(T_{q_1}(R)) = 1$. Then we have*

$$\langle L, \rho_R \rangle = \begin{cases} 0 & \text{if } \mathrm{type}(R,q_1,q_2) = \{\mathrm{irr}\}, \\ -b_{\tau,I} + b_{\tau+2,I} & \text{if } \mathrm{type}(R,q_1,q_2) = \{[\tau,I],[\tau+1,I],[\tau+2,I]\}. \end{cases}$$

(ii) *Let $R \in \overline{\mathcal{M}}_{r+1,0}^{T+}(k)$ be a closed rosary of even length r (which can occur only if $\mathrm{irr} \in T$) and consider the one-parameter subgroup $\rho_R : \mathbb{G}_m \xrightarrow{\cong} \mathrm{Aut}(R)^o$. Then we have*

$$\langle L, \rho_R \rangle = 0.$$

Proof. Let us first prove part (i). Since the weight is linear in L, the result will follow from the following identities:

$$
\begin{cases}
\langle \lambda, \rho_R \rangle = 0, \\
\langle \delta_{\text{irr}}, \rho_R \rangle = 0, \\
\langle \delta_{i,I}, \rho_R \rangle = \begin{cases}
-1 & \text{if } \text{type}(R, q_1, q_2) = \{[i,I], [i+1,I], [i+2,I]\}, \\
1 & \text{if } \text{type}(R, q_1, q_2) = \{[i-2,I], [i-1,I], [i,I]\}, \\
0 & \text{otherwise.}
\end{cases}
\end{cases}
\tag{7.5}
$$

These identities can be proved by adapting the computations in [6], as we now explain.

To compute the weights of the ψ classes, recall that the fibre of ψ_i over a pointed curve $(C, \{p_i\})$ is canonically isomorphic to the cotangent vector space $T_{p_i}(C)^\vee$. Hence, $\langle \psi_i, \rho_R \rangle$ is the weight of the action of $\mathbb{G}_m$, via the one-parameter subgroup ρ_R, on the 1-dimensional k-vector space $T_{p_i}(C)^\vee$. This is not trivial if and only if p_i is either q_1 or q_2, and it is computed in Remark 3.4.

To compute the other weights, we first make the following key remark. The $\mathbb{G}_m$-action on (R, q_1, q_2), which is explicitly described in Remark 3.4, is such that the weights of $\mathbb{G}_m$ on the coordinates (x_1, y_1) that define the first tacnode $t_1 := \{y_1^2 - x_1^4 = 0\}$ are opposite to the weights of $\mathbb{G}_m$ on the coordinates (x_2, y_2) that define the second tacnode $t_2 := \{y_2^2 - x_2^4 = 0\}$. This implies that the contributions that come from the two tacnodes cancel out.

In order to compute the other contributions, consider the formally smooth morphism

$$
\Phi : \text{Def}(C, \{p_i\}) \longrightarrow \text{Def}\left(\widehat{O}_{C,t_1}\right) \times \text{Def}\left(\widehat{O}_{C,t_2}\right) \times \prod_{q_i \text{ node}} \text{Def}\left(\widehat{O}_{C,q_i}\right)
$$

into the product of the (formal) semiuniversal deformation spaces of the two tacnodes a_1 and a_2 of R, and of nodes belonging to $\{q_1, q_2\}$. The group $\text{Aut}(R, q_1, q_2)^o \cong \mathbb{G}_m$ acts on these deformation spaces in such a way that the morphism Φ is equivariant.

Let us now write down explicitly the deformation spaces of the singularities mentioned, together with the action of $\mathbb{G}_m$, using the equation given in Remark 3.4. The semiuniversal deformation space of q_i (for $i = 1, 2$), whenever it is a node, is equal to $\text{Spf } k[b_i]$ and the semiuniversal deformation family is $n_i z_i = b_i$, where z_i is a local coordinate on the branch of the node q_i not belonging to R. The action of $\mathbb{G}_m$ is given by $t \cdot (b_i) = (t b_i)$. The locus of singular deformations of the node q_i is cut out by the equation $\{b_i = 0\}$, which has $\mathbb{G}_m$-weight 1.

On the other hand, the semiuniversal deformation space of the tacnode t_i is equal to $\text{Def}\left(\widehat{O}_{C,p}\right) \cong \text{Spf } k[a_2, a_1, a_0]$ and the semiuniversal deformation family is given by $y^2 = x^4 + a_2 x^2 + a_1 x + a_0$. This forces the action of $\mathbb{G}_m$ to be given by $t \cdot (a_2, a_1, a_0) = \left(t^{-2} a_2, t^{-3} a_1, t^{-4} a_0\right)$. The locus of singular deformations of p is cut out in $\text{Def}\left(\widehat{O}_{C,p}\right)$ by the equation $\{\Delta = 0\}$, where $\Delta := \Delta(a_2, a_1, a_0)$ is the discriminant of the polynomial $x^4 + a_2 x^2 + a_1 x + a_0$. Since the discriminant is a homogeneous polynomial of degree 12 in the roots of this polynomial and $\mathbb{G}_m$ acts on the roots with weight -1 (the same weight

as x), it follows that $\mathbb{G}_{\mathrm{m}}$ acts on the discriminant associated to t_1 with weights -12, and $+12$ on the discriminant associated to t_2.

