NON-SECULAR, LOCALLY COMPACT TRL GROUPS

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(Received 25 October, 1977)

1. Introduction. In [**12**], Loy and Miller proved that a locally compact, eudoxian TR group is algebraically and order-theoretically (and hence, topologically) isomorphic to a finite product of copies of the real numbers. In [**18**], Wirth used their result to describe the subgroup of a locally compact TR group generated by the compact neighbourhoods of zero. The proof of Loy and Miller relied heavily on a result of Mackey (cf. [**10**], p. 390) and either the finite-dimensional case of the Choquet–Kendall Theorem (cf. [**15**], pp. 9–10) or the representation theory of Kakutani (cf. [**11**], Appendix). Below we use only elementary topological results and order-theoretic arguments and a theorem of Conrad [**4**] to characterize all non-secular, locally compact TRL groups (Theorem 3). Our proof of Theorem 3 allows us to deduce algebraically the theorems both of Loy and Miller and of Wirth, in both cases without appealing to the theorem of Conrad.

As general references, we use Birkhoff [**1**], Conrad [**1**], Fuchs [**8**], and Bourbaki [**2**].

2. Locally compact, non-secular TRL groups. Let $(G, \leqslant)$ be a po-group, with *positive cone* $(G,\leqslant)^+ \equiv \{g \in G \mid g \geqslant 0\}$ and *strictly positive cone* $(G,\leqslant)^* = \{g \in G \mid g > 0\}$. An element $g \in G$ is *pseudo-positive* [**9**] if $g + (G,\leqslant)^* \subseteq (G,\leqslant)^*$ and $g \notin (G,\leqslant)^+$; g is a *pseudo-zero* [**9**] of $(G,\leqslant)$ if both g and $-g$ are pseudo-positive. The order $\leqslant$ on G is a *tight Riesz order* [**12**] if $\leqslant$ satisfies the *tight Riesz interpolation property* [**3**]: $a,b < x,y$ implies that there exists $g \in G$ with $a,b < g < x,y$. If $\leqslant$ is a directed tight Riesz order without pseudo-zeros, then $(G,\leqslant)$ is a *TR group* and $(G,\leqslant)^+$, together with the pseudo-positives of $(G,\leqslant)$, forms the positive cone of another directed partial order, $\preccurlyeq$, on G; this order is called the *associated order* [**17, 13**] on $(G,\leqslant)$, and $\leqslant$ is called a *compatible tight Riesz order* (abbreviated: CTRO) [**17**] on $(G,\preccurlyeq)$. A TR group $(G,\preccurlyeq,\leqslant)$ is *non-secular* [**14**] if $b=0$ whenever $b \in G$, $0 < a \in G$, $a \wedge b$ exists in $(G,\preccurlyeq)$, and $a \wedge b = 0$. Of paricular interest will be, of course, the TR groups $(G,\preccurlyeq,\leqslant)$ for which $(G,\preccurlyeq)$ is an l-group. These groups are called TRL *groups* [**14**].

In [**17**], Wirth characterized the CTRO's on a TRL group, and in [**16**], Reilly noted that Wirth's proof remained true for TR groups. Specifically Wirth and Reilly showed that a subset P of a directed po-group $(G,\preccurlyeq)$ with no pseudo-positives is the strictly positive cone of a CTRO on $(G,\preccurlyeq)$ if and only if P satisfies

(a) P is a normal subset of G;
(b) P is a proper dual ideal of $(G,\preccurlyeq)^+$;
(c) $P+P=P$;
(d) $\bigwedge P = 0$.

†The author thanks the Australian Research Grants Committee and Professor J. B. Miller for their support while this research was carried out.

(By a *proper dual ideal* of a poset $(E, \leq)$, we mean a non-empty set D, strictly contained in E, such that if $x, y \in D$ and $z \in E$ are such that $x \leq z$, then $z \in D$ and there exists $w \in D$ such that $w \leq x, y$.) This characterization provides an easy way of determining what convex subgroups of $(G, \preccurlyeq)$ are TR groups in the induced orders.

