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UNIFORM AND TANGENTIAL APPROXIMATIONS BY MEROMORPHIC FUNCTIONS ON CLOSED SETS

ALICE ROTH

1. Let G be an (open) domain in the finite complex plane and F a relatively closed proper subset of G. We denote by $M(G)$ the set of functions meromorphic on G and as usual by $R(K)$ (for a compact set K) the set of uniform limits of rational functions without poles on K.

The problem of approximating uniformly a complex valued function on F by functions in $M(G)$ is reduced by the following Theorem I to the problem of uniform approximation by rational functions on a compact set.

THEOREM I. *A function f can be approximated uniformly on F by functions in $M(G)$ without poles on F if and only if*

(*) $\quad f_{|K} \in R(K)$

for every compact subset K of F.

The necessity of condition (*) is obvious: if m is a meromorphic function which approximates f on F, the restriction $m_{|K}$ can be approximated uniformly on K by rational functions (using Runge's Theorem).

To prove that the condition (*) is sufficient we shall use the following Lemma 1.

LEMMA 1. (Fusion of rational functions). *Let K_1 K_2, and K be compact subsets of the extended plane with K_1 and K_2 disjoint. If r_1 and r_2 are any two rational functions satisfying, for some $\epsilon > 0$,*

(1) $\quad |r_1(z) - r_2(z)| < \epsilon,$ *for* $z \in K,$

then there is a positive number a, depending only on K_1 and K_2 and a rational function r such that for $j = 1, 2$,

(2) $\quad |r(z) - r_j(z)| < a\epsilon, \quad$ *for* $z \in K_j \cup K.$

We remark that in Lemma 1, r_1 and r_2 are allowed to have poles on the sets in question.

Proof. We may assume $K_2 \backslash K \neq \emptyset$ and $\infty \in K_2$. Thus, we can construct open neighbourhoods U_1 and U_2 of K_1 and K_2 respectively such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ and $\infty \in U_2$. Moreover, we may assume that the boundaries of U_1 and U_2 consist of finitely many disjoint smooth Jordan curves. Let E be the

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complement of $U_1 \cup U_2$ in the extended plane. Then E is compact in $\mathbf{C}$, and thus

$$(3) \qquad I(z) = \int_E \int \frac{d\xi d\eta}{|\zeta - z|}, \text{ where } \zeta = \xi + i\eta,$$

is uniformly bounded for z in the extended plane. Indeed, for $z_0 \neq \infty$, set

$$\zeta - z_0 = \rho e^{i\varphi}.$$

Then

$$I(z_0) = \int_E \int d\rho d\varphi,$$

and so $I(z_0)$ is bounded, for instance, by $2\pi d$, where d is the diameter of E. For $z_0 = \infty$, $I(z_0) = 0$.

We introduce now an auxiliary function $\Phi \in C^1(\mathbf{R}^2)$ with values in $[0, 1]$ such that Φ is 1 on U_1 and Φ is 0 on U_2. Then

$$\frac{\partial \Phi}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y}\right)$$

is uniformly bounded. Hence since (3) is also uniformly bounded, there is a constant $a > 2$ such that

$$(4) \qquad \frac{1}{\pi}\int_E \int \left|\frac{\partial \Phi(\zeta)}{\partial \bar{\zeta}}\right| \frac{1}{|\zeta - z|} d\xi d\eta < a - 2,$$

for $z \in \mathbf{C}$.

We return now to our rational functions r_1 and r_2 and we put

$$q = r_1 - r_2.$$

By (1) we can find a neighbourhood U of K such that

$$|q(z)| < \epsilon, z \in \bar{U}.$$

We replace q by a function q_1 constructed as follows. First set

$$(5) \qquad q_1 = q \text{ on } U_1 \cup U_2 \cup U.$$

Now extend q_1 to E so as to satisfy: q_1 is continuous on E and

$$(6) \qquad |q_1(z)| < \epsilon, \quad z \in E.$$

Set

$$(7) \qquad g(z) = \frac{1}{\pi}\int_E \int \frac{q_1(\zeta)}{\zeta - z}\frac{\partial \Phi}{\partial \bar{\zeta}} d\xi\, d\eta.$$

