BULL. AUSTRAL. MATH. SOC. 53C20, 57S25
VOL. 73 (2006) [1-8]

A RIGIDITY THEOREM FOR DISCRETE GROUPS

WEN-HAW CHEN AND JYH-YANG WU

This work considers the discrete subgroups of group of isometries of an Alexandrov space with a lower curvature bound. By developing the notion of Hausdorff distance in these groups, a rigidity theorem for the close discrete groups was proved.

1. INTRODUCTION

This investigation studied the discrete subgroups of group of isometries $\mathrm{Isom}(X)$ of an Alexandrov space X with a lower curvature bound. Let (X,d) be a metric space with metric d. X is an *Alexandrov space* of curvature $\geqslant k$ is a complete locally compact length space satisfying the following Alexandrov convexity (see [1]): *If for any geodesic triangles $\triangle pqr$ in X and $\triangle \widetilde{p}\widetilde{q}\widetilde{r}$ in the complete simply connected space 2-form M_k^2 of constant curvature k with the same correspondent side lengths ($\overline{p,q} =, \overline{\widetilde{p},\widetilde{q}}$, $\overline{p,r} =, \overline{\widetilde{p},\widetilde{r}}$, $\overline{q,r} =, \overline{\widetilde{q},\widetilde{r}}$), then $\overline{p,s} \geqslant \overline{\widetilde{p},\widetilde{s}}$ for any s on the side qr and $\widetilde{s}$ on the side $\widetilde{q}\widetilde{s}$ with $\overline{q,s} = \overline{\widetilde{q},\widetilde{s}}$.* Perelman indicates in [6] that an Alexandrov space with a lower curvature bound is a locally contractible space.

Let X be one of the standard Euclidean n-space, spherical n-space or hyperbolic n-space. If G, $\Gamma \subset \mathrm{Isom}(X)$ are discrete then it can be shown (see [8]) that X/G is isometric to X/Γ if and only if G is conjugate to Γ in $\mathrm{Isom}(X)$, which means that there exists $\Phi \in \mathrm{Isom}(X)$ such that $\Gamma = \Phi^{-1}G\Phi$. Moreover, for a complete simply connected Riemannian manifold X with constant sectional curvature, two space forms X/G and X/Γ are isometric if and only if G is conjugate to Γ in $\mathrm{Isom}(X)$. In this paper, we consider a more general Alexandrov space and obtain a similar result. Our approach is to extend the notion of the classic Hausdorff distance to these discrete groups.

The notion of classic Hausdorff distance d_H between subsets of metric spaces can be found in [5] or [7]. Let (X,d) be a metric space and A, $B \subseteq X$. Define the distance $\mathrm{dist}(A,B)$ between A and B by

$$\mathrm{dist}(A,B) \equiv \inf\{d(a,b) \mid a \in A,\ b \in B\},$$
$$B(A,\varepsilon) = \{x \in X \mid d(x,A) < \varepsilon\}$$

Received 18th April, 2005
This work was supported in part by a Taiwan NSC Grant #NSC 91-2115-M-212-002.

and the Hausdorff distance between A and B is defined by

$$d_H(A,B) = \inf\{\varepsilon \mid A \subset B(B,\varepsilon),\ B \subset B(A,\varepsilon)\}.$$

This idea can be extended to the group of isometries.

Definition: Let (X,d) be a topological manifold with a metric d. Then the distance d induces a natural *pseudometric* $\overline{d}$ on $\mathrm{Isom}(X)$ as follows. Given any two isometries g and γ in $\mathrm{Isom}(X)$. The pseudometric $\overline{d}$ is defined by

$$\overline{d}(g,\gamma) = \sup_{x\in X} d(gx,\gamma x).$$

Moreover, $\overline{d}$ induces a Hausdorff distance $d_H(G,\Gamma)$ between two subgroups G and Γ of $\mathrm{Isom}(X)$.

Notably, $\overline{d}$ may not be finite (this is why we call it a pseudometric) but the triangle inequality for $\overline{d}$ still holds, and then the Hausdorff distance $d_H(G,\Gamma)$ between two subgroups G and Γ of $\mathrm{Isom}(X)$ can be considered. It will be interesting to investigate the relationship between the Hausdorff distance $d_H(G,\Gamma)$ and the group structure of G and Γ. Before stating the main result, we first review some phrases in group actions.