PROPOSITION 1. *Let $(G, \preccurlyeq, \leq)$ be a TR group, and let $(H, \preccurlyeq)$ be a convex directed subgroup of $(G, \preccurlyeq)$. Then the following statements are equivalent:*

(i) *$(H, \preccurlyeq, \leq)$ is a non-trivial TR group;*

(ii) *$H \cap (G, \leq)^* \neq \square$;*

(iii) *$(G, \leq)^*$ is the dual ideal of $(G, \preccurlyeq)^+$ generated by $(H, \leq)^*$.*

Proof. Note that $H \cap (G, \leq)^* = (H, \leq)^*$. Thus, (iii) implies (ii). That (i) implies (iii) follows easily from (b) and the convexity of H. If (ii) holds, then by the convexity of H, $(H, \leq)^*$ must satisfy (a)–(d) because $(G, \leq)^*$ does. Thus (ii) implies (i).

On any po-group $(G, \leq)$, one may define the *open-interval topology*, $\mathcal{U}(G, \leq)$, by taking as a subbase for the open sets the open intervals $(a, b) \equiv \{g \in G \mid a < g < b\}$. On a TR group $(G, \leq)$, $\mathcal{U}(G, \leq)$ is a Hausdorff group topology whose neighbourhoods of 0 are generated by $\{(-a, a) \mid a > 0\}$ [**12**]. We may thus associate topological notions to any TR group $(G, \leq)$ by applying them to $\mathcal{U}(G, \leq)$. In particular, we will be interested in locally compact TR groups $(G, \preccurlyeq, \leq)$, i.e. those for which $\mathcal{U}(G, \leq)$ is locally compact.

An l-group $(L, \preccurlyeq)$ is a *lexico-extension* [**4**] of an l-group S if S is an l-*ideal* (i.e., a normal subgroup and a convex sublattice) of L and every positive element of $L \backslash S$ exceeds every element of S. If $T_1, \ldots, T_n$ are non-trivial o-groups, then an l-group $(G, \preccurlyeq)$ is a *lexico-sum* [**4**] of the T_i if there exist lexico-extensions L_i of T_i, l-groups G_i, and a permutation α of $\{1, \ldots, n\}$ such that (1) $G = G_{\alpha(n)}$ (2) $G_1 = L_{\alpha(1)}$, and (3) for $1 < k \leq n$, G_k is a lexico-extension of $L_{\alpha(k)} |\times| G_{k-1}$, where $|\times|$ denotes the cardinal product. A subset S of an l-group $(L, \preccurlyeq)$ is said to be *disjoint* if $S \subseteq (L, \preccurlyeq)^*$ and $a \wedge b = 0$ whenever a and b are distinct elements of S. An element $b \in L$ is *basic* [**5**] if $b > 0$ and $\{g \in G \mid 0 \preccurlyeq g \preccurlyeq b\}$ is totally ordered; a subset $B \subseteq L$ is a *basis* [**5**] of $(L, \preccurlyeq)$ if B is a maximal disjoint subset of L, every element of which is basic. The following result was noted in [**6**] on p. 101; it is essentially the main result of [**4**].

THEOREM 2. (Conrad). *Let n be a positive integer. An l-group is a lexico-sum of n o-subgroups if and only if it contains a basis with exactly n elements.*

The main result of this paper may now be stated.

THEOREM 3. *For an l-group $(G, \preccurlyeq)$, the following statements are equivalent.*

(i) *There exists a partial order $\leq$ on G such that $(G, \preccurlyeq, \leq)$ is a locally compact, non-secular TRL group.*

(ii) *There exists a positive integer n such that $(G, \preccurlyeq)$ is a lexico-sum of n copies of the additive real numbers.*

(iii) *There exists a positive integer m such that there are exactly m distinct CTRO's on $(G, \preccurlyeq)$, all of which are locally compact and exactly one of which is non-secular.*

The proof of Theorem 3 requires Theorem 2, and two elementary topological results (Propositions 4 and 5 below).

Let $(G, \preccurlyeq, \leqslant)$ be a TR group. For $a,b \in G$, let $[\![a, b]\!] = \{g \in G \mid a \preccurlyeq g \preccurlyeq b\}$ and for $A \subseteq G$, let A^- be the closure of A with respect to the open interval topology $\mathfrak{U}(G, \leqslant)$. It is easy to see that for all $a, b \in G$, $[\![a, b]\!]^- = [\![a, b]\!]$ and if $a < b$, then $(a, b)^- = [\![a, b]\!]$. Let Com_G be 0 together with all $0 < g \in G$ such that $(0, g)^-$ is compact, and let $C(G)$ be the subgroup of G generated by Com_G. The following is an immediate consequence of [**12**], 2.14°, and [**3**], 2.10°.