If both points q_i are nodes, it follows from this discussion that the only boundary divisor of $\overline{\mathcal{M}}_{g,n}^T$ that can have a nonzero weight against ρ_R is the one whose equation on $\mathrm{Def}(C,\{p_i\})$ is given by $\Phi^*(b_1 b_2) = 0$. This divisor is $2\delta_{\mathrm{irr}}$ if $\mathrm{type}(R,q_1,q_2) = \mathrm{irr}$ and $\delta_{i,I} + \delta_{g-2-i,I^c}$ if $\mathrm{type}(R,q_1,q_2) = \{[i,I],[i+1,I],[i+2,I]\}$. The result now follows from [6, Lemma 3.11] and Remark 3.4. If one of the q_is is a node and the other is a marked point, the result follows by combining the foregoing discussion with argument about ψ-classes. When $(g,n) = (2,2)$, it could be that both q_is are marked points, in which the argument about ψ-classes is enough.

To compute the weight of λ, by combining [6, Cor. 3.3] and the computations in [6, Sec. 3.1.3] for A_3, we deduce that $\langle \lambda, \rho_R \rangle = 0$, as we get $+1$ from one tacnode and -1 from the other tacnode.

Part (ii) can be proven in a similar way – the key remark is that since the length of the rosary is even, all contributions cancel out. $\qquad\square$

As a corollary, we can now determine when $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-factorial or $\mathbb{Q}$-Gorenstein.

Corollary 7.9. *Assume* $(g,n) \neq (2,0),(1,2)$, $\mathrm{char}(k) \gg (g,n)$ *and* $T \subseteq T_{g,n}$. *Then the following are true:*

(i) *If* $(g,n) \neq (2,1)$ *or* $(3,0)$, *then the pullback of the (Weil) divisor* $K_{\overline{\mathrm{M}}_{g,n}^{T+}}$ *via the morphism* $\phi^{T+} : \overline{\mathcal{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^{T+}$ *is equal to*

$$\left(\phi^{T+}\right)^* \left(K_{\overline{\mathrm{M}}_{g,n}^{T+}}\right) = K_{\overline{\mathcal{M}}_{g,n}^{T+}} = 13\lambda - 2\delta + \psi. \tag{7.6}$$

(ii) $\overline{\mathrm{M}}_{g,n}^{T+}$ *is* $\mathbb{Q}$-*factorial if and only if* T *does not contain subsets of the form* $\{[\tau,I],[\tau+1,I],[\tau+2,I]\}$ *with* $[\tau,I],[\tau+2,I] \notin \left\{[1,\emptyset], \bigcup_j [1,\{j\}]\right\}$ *and* $[\tau,I] \neq [\tau+2,I]$.

(iii) $\overline{\mathrm{M}}_{g,n}^{T+}$ *is* $\mathbb{Q}$-*Gorenstein if and only if* T *does not contain subsets of the form* $\{[0,\{j\}],[1,\{j\}],[2,\{j\}]\}$ *for some* $j \in [n]$, *or* $(g,n) = (3,1),(3,2),(2,2)$.

Note the following special cases:

- If T^{adm} is minimal (in the sense of Definition 3.21) or $T^{\mathrm{adm}} = T^{\mathrm{div}}$ (see Definition 6.6), then $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-factorial.
- If $g = 1$, then $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-factorial for any $T \subseteq T_{1,n}$.
- If $n = 0$, then $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-Gorenstein for any $T \subseteq T_{g,0}$.

Proof. Part (i): under the assumptions on the pair (g,n), the morphism $\phi^{T+} : \overline{\mathcal{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^{T+}$ is an isomorphism in codimension 1 when restricted to the open substack $\mathcal{M}_{g,n}$ of smooth curves (see [15, Chap. XII, Prop. 2.15]). Moreover, the generic point in each boundary divisor of $\overline{\mathcal{M}}_{g,n}^{T+}$ does not have any nontrivial automorphisms and is T^+-closed (see Definition 3.26), and hence it is a closed point of the stack $\overline{\mathcal{M}}_{g,n}^{T+}$. This implies that the

morphism ϕ^{T+} is an isomorphism in codimension 1, which implies that $\left(\phi^{T+}\right)^* \left(K_{\overline{\mathrm{M}}_{g,n}^{T+}}\right) = K_{\overline{\mathcal{M}}_{g,n}^{T+}}$. We now conclude using Mumford's formula (see Fact 3.28(3.28)).

Part (ii): by the foregoing discussion, the morphism $\phi^{T+} : \overline{\mathcal{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^{T+}$ is an isomorphism in codimension 1. Hence the pullback map via the morphism ϕ^{T+} induces an isomorphism on the divisor class groups

$$\left(\phi^{T+}\right)^* : \mathrm{Cl}\left(\overline{\mathrm{M}}_{g,n}^{T+}\right)_{\mathbb{Q}} \xrightarrow{\cong} \mathrm{Cl}\left(\overline{\mathcal{M}}_{g,n}^{T+}\right)_{\mathbb{Q}} = \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}^{T}\right)_{\mathbb{Q}},$$

where in the last equality we used the fact that $\overline{\mathcal{M}}_{g,n}^{T}$ is a smooth stack. Hence, Proposition 7.7 implies that $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-factorial – that is, $\mathrm{Pic}\left(\overline{\mathrm{M}}_{g,n}^{T+}\right)_{\mathbb{Q}} = \mathrm{Cl}\left(\overline{\mathrm{M}}_{g,n}^{T+}\right)_{\mathbb{Q}}$ – if and only if any $\mathbb{Q}$-line bundle on $\overline{\mathcal{M}}_{g,n}^{T+}$ is T^+-compatible. An inspection of Definition 7.5 gives the result.