A group G is said to act *properly* on a topological space X if $\{g \in G \mid gK \cap K \neq \phi\}$ is finite for each compact subset $K \subset X$. It is well-known that if G is a discrete subgroup of $\mathrm{Isom}(X)$ then G acts properly on X. G acts *freely* on X if $gx \neq x$ for every $x \in X$ and every $g \in G - \{e\}$. G is a *cocompact group* if there exists a compact set $K \subsetneq X$ such that

$$X = GK = \{gx \mid g \in G \text{ and } x \in K\}.$$

Let $G_x = \{g \in G \mid gx = x\}$ denote the *stabiliser* of G at $x \in X$ and $Gx = \{gx \mid g \in G\}$ denote the G-orbit through x.

Main Theorem. *Let (X,d) be an Alexandrov space with a lower curvature bound. Denoted by d_H the Hausdorff distance in the group of isometries $\mathrm{Isom}(X)$ of X induced by d. Given a discrete and cocompact subgroup G of $\mathrm{Isom}(X)$, there exists $\varepsilon = \varepsilon(G) > 0$ such that if Γ is another discrete and cocompact subgroup of $\mathrm{Isom}(X)$ with $d_H(G,\Gamma) < \varepsilon$, then G and Γ are conjugate in the group of homeomorphisms of X.*

Corollary. *This theorem is obviously true for a simply connected Riemannian manifold with a lower curvature bound since it is an Alexandrov manifold with a lower curvature bound.*

Here are two examples about these spaces. The first example is an application of Mostow's rigidity theorem (see [**8**, Chapter 11]).

Example 1. Let X be a hyperbolic space with dimension$\geqslant 3$. X/G and X/Γ are both compact oriented space form with constant sectional curvature -1. The Mostow's rigidity theorem implies that G and Γ are conjugate in $\mathrm{Isom}(X)$ provided G and Γ are isomorphic.

This means that if G, $\Gamma \subsetneq \mathrm{Isom}(X)$ are both discrete and cocompact groups acting freely on X, then G and Γ are isomorphic implies that there exists $\Phi \in \mathrm{Isom}(X)$ such that $d_H(G, \Phi^{-1}\Gamma\Phi) = 0$.

Consider a simply connected Riemannian manifolds X with negative curvature. The first author showed in [**2**] a rigidity result for the discrete subgroups of $\mathrm{Isom}(X)$, which is in some sense a converse of Example 1.

EXAMPLE 2. Let X be a simply connected Riemannian n-manifold with sectional curvature K satisfying $-1 \leqslant K < 0$, and $\Gamma \subsetneq \mathrm{Isom}(X)$ be a discrete and cocompact subgroup acting freely on X. Denote

$$N_\Gamma \equiv \{a \in \mathrm{Diff}(X) \mid a\Gamma a^{-1} = \Gamma\}$$

to be the *normaliser* of Γ in the diffeomorphism group of X. It can be shown in [**1**] that for a finite extension G of Γ (that is, G/Γ is a finite group) with $G \subset N_\Gamma$ and

$$\sup\{d(gx, \Gamma x) \mid g \in G, x \in X\} \leqslant 4^{-(n+4)},$$

one has $G = \Gamma$. This indicates that if $G, \Gamma \subsetneq \mathrm{Isom}(X)$ are both discrete cocompact groups acting freely on X, then $G = \Gamma$ provided $G \subset N_\Gamma$, G/Γ is finite and $d_H(G, \Gamma) < 4^{-(n+4)}$.

REMARK. Fukaya and Yamaguchi proved in [**4**] that the group of isometries of an Alexandrove space with a lower curvature bound is in fact a Lie group. Moreover, the structure of the group of isometries of a given geometric space depends heavily on the geometric and topological properties of the space itself. For example, Wei showed in [**9**] that there are examples of complete manifolds of positive Ricci curvature with nilpotent isometry groups. Therefore, the main theorem gives an idea to investigate the relationship between group structure and geometric structure of a Lie group, and is helpful for one to study the group of isometries.