From (4) and (6) we have

$$(8) \qquad |g(z)| < (a - 2)(a - 2)\epsilon, \quad z \in \mathbf{C}.$$

Since g is a Cauchy integral, g is holomorphic outside of E. Consequently

(9) $f(z) = \Phi(z)q_1(z) + g(z), z \in \mathbf{C},$

is holomorphic in U_2 (for $q_1(z) = \infty$, set $\Phi(z)q_1(z) = 0$). For $z \in U_1$,

$$f(z) = q_1(z) + g(z)$$

is meromorphic and has the same poles as q_1. To see that f is also holomorphic on U, we invoke the Pompeiu formula

$$\Phi(z) = -\frac{1}{\pi}\int_E\int \frac{\partial\Phi(\zeta)}{\partial\bar{\zeta}}\frac{1}{\zeta - z}\,d\xi\,d\eta, \quad z \in \mathbf{C}.$$

Hence,

$$f(z) = \frac{1}{\pi}\int_E\int \frac{\partial\Phi(\zeta)}{\partial\bar{\zeta}}\frac{q_1(\zeta) - q_1(z)}{\zeta - z}\,d\xi\,d\eta, \quad z \in \mathbf{C}, \quad q_1(z) \neq \infty.$$

For $z \in U$, $q_1 = q$ and

$$\frac{q_1(\zeta) - q_1(z)}{\zeta - z}$$

is holomorphic. Thus f is holomorphic in U, and hence f is meromorphic on $U_1 \cup U_2 \cup U$ with the same poles as q. By Runge's theorem there is a rational function r_3 for which

$$|r_3(z) - f(z)| < \epsilon, \quad z \in K_1 \cup K_2 \cup K.$$

Finally we put $r = r_2 + r_3$, and we have the following estimates: on $K_1 \cup K$

$$\begin{aligned}|r - r_1| &\leqq |f - (r_1 - r_2)| + |r_3 - f| \\ &\leqq |\Phi - 1|\,|q| + |g| + |r_3 - f| \\ &< \epsilon + (a - 2)\epsilon + \epsilon = a\epsilon;\end{aligned}$$

on $K_2 \cup K$

$$\begin{aligned}|r - r_2| &\leqq |f| + |r_3 - f| \leqq |\Phi|\,|q| + |g| + |r_3 - f| \\ &< \epsilon + (a - 2)\epsilon + \epsilon = a\epsilon.\end{aligned}$$

This completes the proof of Lemma 1.

Construction of the approximating function in Theorem I: Let $\{G_n\}$ be an exhaustion of G by domains with

$$\bar{G}_n \subset G_{n+1} \quad \text{and} \bigcup G_n = G.$$

For each $n = 1, 2, 3, \ldots$ we choose a positive number a_n associated with $\bar{G}_n$ and $(\mathbf{C} \cup \infty)\backslash G_{n+1}$ in Lemma 1 (these sets replacing K_1 and K_2), so that

$$1 < a_n < a_{n+1}.$$

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If ϵ is a given positive number we select the positive numbers $\epsilon_1, \epsilon_2, \epsilon_3 \dots$ so that

(10) $\epsilon_{n+1} < \epsilon_n$ and $\sum_{n=1}^{\infty} \epsilon_n < \frac{\epsilon}{2}$.

If condition (*) is fulfilled, there exist rational functions $\{q_n\}$ thus

(11) $|q_n(z) - f(z)| < \frac{\epsilon_n}{2a_n}, \quad z \in F_n = F \cap \bar{G}_{n+1}, \quad n = 1, 2, 3, \dots$

and therefore

(12) $|q_{n+1}(z) - q_n(z)| < \frac{\epsilon_n}{a_n}, \quad z \in F_n, \quad n = 1, 2, 3, \dots$

The functions $q_1, q_2, q_3, \dots$ converge to f on every F_n, but generally they don't converge on the domains G_n; we need a second sequence $\{r_n\}$ of rational functions. We use Lemma 1, applying it to the functions q_n, q_{n+1} and to the sets $\bar{G}_n$, $(\mathbf{C} \cup \infty)\backslash G_{n+1}$ and F_n. For $n = 1, 2, 3, \dots$ there exists a rational function r_n such that

(13) $|r_n(z) - q_n(z)| < \epsilon_n, \quad z \in \bar{G}_n \cup F_n,$

(14) $|r_n(z) - q_{n+1}(z)| < \epsilon_n, \quad z \in (\mathbf{C} \cup \infty)\backslash G_{n+1}.$

The inequalities (13) yield

$$\sum_{n}^{\infty} |r_\nu(z) - q_\nu(z)| < \sum_{n}^{\infty} \epsilon_\nu, z \in \bar{G}_n.$$

As $n \to \infty$, $\sum_n^\infty \epsilon_\nu \to 0$; thus $\sum_n^\infty (r_\nu(z) - q_\nu(z))$ converges uniformly to a holomorphic function on $\bar{G}_n$. Therefore

$$m(z) = q_1(z) + \sum_{1}^{\infty} ((r_\nu(z) - q_\nu(z))$$

is holomorphic on G_n with the possible exception of a finite number of poles. Hence $m(z)$ is meromorphic on $G = \bigcup G_n$.