Part (iii): first of all, in the special cases $(g,n) = (2,1)$ or $(3,0)$, it is easy to check, using part (ii), that $\overline{\mathcal{M}}_{g,n}^{T}$ is $\mathbb{Q}$-factorial for any T. Hence we can assume that $(g,n) \neq (2,1)$ or $(3,0)$, which implies that equation (7.6) for $\left(\phi^{T+}\right)^* \left(K_{\overline{\mathrm{M}}_{g,n}^{T+}}\right)$ holds true. By Proposition 7.7, $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-Gorenstein if and only if

$$13\lambda - 2\delta + \psi = 13\lambda - 2\delta_{\mathrm{irr}} - 2 \sum_{[i,I] \notin \left\{[1,\emptyset], \bigcup_j [1,\{j\}], \bigcup_j [0,\{j\}]\right\}} \delta_{i,I} - \sum_{j=1}^{n} \delta_{0,\{j\}}$$

is T^+-compatible. An inspection of Definition 7.5 gives the result. $\qquad\square$

Remark 7.10. It follows from Corollary 7.9 that the algebraic space $\overline{\mathrm{M}}_{g,n}^{T_{g,n}+}$ is

- $\mathbb{Q}$-factorial if and only if $g \leq 1$, or $(g,n) = (2,1), (3,0), (3,1), (3,2), (4,0), (5,0), (6,0)$;
- $\mathbb{Q}$-Gorenstein if and only if $g \leq 1$ or $n = 0$ or $(g,n) = (2,1), (2,2), (3,1), (3,2)$.

In particular, we recover the result of Alper and Hyeon [12, Sec. 6]: $\overline{\mathrm{M}}_g^{T_g+}$ (which coincides with $\overline{\mathrm{M}}_g\left(\frac{7}{10} - \epsilon\right)$ if $\mathrm{char}(k) = 0$; see Remark 7.14) is $\mathbb{Q}$-factorial if and only if $g \leq 6$.

Note that when $\overline{\mathrm{M}}_{g,n}^{T_{g,n}+}$ is not $\mathbb{Q}$-factorial, it cannot be reached via a sequence of elementary steps (that is, relative Picard number 1 steps) of an MMP of $\overline{\mathrm{M}}_{g,n}$. This shows that there is a difference between flipping the elliptic bridge face in a single step and trying to flip each extremal ray one by one.

Another corollary of Proposition 7.7 is the computation of the Picard number of $\overline{\mathrm{M}}_{g,n}^{T_{g,n}+}$ (which coincides with $\overline{\mathrm{M}}_{g,n}\left(\frac{7}{10} - \epsilon\right)$ if $\mathrm{char}(k) = 0$; see Remark 7.14) and the relative Picard number of the morphism $f_{T_{g,n}}^+$ (using Remark 6.5). We assume that $g \geq 1$, for otherwise we have $\overline{\mathrm{M}}_{0,n}^{T_{0,n}+} = \overline{\mathrm{M}}_{0,n}$.

Corollary 7.11. *Assume* $g \geq 1$, $\mathrm{char}(k) \gg (g,n)$ *and* $(g,n) \neq (2,0),(1,2)$.

(i) *The Picard number of* $\overline{\mathrm{M}}_{g,n}^{T_{g,n}+}$ *is equal to*

$$\dim_{\mathbb{Q}} \mathrm{Pic} \left(\overline{\mathrm{M}}_{g,n}^{T_{g,n}+} \right)_{\mathbb{Q}} = \begin{cases} 3 - \delta_{3,g} & \text{if } n = 0 \text{ and } g \geq 3 \text{ is odd,} \\ 4 - \delta_{4,g} & \text{if } n = 0 \text{ and } g \geq 4 \text{ is even,} \\ 2^n + 2 - (n+2)\delta_{2,g} - (2n+2)\delta_{1,g} & \text{if } g \geq 1 \text{ and } n \geq 1. \end{cases}$$

(ii) *The relative Picard number of* $f_{T_{g,n}}^+$ *is equal to*

$$\rho \left(f_{T_{g,n}}^+ \right) = \begin{cases} 2 - \delta_{3,g} & \text{if } n = 0 \text{ and } g \geq 3 \text{ is odd,} \\ 2 - \delta_{4,g} & \text{if } n = 0 \text{ and } g \geq 4 \text{ is even,} \\ 2^{n-1} + 1 - (n+1)\delta_{2,g} - (n+1)\delta_{1,g} & \text{if } g \geq 1 \text{ and } n \geq 1. \end{cases}$$

We now show that f_T^+ is projective by producing an f_T^+- ample line bundle on $\overline{\mathrm{M}}_{g,n}^{T,+}$.

Proposition 7.12. *Assume* $(g,n) \neq (2,0),(1,2)$ *and* $\mathrm{char}(k) \gg (g,n)$. *The line bundle* $-\widehat{\delta} = -(\delta - \psi)$ *on* $\overline{\mathcal{M}}_{g,n}^{T,+}$ *descends to an* f_T^+-*ample* $\mathbb{Q}$-*line bundle* $\left(-\widehat{\delta} \right)^{T+}$ *on* $\overline{\mathrm{M}}_{g,n}^{T,+}$.
In particular, the morphism f_T^+ *is projective.*

Proof. The fact that $-\widehat{\delta} \in \mathrm{Pic} \left(\overline{\mathcal{M}}_{g,n}^{T+} \right)$ descends to a $\mathbb{Q}$-line bundle $\left(-\widehat{\delta} \right)^{T+}$ on $\overline{\mathrm{M}}_{g,n}^{T+}$ follows from Proposition 7.7. The fact that $\left(-\widehat{\delta} \right)^{T+}$ is f_T^+-ample follows from the same argument of [7, Prop. 7.4] using the fact that the open embeddings

$$\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}} \hookrightarrow \overline{\mathcal{M}}_{g,n}^{T} \hookleftarrow \overline{\mathcal{M}}_{g,n}^{T+}$$

arise from local VGIT with respect to the line bundle $\widehat{\delta}$ on $\overline{\mathcal{M}}_{g,n}^{T}$ by Proposition 4.6. □