Proposition 4. *For a TR group $(G, \preccurlyeq, \leqslant)$, $C(G)$ is a normal convex, directed subgroup of $(G, \preccurlyeq)$ such that $(C(G), \leqslant)^+ = \mathrm{Com}_G$, and $(C(G), \preccurlyeq)$ is a complete, and therefore an archimedean and abelian, l-group.*

Proposition 5. *For a TR group $(G, \preccurlyeq, \leqslant)$, no element $0 \prec g \in C(G)$ exceeds an infinite disjoint set, and hence $(C(G), \preccurlyeq)$ has a basis.*

Proof. Suppose $0 \prec g \in C(G)$ exceeds an infinite disjoint set $\{a_1, \ldots, a_n, \ldots\}$ and let $0 < t \in C(G)$ be such that $g \preccurlyeq t$. For each $n = 1, 2, \ldots$, let k_n be a positive integer such that $k_n a_n \preccurlyeq t$ but $(k_n + 1)a_n \not\preccurlyeq t$ (such an integer exists by Proposition 4). Let $b_n = (k_n + 1)a_n$. Then for each n,

$$b_n \preccurlyeq k_n a_n + k_n a_n \preccurlyeq t + t.$$

By Proposition 4, $[\![0, t+t]\!]$ is complete, and hence $\bigvee b_n$ exists in $(C(G), \preccurlyeq)$. Since by Propositions 1 and 4 $\bigwedge(C(G), \leqslant)^* = 0$ in $(C(G), \preccurlyeq)$, for any $0 \prec d \in C(G)$, we may choose $0 < t(d) \in C(G)$ such that $d \not\preccurlyeq t(d)$. If $0 \prec x \preccurlyeq \bigvee b_n$, then $x - t(x) < x < \bigvee b_n + t$; hence

$$[\![0, \bigvee b_n]\!] \subseteq (-t, t) \cup \bigcup_j [\bigcup \{(d - t(d), \bigvee b_n + t) \mid 0 \prec d \preccurlyeq b_j\}].$$

By Proposition 4 again, $[\![0, t+t]\!]$ is compact. Hence, since $[\![0, \bigvee b_n]\!]$ is (topologically) closed, $[\![0, \bigvee b_n]\!]$ is compact, and thus, there exist $d_1, \ldots, d_m$ such that

$$[\![0, \bigvee b_n]\!] \subseteq (-t, t) \cup \bigcup_i (d_i - t(d_i), \bigvee b_n + t).$$

Pick n so that $d_i \not\preccurlyeq b_n$ for all $i = 1, \ldots, m$. Then, since $\{b_j\}$ is disjoint, $b_n \wedge d_i = 0$ for all i, hence $(b_n + t(d_i)) \wedge (d_i + t(d_i)) = t(d_i)$ for all i, and thus, by our choice of $t(d_i)$, $b_n \not> d_i - t(d_i)$, i.e. $b_n \notin \bigcup_i (d_i - t(d_i), \bigvee b_n + t)$. But also by our original choice of the b_n, $b_n \not< t$. This is a contradiction, and hence $0 \prec g \in C(G)$ cannot exceed an infinite disjoint set. The usual argument (e.g. [**7**], p. 3.31) shows that therefore every $0 \prec g \in C(G)$ must exceed a basic element, and this is clearly equivalent to $(C(G), \preccurlyeq)$ possessing a basis.