2. Proof of the Main Theorem

A proof of the main theorem will be presented. Recall some properties about a discrete and cocompact group $G \subset \mathrm{Isom}(X)$ acting on an Alexandrov space (see [**8**]). Let $P : X \to X/G$ be the quotient map and $B(x, r)$ denote the open ball centred at x with radius r. Then for each $x \in X$ with $P(x) = \overline{x}$, the map P induces a homeomorphism from $B(x, r)/G_x$ onto $B(\overline{x}, r)$ for all r such that

$$0 < r \leqslant (1/2)\,\mathrm{dist}(x, Gx - \{x\}).$$

Moreover, P induces an isometry from $B(x, r)/G_x$ onto $B(\overline{x}, r)$ for all r such that

$$0 < r \leqslant (1/4)\,\mathrm{dist}(x, Gx - \{x\}).$$

https://doi.org/10.1017/S0004972700038569 Published online by Cambridge University Press

A point $x \in X$ is called a *regular* point if $G_x = \{e\}$ is the trivial group, otherwise x is called a *singular* point of X. The set of all regular points of X is a connected, open, dense subset and then the set of all singular points of X is a closed nowhere dense subset. A point $\overline{x} = P(x) \in X/G$ is said to be a regular (or singular) point if x is a regular (or singular) point in X.

The *closed Dirichlet domain* for a regular point $x_0 \in X$ is the set

$$D_G(x_0) \equiv \{u \in X \mid d(u, x_0) \leqslant d(u, gx_0) \text{ for all } g \in G\}.$$

The open interior $D_G^0(x_0)$ of D_G is called the *open Dirichlet domain* for the regular orbit Gx_0. Since $X = \bigcup_{g \in G} gD_G(x_0)$, there is a *fundamental set* F_G in X containing x_0, which means that F_G meets each orbit in exactly one point, for the action of G satisfying $D_G^0(x_0) \subseteq F_G \subset D_G(x_0)$. Note that each point in $D_G^0(x_0)$ is a regular point and then $gD_G^0(x_0)$ consists of regular points for all $g \in G$. So the set of all singular points of X is contained in the set

$$\left\{g\big(D_G(x_0) - D_G^0(x_0)\big) \mid g \in G\right\}.$$

LEMMA 1. *Consider two discrete and cocompact subgroups G and Γ of* $\mathrm{Isom}(X)$ *of an Alexandrov space with a lower curvature bound. Let $P_1 : X \to X/G$ and $P_2 : X \to X/\Gamma$ be the quotient maps. Then $d_H(X/G, X/\Gamma) < \varepsilon$ provides $d_H(G, \Gamma) < \varepsilon$.*

PROOF: Since the groups G and Γ are both discrete and cocompact subgroups in $\mathrm{Isom}(X)$, there exist compact subsets D_G and D_Γ of X containing the same point x_0 such that $GD_G = X$ and $\Gamma D_\Gamma = X$. For each $x \in X$ the two sets $Gx \cap D_G$ and $\Gamma x \cap D_\Gamma$ are both finite. To prove this lemma, it suffices to show that there exists $g \in G$ such that $d_H(gD_\Gamma, D_G) < \varepsilon$.

Suppose there were $x \in D_G - D_\Gamma$ such that $d(x, gy) \geqslant \varepsilon$ for all $g \in G$ and $y \in D_\Gamma$. Since $x \notin D_\Gamma$, there exist $y \in D_\Gamma$ and $\gamma \in \Gamma$ such that $\gamma y = x$. This implies that $d(\gamma y, gy) = d(x, gy) \geqslant \varepsilon$ for all $g \in G$. It contradicts to the assumption that $d_H(G, \Gamma) < \varepsilon$. ∎

The quotient spaces X/G and X/Γ can be shown to be compact Alexandrov spaces with lower curvature bound. Perelman indicates in [**6**] that there exists $\varepsilon_1 = \varepsilon_1(G) > 0$ depending only on the group G such that if $d_H(X/G, X/\Gamma) < \varepsilon_1$ then there exists a homeomorphism $\phi : X/G \to X/\Gamma$, and ϕ is also an ε_1-*Hausdorff approximation*, which means that

$$\left|d\big(\phi(\overline{x}_1), \phi(\overline{x}_2)\big) - d(\overline{x}_1, \overline{x}_2)\right| < \varepsilon_1$$

for all $\overline{x}_1, \overline{x}_2 \in X/G$. Similar argument also applies to the inverse $\phi^{-1} : X/\Gamma \to X/G$ of ϕ.