Corollary 7.13. *Assume* $(g,n) \neq (2,0),(1,2)$ *and* $\mathrm{char}(k) = 0$. *Then* $\overline{\mathrm{M}}_{g,n}^{T+}$ *is projective.*

Proof. $\overline{\mathrm{M}}_{g,n}^{T}$ is projective if $\mathrm{char}(k) = 0$, by Theorem 6.1; the corollary now follows from the projectivity of f_T^+ proven in Proposition 7.12. □

Remark 7.14. If $T = T_{g,n}$ (and $(g,n) \neq (2,0),(1,2)$), then the projectivity of $\overline{\mathrm{M}}_{g,n}^{T_{g,n}+}$ follows from Remark 6.5 and Proposition 7.12. Furthermore, if $\mathrm{char}(k) = 0$ then it follows from [7, Thm. 1.1] that $\overline{\mathrm{M}}_{g,n}^{T_{g,n}+}$ is identified with a log canonical model of $\overline{\mathrm{M}}_{g,n}$:

$$\overline{\mathrm{M}}_{g,n}^{T_{g,n}+} \cong \overline{\mathrm{M}}_{g,n}(7/10 - \epsilon) := \mathrm{Proj} \bigoplus_{m \geq 0} H^0 \left(\overline{\mathcal{M}}_{g,n}, \left\lfloor m \left(K_{\overline{\mathcal{M}}_{g,n}} + \psi + \left(\frac{7}{10} - \epsilon \right) (\delta - \psi) \right) \right\rfloor \right),$$

$$(7.7)$$

extending the previous result of Hassett and Hyeon [34] for $n = 0$.

Next, we study the fibres and the exceptional loci of the morphism f_T^+.

Proposition 7.15. *Assume $(g,n) \neq (2,0),(1,2)$, and $\mathrm{char}(k) \gg (g,n)$.*

(i) *The morphism f_T^+ is a contraction – that is, $\left(f_T^+\right)_* \left(\mathcal{O}_{\overline{\mathrm{M}}_{g,n}^{T+}}\right) = \mathcal{O}_{\overline{\mathrm{M}}_{g,n}^T}$.*

(ii) *The exceptional locus of f_T^+ is the union of the irreducible closed subsets*

$$\mathrm{Tac}([\tau,I],[\tau+1,I])$$
$$:= \overline{\left\{ (C,\{p_i\}) \in \overline{\mathrm{M}}_{g,n}^{T+} : (C,\{p_i\}) \text{ has a tacnode of type } \{[\tau,I],[\tau+1,I]\} \right\}}$$

for every $\{[\tau,I],[\tau+1,I]\} \subseteq T - \{[1,\emptyset]\}$ which is not of the form $\{[0,\{i\}],[1,\{i\}]\}$ for some $1 \leq i \leq n$, and

$$\mathrm{Tac}(\mathrm{irr})$$
$$:= \overline{\left\{ (C,\{p_i\}) \in \overline{\mathrm{M}}_{g,n}^{T+} : (C,\{p_i\}) \text{ has a tacnode of type } \{\mathrm{irr}\} \right\}} \quad \text{if } \mathrm{irr} \in T \text{ and } g \geq 2.$$

All these closed subsets have codimension 3, so that the morphism f_T^+ is small.

Proof. Part (i) follows from the Zariski main theorem using the fact that f_T^+ is a proper morphism between irreducible normal algebraic spaces (see Theorem 4.4), which is moreover birational because it is an isomorphism when restricted to the dense open subset of smooth curves.

Part (ii): first of all, the closed subsets in the statement are irreducible and have codimension 3, since the semiuniversal deformation space of a tacnode has dimension 3 (since $\mathrm{char}(k) \neq 2$). By Proposition 3.24, the morphism f_T^+ sends a T^+-closed curve $(C,\{p_i\})$ into the T-closed curve $f_T^+((C,\{p_i\}))$, which is the stabilisation of the n-pointed curve obtained from $(C,\{p_i\})$ by replacing each tacnode (necessarily of type contained in $T - \{[1,\emptyset]\}$, since $(C,\{p_i\})$ cannot have A_3-attached elliptic tails) by an attached rosary of length 2. Now observe that a tacnode has local moduli isomorphic to $\mathbb{G}_\mathrm{m}$, because it is constructed from the normalisation by gluing together the two tangent spaces at the two smooth branches (see [33, Sec. 4.1] for details). Since $\omega_C\left(\sum p_i\right)$ is ample, these local moduli do not give rise to global moduli if and only if one of the two branches of the tacnode belongs to a rational curve with only one other marked point (which always happen if the type of the tacnode is equal to $\{[0,\{i\}],[1,\{i\}]\}$ for some $1 \leq i \leq n$), in which case the automorphism group of the 2-pointed rational curve cancels out the local moduli. The curve $f_T^+((C,\{p_i\}))$ does not depend on the global moduli given by the tacnodes of $(C,\{p_i\})$. By putting everything together, we deduce that the exceptional locus of f_T^+ is equal to the union of the closed subsets described in the statement. $\square$

As a corollary of this proposition, we can determine when f_T^+ is an isomorphism.