Proof of Theorem 3. Suppose (i) holds, and let $0 < g \in G$ be such that $(0, g)^-$ is compact. If $\{a_n\}$ is an infinite disjoint subset of $(G, \preccurlyeq)$, then $\{g \wedge a_n\}$ is also an infinite disjoint set because $(G, \preccurlyeq, \leqslant)$ is non-secular. But $g \succcurlyeq g \wedge a_n$ for all n, a contradiction of Proposition 5. Thus, $(G, \preccurlyeq)$ contains no infinite disjoint subsets. Again the usual argument (e.g. [**7**], p. 3.31) shows that $(G, \preccurlyeq)$ must have a basis, and since a basis is a disjoint set, it too must be finite. Thus, we may apply Theorem 2 and conclude that $(G, \preccurlyeq)$ is a lexico-sum of n o-groups $S_1, \ldots, S_n$. Since $(G, \preccurlyeq, \leqslant)$ is non-secular, and since each S_i is a convex o-subgroup, $T_i \equiv S_i \cap C(G) \neq \{0\}$ for all i. By Proposition 4, and the definition of lexico-sum, each T_i is normal in G, and hence $(G, \preccurlyeq)$ is also a lexico-sum of the n o-groups $T_1, \dots, T_n$. By Proposition 4, each $(T_i, \preccurlyeq)$ is a complete o-group and hence isomorphic to either the real numbers or the integers. If $(T_i, \preccurlyeq)$ is isomorphic to the integers for some i, then there exists an element $t \in T_i$ which covers 0. But there exists $0 < g \in G$ such that $g \wedge t < t$, i.e., such that $g \wedge t = 0$. This contradicts the non-secularity of $(G, \preccurlyeq, \leqslant)$, and hence each $(T_i, \preccurlyeq)$ is isomorphic to the real numbers. Therefore, (ii) holds.

Next, suppose that (ii) holds, and let $R_1, \dots, R_n$ be the n copies of the real numbers of which $(G, \preccurlyeq)$ is a lexico-sum. Let H be the subgroup of G generated by the R_i. From the definition of the lexico-sum, it is clear that each R_i is a convex o-subgroup of $(G, \preccurlyeq)$ and that $R_i \cap R_j = \{0\}$ if $i \neq j$. From this, it follows easily that H is a convex l-subgroup of $(G, \preccurlyeq)$ which is isomorphic to the cardinal product of the R_i. For $1 \leqslant i \leqslant n$, let

$$M_i \equiv \{g \in G \mid \text{for some } 0 < r_i \in R_i,\ g \succcurlyeq r_i\},$$

and for each non-empty subset S of $\{1, \dots, n\}$, let $P_S = \bigcap_S M_i$. Clearly, we have defined $m \equiv 2^n - 1$ distinct subsets P_S of G, and it is straightforward to show that each P_S is the strictly positive cone of a CTRO on $(G, \preccurlyeq)$. It is also clear that the order defined by P_S is secular if S is strictly contained in $\{1, \dots, n\}$ and non-secular if $S = \{1, \dots, n\}$. Suppose that P is the strictly positive cone of a CTRO $\leqslant$ on $(G, \preccurlyeq)$. For $i = 1, \dots, n$, let $p_i \in P$ be such that $p_i \wedge r_i < r_i$ for some $r_i \in R_i$. Then $\bigwedge p_i \not\succcurlyeq r_j$ for all $j = 1, \dots, n$, and hence by definition of the lexico-sum, $\bigwedge p_i \in H$. Since $\bigwedge p_i \in (G, \leqslant)^*$, $(H, \preccurlyeq, \leqslant)$ is a TRL group by Proposition 1. Since $(H, \preccurlyeq)$ is the cardinal product of the R_i, we must have $(H, \leqslant)^* = H \cap P_S$ for some non-empty subset S of $\{1, \dots, n\}$, and hence by Proposition 1, we must have $P = P_S$ for some non-empty subset S of $\{1, \dots, n\}$. Thus, (iii) holds.

Clearly, (iii) implies (i), and this proves Theorem 3.

Let $\mathbf{R}$ be the additive o-group of real numbers, and for any positive integer n, let $(\mathbf{R}^n, \preccurlyeq, \leqslant)$ denote the TRL group whose underlying group is $\mathbf{R}^n$, whose lattice-order $\preccurlyeq$ is the cardinal order, and whose CTRO $\leqslant$ is the *strong pointwise order*: $(r_1, \dots, r_n) > (0, \dots, 0)$ if and only if $r_i > 0$ for all i. A po-group $(G, \leqslant)$ is *eudoxian* [**14**] if and only if whenever $a, b \in (G, \leqslant)^*$, there exists a positive integer n such that $na \geqslant b$.