LEMMA 2. *There exists $\varepsilon = \varepsilon(G) > 0$ such that if $d_H(G, \Gamma) < \varepsilon$ then for each $g \in G$ there is a unique $\gamma \in \Gamma$ such that $\overline{d}(g, \gamma) < \varepsilon$ for the pseudometric $\overline{d}$. Moreover, if $g_1, g_2 \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ satisfy $\overline{d}(g_1, \gamma_1) < \varepsilon$ and $\overline{d}(g_2, \gamma_2) < \varepsilon$ then $\overline{d}(g_1 g_2, \gamma_1 \gamma_2) < \varepsilon$.*

PROOF: Denote x_0 be a regular point in X, $\overline{x}_0 = P_1(x_0)$, $\overline{y}_0 = \phi(\overline{x}_0)$, $P_2(y_0) = \overline{y}_0$ and $D_G^0(x_0)$ be the open Dirichlet domain for x_0. The point x_0 can be chosen such that the open ball $B(x_0, r_0)$ with

$$r_0 = (1/2)\operatorname{dist}\bigl(x_0, Gx_0 - \{x_0\}\bigr)$$

is the largest ball contained in $D_G^0(x_0)$. Here the radius r_0 depends on the group G. Let

$$\varepsilon = \varepsilon(G) = min\{\varepsilon_1, (1/10)r_0\}$$

depending on the group G and $d(G, \Gamma) < \varepsilon$. Then $P_1\bigl(B(x_0, 5\varepsilon)\bigr)$ is isometric to the ball $B(\overline{x}_0, 5\varepsilon)$. Since by Lemma 1 ϕ is an ε-Hausdorff approximation and a homeomorphism,

$$B(\overline{y}_0, 3\varepsilon) \subset \phi\bigl(B(\overline{x}_0, 5\varepsilon)\bigr)$$

and $B(\overline{y}_0, 3\varepsilon)$ consists of regular points. This implies that $d(y_0, \gamma y_0) > 2\varepsilon$ for each nontrivial $\gamma \in \Gamma$.

Let γ_1, $\gamma_2 \in \Gamma$ such that $\overline{d}(g, \gamma_1) < \varepsilon$ and $\overline{d}(g, \gamma_2) < \varepsilon$. Then $\overline{d}(\gamma_1, \gamma_2) \leqslant 2\varepsilon$ by the triangle inequality. Denote $\gamma = \gamma_1^{-1}\gamma_2$ and then $\overline{d}(e, \gamma) \leqslant 2\varepsilon$. However, $d(y_0, \gamma y_0) > 2\varepsilon$ for each nontrivial $\gamma \in \Gamma$. Therefore γ is the identity and then $\gamma_1 = \gamma_2$. This proves the first part of Lemma 2.

Moreover, the triangle inequality implies that

$$\begin{aligned}\overline{d}(g_1g_2, \gamma_1\gamma_2) &\leqslant \overline{d}(g_1g_2, g_1\gamma_2) + \overline{d}(g_1\gamma_2, \gamma_1\gamma_2)\\ &= \overline{d}(g_2, \gamma_2) + \overline{d}(g_1, \gamma_1)\\ &\leqslant 2\varepsilon.\end{aligned}$$

Therefore, by the proof proposed in the first part, $\gamma_1\gamma_2$ is the unique element in Γ with $\overline{d}(g_1g_2, \gamma_1\gamma_2) < \varepsilon$. ∎

It can be shown that the homomorphism $\phi : X/G \to X/\Gamma$ between the quotient spaces can be lifted to a homeomorphism $\Phi : X \to X$.

LEMMA 3. *Suppose that $\phi : X/G \to X/\Gamma$ with $\phi(\overline{x}_0) = \overline{y}_0$ is a homeomorphism as above. Then there exists a homeomorphism $\Phi : X \to X$ with $\Phi(x_0) = y_0$ such that the following diagrams commute.*

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{\Phi} & (X, y_0) \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\ (X/G, \overline{x}_0) & \xrightarrow{\phi} & (X/\Gamma, \overline{y}_0) \end{array} \qquad \begin{array}{ccc} (X, x_0) & \xleftarrow{\Phi^{-1}} & (X, y_0) \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\ (X/G, \overline{x}_0) & \xleftarrow{\phi^{-1}} & (X/\Gamma, \overline{y}_0) \end{array}$$

PROOF: First note that X is a length space and has the property of local contractibility. Then, since $\phi : X/G \to X/\Gamma$ is a homeomorphism, ϕ lifts a homeomorphism