Corollary 7.16. *Assume $(g,n) \neq (2,0),(1,2)$ and $\mathrm{char}(k) \gg (g,n)$. Then $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T+} \to \overline{\mathrm{M}}_{g,n}^T$ is an isomorphism if and only if $T^{\mathrm{adm}} = T^{\mathrm{div}}$.*

Proof. Proposition 7.15(i) implies that the exceptional locus of f_T^+ is empty – that is, f_T^+ is an isomorphism – if and only $T^{\mathrm{adm}} = T^{\mathrm{div}}$. $\square$

The final ingredient we need is a description of the relative Mori cone of the morphism f_T^+. With this in mind, we introduce the following curves, which were already considered in [34, Propositions 4.1 and 4.2]:

Definition 7.17 (Tacnodal curves, see Figure 9). Let $(g,n) \neq (2,0),(1,2)$ be a hyperbolic pair. Consider the following irreducible curves (well defined up to numerical equivalence) in $\overline{\mathcal{M}}_{g,n}^{T+}$, which we call *tacnodal curves*:

1. If irr $\in T$ and $g \geq 2$, then let $D(\text{irr})^o \cong \mathbb{G}_m$ be the curve in $\overline{\mathcal{M}}_{g,n}^{T+}$ which parametrises T^+-semistable curves obtained from a fixed smooth irreducible curve E of genus $g-2$ with $n+2$ marked points by gluing the last two marked points, which we call a and b, to form a tacnode of type irr using the identification of $T_a E$ and $T_b E$ provided by the elements of $\mathbb{G}_m$. We denote by $D(\text{irr})$ the closure of $D(\text{irr})^o$ in $\overline{\mathcal{M}}_{g,n}^{T+}$. The curve $D(\text{irr})$ is isomorphic to $\mathbb{P}^1$; the two points on the closure parametrise the two curves formed by gluing a and b with a $\mathbb{P}^1$ which is attached nodally at a and tacnodally at b (or the other way around).

2. For any pair $\{[\tau,I],[\tau+1,I]\} = \{[\tau,I],[g-1-\tau,I^c]\} \subset T - \{[1,\emptyset],\bigcup_j[1,\{j\}],\text{irr}\}$, we let $D([\tau,I],[\tau+1,I])^o \cong \mathbb{G}_m$ be the curve in $\overline{\mathcal{M}}_{g,n}^{T+}$ which parametrises T^+-semistable curves obtained from two fixed irreducible curves A and B, the first of genus τ with $I \cup \{a\}$ marked points and the second one of genus $g-1-\tau$ with $I^c \cup \{b\}$ marked points, by gluing the points a and b to form a tacnode of type $\{[\tau,I],[\tau+1,I]\}$, using the identification of $T_a A$ and $T_b B$ provided by the elements of $\mathbb{G}_m$. We denote by $D([\tau,I],[\tau+1,I])$ the closure of $D([\tau,I],[\tau+1,I])^o$ in $\overline{\mathcal{M}}_{g,n}^{T+}$. The curve $D([\tau,I],[\tau+1,I])$ is isomorphic to $\mathbb{P}^1$; the two points on the closure parametrise the two curves formed by gluing a and b with a $\mathbb{P}^1$ which is attached nodally at a and tacnodally at b (or the other way around).

The *type* of a tacnodal curve is defined as follows: $D(\text{irr})$ has type $\{\text{irr}\} \subset T_{g,n}$, while $D([\tau,I],[\tau+1,I])$ has type equal to $\{[\tau,I],[\tau+1,I]\} \subset T_{g,n}$. It is straightforward to see that the tacnodal curves parametrises T^+-closed points of $\overline{\mathcal{M}}_{g,n}^{T+}$ (see Definition 3.26); hence they descend to integral curves (which we will continue to call tacnodal curves and denote with the same notation) in the good moduli space $\overline{\mathrm{M}}_{g,n}^{T+}$ by Proposition 3.27(ii).

Remark 7.18. Notice that we have not defined the tacnodal curves $D([0,\{i\}],[1,\{i\}])$ and $D([1,\{i\}],[2,\{i\}])$ for $1 \leq i \leq n$. This is for the following reasons:

- If we define $D([0,\{i\}],[1,\{i\}])^o$ as in Definition 7.17, then $D([0,\{i\}],[1,\{i\}])^o$ is a point and not a curve inside $\overline{\mathcal{M}}_{g,n}^{T+}$, since the continuous automorphism group of the curve A of genus and with two marked points kills the gluing data that are needed to construct the tacnode.
- The curve $D([1,\{i\}],[2,\{i\}])$, defined as the closure of the curve $D([1,\{i\}],[2,\{i\}])^o$ defined as before, is contracted when mapped into $\overline{\mathrm{M}}_{g,n}^{T+}$ via the morphism ϕ^{T+}, since its generic point is not T^+-closed (because it contains an A_1/A_3-attached elliptic bridge of type $\{[1,\{i\}],[2,\{i\}]\} \subseteq T$; see Proposition 3.27(i)).
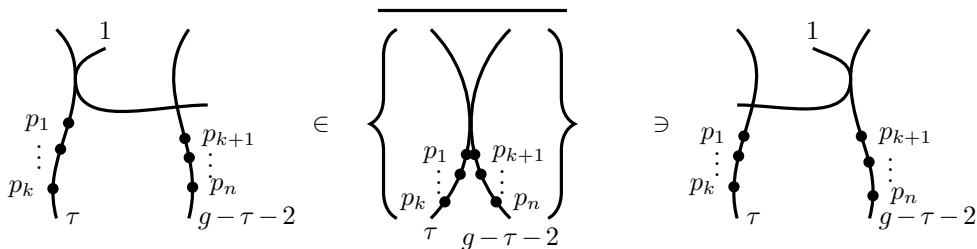
Figure 9. The tacnodal curve $D([\tau,I],[\tau+1,I])$ with two limit points, where $I=\{1,\dots,k\}$.