Corollary 6. *Let $(G, \preccurlyeq, \leqslant)$ be a TR group such that $G = C(G)$. If $(G, \preccurlyeq, \leqslant)$ is non-secular, then $(G, \preccurlyeq, \leqslant)$ is isomorphic to $(\mathbf{R}^m, \preccurlyeq, \leqslant)$ for some integer $m > 0$.*

Proof. Proposition 4 and the proof of Theorem 3 ((i) implies (ii)) show—without using Theorem 2—that $(G, \preccurlyeq)$ is a complete archimedean l-group with a finite basis such that no element of $(G, \preccurlyeq)^*$ covers 0. From this, it follows easily that $(G, \preccurlyeq)$ is isomorphic to $(\mathbf{R}^m, \preccurlyeq)$ for some $m > 0$. As in the proof of Theorem 3 ((ii) implies (iii), it is clear that $\leqslant$ defined above is the only non-secular CTRO on $(\mathbf{R}^m, \preccurlyeq)$, and hence $(G, \preccurlyeq, \leqslant)$ must be isomorphic to $(\mathbf{R}^m, \preccurlyeq, \leqslant)$.

Corollary 7 (Loy and Miller [**12**], Theorem 5.1°). *Let $(G, \preccurlyeq, \leqslant)$ be a locally compact TR group and suppose that $(G, \leqslant)$ is eudoxian. Then $(G, \preccurlyeq, \leqslant)$ is isomorphic to $(\mathbf{R}^m, \preccurlyeq, \leqslant)$ for some integer $m > 0$.*

https://doi.org/10.1017/S0017089500003918 Published online by Cambridge University Press

Proof. Since $(G, \leq)$ is eudoxian, $G = C(G)$, and hence by Corollary 6, it suffices to show that $(G, \preccurlyeq, \leq)$ is non-secular. If $0 \prec g \in G$ and $0 < t \in G$ are such that $g \wedge t = 0$ in $(G, \preccurlyeq)$, then $g \wedge nt = 0$ for all positive integers n. But $0 < g + t$ and by the eudoxian property $nt > g + t$ for some n, i.e., $(n-1)t > g$. This is a contradiction and hence $(G, \preccurlyeq, \leq)$ is non-secular. Note that since Corollary 6 did not require Theorem 2, neither does Corollary 7.

3. Consequences of the characterization. Theorem 3 dealt with non-secular TRL groups. Dropping the requirement of non-secularity allows a great deal of leeway—in fact, by Corollary 10 below, any l-group with a convex l-subgroup isomorphic to the real numbers possesses a locally compact CTRO. However, Propositions 1 and 4 do clearly imply the following.

PROPOSITION 8. *For a TR group $(G, \preccurlyeq, \leq)$, the following statements are equivalent*

(i) *$\mathcal{U}(G, \leq)$ is locally compact:*
(ii) *$C(G) \neq \{0\}$;*
(iii) *$(G, \leq)^*$ is the dual ideal of $(G, \preccurlyeq)^+$ generated by $(C(G), \leq)^*$.*

In view of Proposition 8, it seems reasonable in the general case to turn our attention to the structure of $C(G)$, and in this spirit, we use Proposition 5 to prove Theorem 11 below.

PROPOSITION 9. *Every TRL group $(G, \preccurlyeq, \leq)$ contains a convex l-subgroup $N(G)$ of $(G, \preccurlyeq)$ satisfying:*

(i) *$(N(G), \preccurlyeq, \leq)$ is a non-secular TRL group:*

(ii) *if K is a convex l-subgroup of $(G, \preccurlyeq)$ such that $(K, \preccurlyeq, \leq)$ is a non-secular TRL group, then $K \subseteq N(G)$.*

Proof. Let S_G be the set of all $g \in G$ such that for some $0 < t \in G$, $g \wedge t = 0$ in $(G, \preccurlyeq)$ (S_G is essentially T^- of [**13**]), and let $N(G)$ be the set of all $y \in G$ such that there exists $0 \leq f \in G$ such that $-f \preccurlyeq y \preccurlyeq f$ and, for all $g \in S_G$, $f \wedge g = 0$ in $(G, \preccurlyeq)$. That $N(G)$ is a convex l-subgroup of $(G, \preccurlyeq)$ follows from the elementary theory of l-groups (see, e.g., [**8**], p. 70, B)). By definition, either $N(G) = 0$ or $N(G) \cap (G, \leq)^* \neq \square$. Thus, by Proposition 1, $(N(G), \preccurlyeq, \leq)$ is a TRL group, and it is clear, again from the definition, that $(N(G), \preccurlyeq, \leq)$ is non-secular. If K is as described in (ii), then either $K = \{0\} \subseteq N(G)$ or there exists $0 < k \in K$. If $K \nsubseteq N(G)$, then there exists $0 \prec s \in S_G \cap K$. Let $0 < t \in G$ be such that $t \wedge s = 0$. Since $0 < k$, $0 < k \wedge t \in K$. Then $(k \wedge t) \wedge s = 0$, which contradicts the non-secularity of $(K, \preccurlyeq, \leq)$. Thus, $K \subseteq N(G)$.