Φ_1 from the open Dirichlet fundamental domain $D_G^0(x_0)$ to an open set D_Γ^0 containing y_0 in X. Choose a fixed fundamental set F_G with $D_G^0(x_0) \subseteq F_G \subset D_G(x_0)$. For each $x' \in F_G - D_G^0(x_0)$, there is a sequence $\{x_i\}$ in $D_G^0(x_0)$ such that $x_i \to x'$ as $i \to \infty$. Define a new map Φ_2, which extends Φ_1, by $\Phi_2(x) \equiv \Phi_1(x)$ if $x \in D_G^0(x_0)$; and $\Phi_2(x') \equiv \lim_{i\to\infty} \Phi_1(x_i)$. Set $y' \equiv \Phi_2(x')$ and $F_\Gamma \equiv \Phi_2(F_G)$. Then we claim that the map $\Phi_2 : F_G \to F_\Gamma$ is a homeomorphism and F_Γ is a fundamental set for Γ.

Let $\{u_i\}$ be another sequence in $D_G^0(x_0)$ such that $\Phi_1(u_i) \to \gamma y'$ in X as $i \to \infty$ for some nontrivial $\gamma \in \Gamma$ with $\gamma y' \neq y'$. Without lose of generality, both of the sequences $\{\Phi_1(x_i)\}$ and $\{\Phi_1(u_i)\}$ can be assumed to consist of regular points. Let α_i denote a minimal geodesic from x_i to u_i, $\beta_i = \Phi_1(\alpha_i)$, $\overline{\alpha}_i = P_1(\alpha_i)$ and $\overline{\beta}_i = P_2(\widetilde{\beta}_i)$. Then ϕ maps $\overline{\alpha}_i$ homeomorphically to $\overline{\beta}_i$ for each i. However, $\overline{\alpha}_i$ tends to a point however $\overline{\beta}_i$ tends to a loop. It is impossible. Therefore the map Φ_2 is well-defined. Since F_G is a fundamental set for G, F_Γ is a fundamental set for Γ and hence $\Phi_2 : F_G \to F_\Gamma$ is a homeomorphism.

Next, Φ_2 can be extended to a map Φ defined on the whole X. For each $x \in X$ there is a unique $g \in G$ such that $x \in gF_G$. By Lemma 2, there is a unique $\gamma \in \Gamma$ such that $\overline{d}(g,\gamma) < \varepsilon$. So we define the map $\Phi : X \to X$ by

$$\Phi(x) \equiv \gamma \circ \Phi_2 \circ g^{-1}(x).$$

It is clear that ϕ is a bijection. Moreover, Φ maps homeomorphically the open dense set $\{gD_G^0(x_0) \mid g \in G\}$ in X to the open set $\{\gamma D_\Gamma^0 \mid \gamma \in \Gamma\}$ in X. It can be shown that the map $\Phi : X \to X$ is in fact a homeomorphism by the following argument. Let $x_i \in gF_G$ for all i and $x_i \to x''$ as $i \to \infty$. If $x'' \in gF_G$ then, by the above argument, we have

$$\Phi(x_i) = \gamma \circ \Phi_2 \circ g^{-1}(x_i) \to \gamma \circ \Phi_2 \circ g^{-1}(x'') = \Phi(x'')$$

as $i \to \infty$. On the other hand, it suffices to show that if $x_i \in F_G$ for all i, $x_i \to x''$ $\in \overline{F}_G - F_G$ and $x'' \in g'F_G$ for some nontrivial $g' \in G$, then $\Phi(x_i) \to \Phi(x'') \in \overline{F}_\Gamma - F_\Gamma$ and $\Phi(x'') \in \gamma' F_G$ with $\overline{d}(g',\gamma') < \varepsilon$.

Let $\overline{d}(g',\gamma') < \varepsilon$. Then $g'\overline{F}_G \cap \overline{F}_G \neq \emptyset$ if and only if $\gamma'\overline{F}_\Gamma \cap \overline{F}_\Gamma \neq \emptyset$. Therefore $\Phi(x'') \in \overline{F}_\Gamma - F_\Gamma$ Since $\Phi(x_i) \in F_\Gamma$ for all i, there exists $y'' \in \overline{F}_\Gamma - F_\Gamma$ such that $\Phi(x_i) \to y''$ as $i \to \infty$. Moreover,

$$P_2(y'') = P_2\big(\Phi(x'')\big) = \Phi\big(P_1(x'')\big).$$

Hence $y'' = \Phi(x'')$ and the proof of Lemma 3 is complete. ∎

Proof of the Main Theorem: Denote

$$\Phi^{-1}\Gamma\Phi \equiv \{\Phi^{-1} \circ \gamma \circ \Phi \mid \gamma \in \Gamma\}.$$

https://doi.org/10.1017/S0004972700038569 Published online by Cambridge University Press