Proposition 7.19. *Assume* $(g,n)\neq(2,0),(1,2)$ *and* $\mathrm{char}(k)\gg(g,n)$.

(i) *The relative Mori cone of the morphism* f_T^+ *is the subcone of* $\overline{\mathrm{NE}}\left(\overline{\mathrm{M}}_{g,n}^{T+}\right)$ *spanned by the tacnodal curves of type contained in* T.

(ii) *Given a* $\mathbb{Q}$-*line bundle*

$$L=a\lambda+b_{\mathrm{irr}}\delta_{\mathrm{irr}}+\sum_{[i,I]\in T_{g,n}^*-\left\{[1,\emptyset],\bigcup_j[1,\{j\}]\right\}}b_{i,I}\delta_{i,I}$$

on $\overline{\mathcal{M}}_{g,n}^{T+}$, *we have the following intersection formulas:*

$$\begin{cases}D([\tau,I],[\tau+1,I])\cdot L=-a-12b_{\mathrm{irr}}+b_{\tau,I}+b_{\tau+1,I},\\ D(\mathrm{irr})\cdot L=-a-10b_{\mathrm{irr}}.\end{cases}$$

Proof. Part (i): let D be an integral curve inside $\overline{\mathrm{M}}_{g,n}^{T+}$ that is contracted by the morphism f_T^+. By Proposition 3.24(i), the geometric generic point of D parametrises a T^+-closed curve C (by Proposition 3.27(ii)) with a tacnode t of type contained in T and having some nontrivial global gluing data, which happens if and only if type(t) is not equal to $\{[0,\{i\}],[1,\{i\}]\}$ for some $1\leq i\leq n$. Moreover, since C is T^+-closed curve, type(t) cannot be equal to $\{[1,\emptyset],[2,\emptyset]\}$ (otherwise C would contain an A_3-attached elliptic tail) or to $\{[1,\{i\}],[2,\{i\}]\}$ for some $1\leq i\leq n$ (otherwise C would contain an A_1/A_3-attached elliptic bridge of type contained in T). From this discussion, it follows that D is numerically equivalent to a tacnodal curve of type contained in T, and part (i) follows.

Part (ii): let $D\cong\mathbb{P}^1\subset\overline{\mathcal{M}}_{g,n}^{T+}$ be a tacnodal curve and let $\pi:\mathcal{X}\to D$ be the associated (flat and projective) family of n-pointed T^+-semistable curves of genus g. The family $\mathcal{X}\to D$ has a tacnodal section τ (which is also the only singularity of each fibre over $\mathbb{G}_{\mathrm{m}}\subset\mathbb{P}^1$) and two nodes over 0 and ∞ that are of type $[\tau,I]$ and $[\tau+1,I]$ if $D=D([\tau,I],[\tau+1,I])$ or both of type $\{\mathrm{irr}\}$ if $D=D(\mathrm{irr})$. This implies that the only boundary divisor that

contains D is δ_{irr} and that for any $[i,J] \in T_{g,n} - \{\mathrm{irr}\}$, we have

$$\begin{cases} \delta_{i,J} \cdot D(\mathrm{irr}) = 0, \\ \delta_{i,J} \cdot D([\tau,I],[\tau+1,I]) = \begin{cases} 1 & \text{if } [i,J] = [\tau,I] \text{ or } [\tau+1,I], \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{7.8}$$

Consider now the normalisation $\widetilde{\pi} : \mathcal{Y} \to D$ of the family $\mathcal{X} \to D$ along the tacnodal section τ. The (flat and projective) family $\mathcal{Y} \to D$ has $n+2$ sections, the first n of which are the pullback of the n sections of the family $\mathcal{X} \to D$, and the last two of which, call them σ_a and σ_b, are the inverse image of the tacnodal section τ along the normalisation morphism $\mathcal{Y} \to \mathcal{X}$. We can apply [7, Prop.6.1] in order to get

$$\begin{cases} \lambda \cdot D = \deg_D(\lambda_{\mathcal{Y}/D}) - \dfrac{\deg_D(\psi_a + \psi_b)}{2}, \\ \delta \cdot D = \deg_D(\delta_{\mathcal{Y}/D}) - 6\deg_D(\psi_a + \psi_b), \end{cases} \tag{7.9}$$

where $\delta_{\mathcal{Y}/D}$ is the total boundary of the family $\widetilde{\pi} : \mathcal{Y} \to D$, $\lambda_{\mathcal{Y}/D} := \det \widetilde{\pi}_*(\omega_{\mathcal{Y}/D})$, $\psi_a = \sigma_a^*(\omega_{\mathcal{Y}/D})$ and $\psi_b = \sigma_b^*(\omega_{\mathcal{Y}/D})$. By the definition of the tacnodal curve D, it follows that the family $\mathcal{Y} \to D \cong \mathbb{P}^1$ together with the two sections σ_a and σ_b are obtained from a constant family $F \times \mathbb{P}^1 \to \mathbb{P}^1$ (where, using the notations of Definition 7.17, $F = E$ if $D = D(\mathrm{irr})$ or $F = A \coprod B$ if $D = D([\tau,I],[\tau+1,I])$), together with two constant sections $\{a\} \times \mathbb{P}^1$ and $\{b\} \times \mathbb{P}^1$, by blowing up the points $\{a\} \times \{0\}$ and $\{b\} \times \{\infty\}$ and taking the strict transform of the two constant sections. Therefore, the family $\widetilde{\pi} : \mathcal{Y} \to D$ has two singular fibres, namely $\widetilde{\pi}^{-1}(0)$ and $\widetilde{\pi}^{-1}(\infty)$, which are formed by F and the exceptional divisors E_0 and E_∞, respectively, meeting in one node; hence we have