(The following example shows that it need not be the case that $N(G) = \{g \mid |g| \wedge s = 0$ for all $s \in S_G\}$: Let $(G, \preccurlyeq)$ be the cardinal product of a countable number of copies of the real numbers. Define a CTRO $\leq$ on $(G, \preccurlyeq)$ by letting $f > 0$ if and only if $f_1 > 0$ and there exists m such that $f_n > 0$ for all $n \geq m$. Then $S_G = \{f \mid f_1 = 0$ and $f_n = 0$ for all but a finite number of $n > 1\}$, but $N(G) = \{0\}$.)

LEMMA 10. *Let $(G, \preccurlyeq, \leq)$ be a locally compact TR group. If T is an o-subgroup of $(C(G), \preccurlyeq)$ such that $\bigwedge (T, \preccurlyeq)^* = 0$, then $T \subseteq N(C(G))$.*

Proof. If $T \nsubseteq N(C(G))$, then there exists $t \in T \cap S_{C(G)}$. Let $0 < g \in C(G)$ be such that $g \vee t = 0$. Since $\bigwedge(T, \preccurlyeq)^* = 0$, it is easy to see that $\{(x-g, x+g) \mid 0 \preccurlyeq x \preccurlyeq t\}$ is an open cover of $[\![0, t]\!]$ which has no finite subcover. Thus, $[\![0, t]\!]$ is not compact. But $[\![0, t]\!]$ is a closed bounded subset of $(C(G), \preccurlyeq)^+$, and hence by Proposition 4, $[\![0, t]\!]$ is compact. This is a contradiction. Thus, $T \subseteq N(C(G))$.

Let $(G, \preccurlyeq, \leqslant)$ be a TRL group, and let $\mathbf{Z}$ be the o-group of integers. For any cardinal $\aleph$, let $(G \times \sum_{\aleph} \mathbf{Z}, \preccurlyeq, \leqslant)$ be the TRL group whose underlying group is $G \times \sum_{\aleph} \mathbf{Z}$, whose lattice-order $\preccurlyeq$ is the cardinal order, and whose CTRO $\leqslant$ is generated by $(G, \leqslant)^*$: $(g, f) > (0, 0)$ if and only if $g > 0$ and $f \succcurlyeq 0$ in the cardinal order of $\sum_{\aleph} \mathbf{Z}$. Note that $\mathcal{U}(G \times \sum_{\aleph} \mathbf{Z}, \leqslant)$ is just the product of $\mathcal{U}(G, \leqslant)$ and the discrete topology on $\sum_{\aleph} \mathbf{Z}$.

The following result, which, with Proposition 8, characterizes $(C(G), \preccurlyeq, \leqslant)$ for all TR groups $(G, \preccurlyeq, \leqslant)$, is essentially a theorem of Wirth [**18**] (cf. §1). Besides the proof, the main novelty in the theorem below is the introduction of $N(C(G))$.

THEOREM 11. *Let $(G, \preccurlyeq, \leqslant)$ be a locally compact TR group. Then there exist a positive integer n and a cardinal $\aleph$ such that*

(1) *$(N(C(G)), \preccurlyeq, \leqslant)$ is isomorphic to $(\mathbf{R}^n, \preccurlyeq, \leqslant)$, and*

(2) *$(C(G), \preccurlyeq, \leqslant)$ is isomorphic to $(N(C(G)) \times \sum_{\aleph} \mathbf{Z}, \preccurlyeq, \leqslant)$.*

Proof. By Propositions 4 and 5, $(C(G), \preccurlyeq, \leqslant)$ is a complete archimedean l-group with a basis, say $\{b_\alpha\}$. For each α, let B_α be the maximal convex o-subgroup of $(C(G), \preccurlyeq)$ containing b_α. Then $(C(G), \preccurlyeq)$ is isomorphic to the cardinal sum of the B_α by [**5**], Theorem 7.2, or the following direct proof.