From (11), (13) and (10) follows for $z \in F_1$

$$|m(z) - f(z)| \leqq |q_1(z)| - f(z)| + \sum_{1}^{\infty} |r_\nu(z) - q_\nu(z)| < \frac{\epsilon_1}{2a_1} + \sum_{1}^{\infty} \epsilon_\nu < \epsilon.$$

From (11), (13), (14) and (10) and because

$$F_n\backslash F_{n-1} \subset (\mathbf{C} \cup \infty)\backslash G_k, \ k = 1, 2, \dots n,$$

we have

$$|m(z) - f(z)| \leqq \sum_{1}^{n-1} |r_\nu(z) - q_{\nu+1}(z)| + |q_n - f| + \sum_{n}^{\infty} |r_\nu(z) - q_\nu(z)|$$

$$< \sum_{1}^{n-1} \epsilon_\nu + \frac{\epsilon_n}{2a_n} + \sum_{n}^{\infty} \epsilon_\nu < \epsilon \quad \text{for } z \in F_n\backslash F_{n-1}, n = 2, 3, \dots$$

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Thus $|m(z) - f(z)| < \epsilon$ for $z \in F$; i.e. f can be approximated uniformly on F by meromorphic functions.

Remark. Condition (*) in Theorem I can be replaced by a simpler condition, namely that for each $z \in F$ there exists a closed disc D_z with center z such that

$$f_{|F \cap D_z} \in R(F \cap D_z).$$

This is an immediate consequence of the Localization Theorem of Bishop [**7**, p. 97], which can be proved by applying Lemma 1.

2. We denote by $A(F)$ the set of continuous functions from F to $\mathbf{C}$ whose restrictions to the interior F^0 are holomorphic. We seek to characterize those sets F having the property that *every* function $f, f \in A(F)$, can be uniformly approximated by functions in $M(G)$.

THEOREM II. *A necessary and sufficient condition in order that every function in $A(F)$ can be approximated uniformly on F by functions in $M(G)$ is that*

$$(**) \quad R(F \cap \bar{G}_1) = A(F \cap \bar{G}_1)$$

for every domain G_1, $\bar{G}_1 \subset G$.

By the Localization-Theorem of Bishop we may replace the closed domains G_1 by closed discs.

Theorem II was stated by Nersesian [**4**] and proved for the special case $G = \mathbf{C}$.

The sufficiency of condition (**) follows immediately from the proof of Theorem I. The construction we employed (and which we found before learning of [**4**]) to prove Theorem I is different from Nersesian's method. Perhaps his method (especially with the modifications necessary for applying it to general domains) is more complicated than our method. This may serve as a small justification for publishing the present work.

The proof that condition (**) is necessary is very simple in case F is nowhere dense ($F^0 = \emptyset$) and hence $A(F) = C(F)$: indeed any continuous function on $F \cap \bar{G}_1$ may be extended to a continuous function on all of F.

It seems that at the current state of the subject, the necessity of (**) in the case $F^0 \neq \emptyset$ can only be shown using the results of Vitushkin on continuous analytic capacity [**7**, p. 104].

3. The problem of characterizing a set F having the property, that every function in $A(F)$ can be uniformly approximated by functions *holomorphic* on G was treated in a special case by [**3**] and [**5**] and solved completely by Arakeljan [**1**]: a necessary and sufficient condition on F is that $G^*\backslash F$ is connected and locally connected (G^* is the one-point compactification of G). In [**6**] we pointed out that Arakeljan's Theorem can be proved using Theorem II (at that time only a conjecture).

4. In order to treat *tangential approximations* the following lemma is useful.

LEMMA 2. *If condition (**) is satisfied and* $f, h \in A(F)$, *with*

$$0 < |h(z)| < 1, z \in F,$$

then there is an $m \in M(G)$, *for which*

$$|m(z) - f(z)| < |h(z)|, \quad z \in F.$$

Proof. Since $2h^{-1} \in A(F)$, there is by Theorem II a function $m_1, m_1 \in M(G)$:

$$\left| m_1(z) - \frac{2}{h(z)} \right| < 1, \quad z \in F.$$

Thus

$$|m_1(z)| > \frac{2}{|h(z)|} - 1 > \frac{1}{|h(z)|}, \quad z \in F.$$

A further application of Theorem II yields the existence of a second function $m_2 \in M(G)$:

$$|m_2(z) - m_1(z)f(z)| < 1, \quad z \in F.$$

Set

$$m = m_2/m_1;$$

then $m \in M(G)$ and

$$|m(z) - f(z)| < \frac{1}{|m_1(z)|} < |h(z)|, \quad z \in F.$$

The following Theorems III, IV and V are consequences of Theorem II and Lemma 2.