Then, for given $\gamma \in \Gamma$ and $x \in X$, one has by Lemma 3 that

$$\begin{aligned} P_1 \circ \Phi^{-1} \circ \gamma \circ \Phi(x) &= \phi^{-1} \circ P_2 \circ \gamma \circ \Phi(x) \\ &= \phi^{-1} \circ P_2 \circ \Phi(x) \\ &= \phi^{-1} \circ \phi \circ P_1(x) \\ &= P_1(x). \end{aligned}$$

This shows that $\Phi^{-1} \circ \gamma \circ \Phi(x) = g_x x$ for some $g_x \in G$ and g_x depends on the point x. Note that $g_x = g$ for all $x \in F_G$, where $\overline{d}(g, \gamma) < \varepsilon$. Let $x_k = g_k x$ for $g_k \in G$, $\overline{d}(g_k, \gamma_k) < \varepsilon$ and $\Phi(x) = y$. Then by Lemma 2,

$$\begin{aligned} \Phi^{-1} \circ \gamma \circ \Phi(x_k) &= \Phi \circ \gamma\gamma_k(y) \\ &= g g_k(x) \\ &= g(x_k). \end{aligned}$$

This indicates that for each $\gamma \in \Gamma$, $\Phi^{-1} \circ \gamma \circ \Phi \in G$ and then $\Phi^{-1}\Gamma\Phi$ is a subgroup of G. Also, $\Phi G \Phi^{-1} \subseteq \Gamma$ is a subgroup of Γ. Therefore,

$$G = \Phi^{-1}\Phi G \Phi^{-1}\Phi \subseteq \Phi^{-1}\Gamma\Phi \subseteq G.$$

Hence $\Phi^{-1}\Gamma\Phi = G$ and the proof of the main theorem is complete. $\square$

REMARK. The approach can apply to manifolds with lower Ricci curvature bound. Consider a discrete and cocompact subgroup G of $\mathrm{Isom}(X)$ acting freely on a simply connect Riemannian n-manifold with Ricci curvature $\mathrm{Ric}_X \geqslant -(n-1)$. Cheeger and Colding proposed in ([**3**, Theorem A.1.2]) that compact Riemannian n-manifolds with lower Ricci curvature bound $-(n-1)$ and close Hausdorff distance will be diffeomorphic. Therefore there exists $\varepsilon = \varepsilon(G) > 0$ such that if Γ is another discrete and cocompact subgroup of $\mathrm{Isom}(X)$ acting freely on X with $d_H(G, \Gamma) < \varepsilon$, then G and Γ are conjugate in the group of homeomorphisms of X.

REFERENCES

[1] Y. Burago, M. Gromov and G. Perelman, 'A.D. Aleksandrov spaces with curvature bounded below', *Uspehi Mat. Nauk* **47** (1992), 3–51. Translation in *Russian Math. Surveys* **47** (1992) 1–58.

[2] W.-H. Chen, 'Finite group actions on compact negatively curved manifolds', *Arch. Math.* **77** (2001), 430–433.

[3] J. Cheeger and T. H. Colding, 'On the structure of spaces with Ricci curvature bounded from below', *J. Diff. Geom.* **46** (1997), 406–480.

[4] K. Fukaya and T. Yamaguchi, 'Isometry groups of singular spaces', *Math. Z.* **216** (1994), 31–44.

[5] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics **152** (Birkhauser Boston Inc., Boston, 1999).

[6] G. Perelman, 'Aleksandrov spaces with curvature bounded from below II', (preprint).

[7] P. Petersen V, 'Gromov-Hausdorff convergence of metric spaces', in *Differential Geometry*, (S.-T Yau and R. Green, Editors), Proc. Symp. Pure. Math. **54** (American Mathematical Society, Providence, RI, 1993), pp. 489–504.

[8] J. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics **149** (Springer-Verlag, New York, 1994).

[9] G. Wei, 'Examples of complete manifolds of positive Ricci curvature with nilpotent isometry groups', *Bull. Amer. Math. Soc.* **19** (1988), 311–313.

Department of Mathematics
Tunghai University
Taichung 40704
Taiwan
e-mail: whchen@thu.edu.tw

Department of Mathematics
National Chung-Cheng University
Chai-Yi 621
Taiwan
e-mail: jywu@math.ccu.edu.tw