It is easy to show (cf. [**7**], p. 3.14) that $(B, \preccurlyeq)$, the convex l-subgroup of $(G, \preccurlyeq)$ generated by the B_α, is isomorphic to the cardinal sum of the B_α. Let $0 \prec g \in C(G)$. By Proposition 5, there exist $b_1, \ldots, b_n \in \{b_\alpha\}$ such that $g \wedge b_\beta \succ 0$ if and only if $b_\beta = b_i$ for some i. Since $(C(G), \preccurlyeq)$ is archimedean, there exists, for each i, $0 \prec k_i \in B_i$ such that $g \not\succcurlyeq k_i$. Since $k_i \not\preccurlyeq g \wedge (k_i + k_i) \in B_i$, $g \wedge k_i \preccurlyeq g \wedge (k_i + k_i) \preccurlyeq k_i$, and hence $g \wedge (k_i + k_i) = g \wedge k_i$. Thus, $g \wedge \bigvee (k_i + k_i) = g \wedge \bigvee k_i$, i.e., $(g - \bigvee k_i) \wedge \bigvee k_i = (g - \bigvee k_i) \wedge 0$. Then

$$(g - g \wedge \bigvee k_i) \wedge \bigvee k_i = 0 \vee ((g - \bigvee k_i) \wedge 0) = 0,$$

and thus, since $g \succcurlyeq g - g \wedge \bigvee k_i \succcurlyeq 0$, $(g - g \wedge \bigvee k_i) \wedge b_\alpha = 0$ for all α. Since $\{b_\alpha\}$ is a basis, $g = g \wedge \bigvee k_i \in B$. We conclude that $C(G) = B$, i.e., that $(C(G), \preccurlyeq)$ is isomorphic to the cardinal product of the B_α.

Therefore, if no B_α is such that $\bigwedge(B_\alpha, \preccurlyeq)^* = 0$ then for all $0 \prec g \in C(G)$, the cardinality of $[\![0, g]\!]$ is finite, and hence $(C(G), \preccurlyeq)^+$ contains no proper dual ideal whose greatest lower bound is 0. Thus, at least one B_α satisfies $\bigwedge(B_\alpha, \preccurlyeq)^* = 0$, and hence by Lemma 10, $N(C(G)) \neq \{0\}$. Since clearly $C(N(C(G))) = N(C(G))$, Corollary 6 implies that there exists an integer $m > 0$ such that $(N(C(G)), \preccurlyeq, \leqslant)$ is isomorphic to $(\mathbf{R}^m, \preccurlyeq, \leqslant)$. (Note that Theorem 2 was not used to prove Corollary 6). By Lemma 10, every B_α not contained in $N(C(G))$ must be isomorphic to $\mathbf{Z}$, and hence by Proposition 1, $(C(G), \preccurlyeq, \leqslant)$ must be isomorphic to $(N(C(G)) \times \sum_{\aleph} \mathbf{Z}, \preccurlyeq, \leqslant)$, where $\aleph$ is the cardinality of $\{\alpha \mid B_\alpha$ is isomorphic to $\mathbf{Z}\}$.

COROLLARY 12. *A directed po-group* $(G, \preccurlyeq)$ *without pseudo-positives possesses a partial order* $\leqslant$ *such that* $(G, \preccurlyeq, \leqslant)$ *is a locally compact TR group if and only if there is a convex subgroup of* $(G, \preccurlyeq)$ *isomorphic to the additive o-group of real numbers.*

Proof. If $(G, \preccurlyeq)$ possesses such a subgroup, say H, then the dual ideal of $(G, \preccurlyeq)^+$ generated by $(H, \preccurlyeq)^*$ is easily seen to be the strictly positive cone of a CTRO $\leqslant$ on $(G, \preccurlyeq)$ (that is, it obviously satisfies conditions (a)–(d) of §2). Clearly $\mathfrak{U}(G, \leqslant)$ is a locally compact topology. The converse follows from Theorem 11.

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