THEOREM III. *If F is a proper closed subset of* $\mathbf{C}$ *satisfying condition (**) for every disc and* $f \in A(F)$, *then for every* $\epsilon > 0$, *there exists a function m meromorphic on* $\mathbf{C}$ *for which*

$$|m(z) - f(z)| < \epsilon, \quad z \in F,$$

and moreover

$$\lim\ (m(z) - f(z)) = 0$$

uniformly as $z \to \infty$ *on* F.

Proof. Choose $z_1, z_1 \in \mathbf{C}\backslash F$, $n \in \mathbf{N}$ and then η so that

$$0 < \eta < |z - z_1|^n \quad \text{for } z \in F.$$

In Lemma 2 set

$$h(z) = \epsilon\eta(z - z_1)^{-n}.$$

The approximation of Theorem III is "best-possible" in some sense, [**6**, p. 164].

If $F^0 = \emptyset$, then $A(F) = C(F)$ and so from Theorem II and Lemma 2 follows

THEOREM IV. *Let N be a relatively closed nowhere dense subset of the domain G. Then the condition that*

$$R(N_1) = C(N_1)$$

for every compact subset N_1 of N is necessary and sufficient in order that for every $f \in C(N)$, and for every $\epsilon(z) \in C(N)$, $\epsilon(z) > 0$, there is a function m meromorphic on G for which

$$|m(z) - f(z)| < \epsilon(z), \quad z \in N.$$

Since the function $\epsilon(z)$ can tend arbitrarily fast to 0 as z approaches the boundary of G, we have a so called "*Carleman-approximation*". Theorem IV was proved in [**6**] by a different method.

A particularly useful auxiliary function h was introduced by Brown and Gauthier [**2**] for approximations by holomorphic functions. Namely h is a continuous function on F which is constant on every component of F^0 (and hence $h \in A(F)$). Such a function h allows the possibility of *simultaneous uniform approximation on all of F and a Carleman-approximation on a certain subset of F*. The following Theorem V contains both Theorem II as well as Theorem IV.

THEOREM V. *Let F be a closed subset of the domain G and $\hat{N}$ a closed subset of the nowhere dense set $N = F \backslash F^0$ (where "closed" means closed in G). Then condition (**) is necessary and sufficient in order that for every $f \in A(F)$, for every $\eta > 0$ and for every $\epsilon(z) \in C(\hat{N})$, $\epsilon(z) > 0$, there is a function $m \in M(G)$, for which*

$$|m(z) - f(z)| < \eta, \quad z \in F,$$
$$|m(z) - f(z)| < \epsilon(z), \quad z \in \hat{N}.$$

The necessity of condition (**) follows from Theorem II. The proof that (**) is sufficient follows from Theorem II and Lemma 2. We can suppose $\eta < 1$ and $\epsilon(z) < \eta$. Then we choose the auxiliary function h by setting $h_{|\overline{F^0}} = \eta$, $h_{|\hat{N}} = \epsilon(z)$ and extend this function (by Tietze's theorem) to a function h continuous and positive on F and for which $h(z) \leqq \eta$ for $z \in F$.

5. The function f of Theorems I–V is in $A(F)$. Instead of $A(F)$ we may consider a larger set of functions if we admit as approximating functions all functions in $M(G)$ with or without poles on F. Then a necessary condition for f is that for every compact subset K of F the restriction $f_{|K}$ is the sum of a function in $A(K)$ and a rational function. Let us denote by $M(F)$ (generalizing the notation $M(G)$) a function with that property.

Theorem I is valid if we admit all functions of $M(G)$ as approximating functions and replace the condition (*) by the condition that for every compact subset K of F the restriction $f_{|K}$ can be approximated uniformly by rational functions (with or without poles on K). The proof needs no modifications.

An immediate consequence is that *in Theorems II–V we can suppose* $f \in M(F)$.

Remark. The theorem of Mittag-Leffler (concerning the existence of a meromorphic function with given principal parts) follows easily from the modified Theorem II. Vice-versa: to see that in Theorems II–V we may suppose $f \in M(F)$, we can prove with Mittag-Leffler's theorem that such a function f is the sum of a function in $A(F)$ and a function in $M(G)$.

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Willadingweg 34, CH-3006 Berne,
Switzerland