$$\deg_D(\delta_{\mathcal{Y}/D}) = 2. \tag{7.10}$$

Moreover, since there is no variation of moduli in the fibres of the family $\widetilde{\pi} : \mathcal{Y} \to D$, we have

$$\deg_D(\lambda_{\mathcal{Y}/D}) = 0. \tag{7.11}$$

Finally, since $\sigma_a^*(\omega_{\mathcal{Y}/D}) = \sigma_a^*(\mathcal{O}_{\mathcal{Y}}(-\mathrm{Im}(\sigma_a)))$, we have $\deg_D(\psi_a) = -(\mathrm{Im}\,\sigma_a)^2$. Since the pullback of the constant section $\{a\} \times \mathbb{P}^1$ to the blowup family $\widetilde{\pi} : \mathcal{Y} \to D$ is equal to $E_0 + \mathrm{Im}\,\sigma_a$, we get

$$0 = (E_0 + \mathrm{Im}\,\sigma_a)^2 = E_0^2 + 2E_0 \cdot \mathrm{Im}\,\sigma_a + (\mathrm{Im}\,\sigma_a)^2 = -1 + 2 + (\mathrm{Im}\,\sigma_a)^2 \Rightarrow \deg_D(\psi_a)$$
$$= -(\mathrm{Im}\,\sigma_a)^2 = 1. \tag{7.12}$$

Similarly, we have

$$\deg_D(\psi_b) = 1. \tag{7.13}$$

Substituting equations (7.10)–(7.13) into equation (7.9), we get

$$\lambda \cdot D = -1 \quad \text{and} \quad \delta \cdot D = -10. \tag{7.14}$$

By combining equations (7.8) and (7.14), we conclude the proof of part (ii). $\qquad \square$

We are now ready, by combining these propositions, to give a proof of Theorem 7.4.

Proof.[Proof of Theorem 7.4] Note that the algebraic space $\overline{\mathrm{M}}_{g,n}^{T+}$ is normal by Theorem 4.4 and the morphism f_T^+ is a small contraction by Proposition 7.15. Hence the first two conditions of Definition 7.1 are always satisfied. Moreover, in order for f_T^+ to be the L-flip of f_T, we need the fact that L is f_T-antiample (see Definition 7.1).

It remains to check the last condition of Definition 7.1 with respect to the rational morphism

$$\eta := \left(f_T^+\right)^{-1} \circ f_T : \overline{\mathrm{M}}_{g,n}^T \dashrightarrow \overline{\mathrm{M}}_{g,n}^{T+}$$

and any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on $\overline{\mathrm{M}}_{g,n}^T$ whose associated $\mathbb{Q}$-line bundle is L. If the restriction of L to $\overline{\mathcal{M}}_{g,n}^{T+}$ (which we denote again by L) is T^+-compatible, it will descend to a $\mathbb{Q}$-line bundle L^{T+} on $\overline{\mathrm{M}}_{g,n}^{T+}$ by Proposition 7.7. By the commutativity of diagram (4.1), we have that the linear equivalence class of the $\mathbb{Q}$-divisor $\eta_*(D)$ is L^{T+}, which implies that $\eta_*(D)$ is $\mathbb{Q}$-Cartier. Conversely, if $\eta_*(D)$ is $\mathbb{Q}$-Cartier, then its linear equivalence class is a $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^{T+}$ whose pullback to $\overline{\mathcal{M}}_{g,n}^{T+}$ is the restriction of L to $\overline{\mathcal{M}}_{g,n}^{T+}$, and this again implies that L is T^+-compatible, by Proposition 7.7.

It remains to show that if L is f_T-antiample, then L^{T+} is f_T^+-ample. Since f_T^+ is projective by Proposition 7.12 and the relative Mori cone of f_T^+ is generated by the tacnodal curves of type contained in T by Proposition 7.19(i), it is enough to show, by the relative Kleiman ampleness criterion [42, Thm. 1.44], that L negatively intersects these curves. By combining Proposition 7.19(ii) with Lemma 5.8 and using the fact that the intersection of L with all the elliptic bridge curves of type contained in T is negative because L is f_T-antiample, we get

$$\begin{cases} D(\mathrm{irr}) \cdot L = -C(\mathrm{irr}) \cdot L > 0 \text{ if } \mathrm{irr} \in T, \\ D([\tau,I],[\tau+1,I]) \cdot L = -C([\tau,I],[\tau+1,I]) \cdot L > 0 \end{cases}$$

for any $\left\{[\tau,I],[\tau+1,I]\right\} \subset T - \left\{[1,\emptyset], \bigcup_j [1,\{j\}]\right\}$, and this concludes the proof. $\qquad \square$

We now describe two important special cases of Theorem 7.4.

Corollary 7.20. *Assume* $(g,n) \neq (2,0),(1,2)$ *and* $\mathrm{char}(k) \gg (g,n)$.

(i) *The morphism* $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T,+} \to \overline{\mathrm{M}}_{g,n}^T$ *is the* $\left(K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi\right)$-*flip of* f_T.

(ii) *The morphism* $f_T^+ : \overline{\mathrm{M}}_{g,n}^{T,+} \to \overline{\mathrm{M}}_{g,n}^T$ *is the* $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-*flip of* f_T *if and only if* $\overline{\mathrm{M}}_{g,n}^{T,+}$ *is* $\mathbb{Q}$-*Gorenstein – that is, if and only if* T *does not contain subsets of the form* $\{[0,\{j\}],[1,\{j\}],[2,\{j\}]\}$ *for some* $j \in [n]$ *or* $(g,n) = (3,1),(3,2),(2,2)$.

Proof. Since the relative Mori cone of f_T is generated by the elliptic bridge curves of type contained in T, by Proposition 6.2(ii), and the elliptic bridge curves are both $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$- and $\left(K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi\right)$-negative, by Proposition 5.9(i), the relative Kleiman ampleness criterion (which can be applied, since f_T^+ is projective by Proposition 7.12) implies that $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$ and $\left(K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi\right)$ are f_T-antiample. By Mumford's formula (see Fact 3.28(3.28)), we have

that $K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi = 13\lambda - 2\widehat{\delta}$ and the restriction of $13\lambda - 2\widehat{\delta}$ to $\overline{\mathcal{M}}_{g,n}^{T+}$ is T^+-compatible (see Definition 7.5). Hence we conclude that f_T^+ is the $\left(K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}} + \psi\right)$-flip of f_T, by Theorem 7.4.

In order to prove part (ii), observe first that

$$\left(\left(f_T^+\right)^{-1} \circ f_T\right)_* \left(K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}\right) = K_{\overline{\mathrm{M}}_{g,n}^{T+}}. \tag{7.15}$$

Therefore, if f_T^+ is the $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-flip of f_T, then $K_{\overline{\mathrm{M}}_{g,n}^{T+}}$ is $\mathbb{Q}$-Cartier – that is, $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-Gorenstein, which happens if and only if T does not contain subsets of the form $\{[0,\{j\}],[1,\{j\}],[2,\{j\}]\}$ for some $j \in [n]$ or $(g,n) = (3,1),(3,2),(2,2)$, by Corollary 7.9(iii). Conversely, if $K_{\overline{\mathrm{M}}_{g,n}^{T+}}$ is $\mathbb{Q}$-Cartier, then by diagram (4.1) we deduce that the restriction of the $\mathbb{Q}$-line bundle $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$ (seen as a $\mathbb{Q}$-line bundle on $\overline{\mathcal{M}}_{g,n}^T$ by Corollary 3.29 and Proposition 5.1(ii)) to $\overline{\mathcal{M}}_{g,n}^{T+}$ descends to the $\mathbb{Q}$-line bundle $K_{\overline{\mathrm{M}}_{g,n}^{T+}}$ and hence is T^+-compatible. Hence, we conclude that f_T^+ is the $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-flip of f_T by Theorem 7.4. □

Theorem 7.4 implies that when $\overline{\mathrm{M}}_{g,n}^{T+}$ is $\mathbb{Q}$-factorial (comopare Corollary 7.9(ii)), the morphism f_T^+ is the L-flip of f_T with respect to any $\mathbb{Q}$-line bundle L on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ which is f_T-antiample. Under these assumptions, and assuming furthermore that f_T is small (compare Proposition 6.7(6.7)), we will now prove that f_T^+ is the composition of elementary L-flips.

Corollary 7.21. *Assume $(g,n) \neq (2,0),(1,2)$ and $\mathrm{char}(k) = 0$. Let $T \subset T_{g,n}$ be such that $f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^T$ is small and $\overline{\mathrm{M}}_{g,n}^{T,+}$ is $\mathbb{Q}$-factorial (compare Proposition 6.7(6.7) and Corollary 7.9(ii)). Let L be a $\mathbb{Q}$-line bundle on $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ which is f_T-antiample.*

Then the rational map $\left(f_T^+\right)^{-1} \circ f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \dashrightarrow \overline{\mathrm{M}}_{g,n}^{T,+}$ can be decomposed (up to isomorphism) as a sequence of elementary L-flips.

Proof. The morphism $f_T : \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} \to \overline{\mathrm{M}}_{g,n}^T$ is a relative Mori dream space because it is $K_{\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}}$-negative (by Theorem 6.1) and $\overline{\mathrm{M}}_{g,n}^{\mathrm{ps}}$ is klt and $\mathbb{Q}$-factorial (by Proposition 5.1) with a discrete Picard group (by Corollary 3.29 and Proposition 5.1(i)). Hence, we can run an MMP for L over $\overline{\mathrm{M}}_{g,n}^T$ and obtain a relative minimal model

$$\begin{array}{ccc} \overline{\mathrm{M}}_{g,n}^{\mathrm{ps}} & \xdashrightarrow{\quad\eta\quad} & X \\ & {\scriptstyle f_T}\searrow \quad \swarrow{\scriptstyle g} & \\ & \overline{\mathrm{M}}_{g,n}^T. & \end{array} \tag{7.16}$$

Since f_T is small, g is also small and η is a composition of flips. Moreover, since $\overline{\mathrm{M}}_{g,n}^{T,+}$ is the ample model of L over $\overline{\mathrm{M}}_{g,n}^T$, there is a birational morphism $X \to \overline{\mathrm{M}}_{g,n}^{T,+}$ over $\overline{\mathrm{M}}_{g,n}^T$, which is again small. Since both spaces are $\mathbb{Q}$-factorial, we conclude that the morphism $X \to \overline{\mathrm{M}}_{g,n}^{T,+}$ is an isomorphism, as wanted. □

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