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CORRIGENDUM TO "CLUSTER CATEGORIES FROM GRASSMANNIANS AND ROOT COMBINATORICS"

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Abstract. In this note, we correct an oversight regarding the modules from Definition 4.2 and proof of Lemma 5.12 in Baur *et al.* (Nayoga Math. J., 2020, **240**, 322–354). In particular, we give a correct construction of an indecomposable rank 2 module $\mathbb{L}(I,J)$, with the rank 1 layers I and J tightly 3-interlacing, and we give a correct proof of Lemma 5.12.

§1. Indecomposable rank 2 modules with tightly 3-interlacing layers

In [1], we studied the category $\mathrm{CM}(B_{k,n})$ of Cohen–Macaulay modules over the completion of an algebra $B_{k,n}$, which is a quotient of the preprojective algebra of type A_{n-1}. The category $\mathrm{CM}(B_{k,n})$ is important in a categorification of the cluster algebra structure on the homogeneous coordinate ring $\mathbb{C}[Gr(k,n)]$ of the Grassmannian variety of k-dimensional subspaces in $\mathbb{C}^n$ (see [3]–[5]).

For the notation and background results used in this note, we refer the reader to [1 , Sect. 1]. We thank Karin Erdmann and Alastair King for useful conversations about indecomposable modules.

In [1 , Def. 4.2], we constructed a Cohen–Macaulay module of an arbitrary rank. In the case of rank 2, in [1 , Lem. 5.12], we claimed that the constructed module is indecomposble. In fact, the rank 2 module from this lemma is not indecomposable. The aim of this note is to correct this mistake, that is, for given k-subsets I and J that are tightly 3-interlacing, to construct explicitly an indecomposable rank 2 Cohen–Macaulay module with filtration $L_I \mid L_J$.

We show that this module is indecomposable by proving that its endomorphism ring does not have nontrivial idempotents.

Assume that we are in the case when I and J are tightly 3-interlacing k-subsets ($|I\setminus J| = |J\setminus I| = 3$ and noncommon elements of I and J interlace). Write $I\setminus J$ as $\{i_1,i_2,i_3\}$ and $J\setminus I = \{j_1,j_2,j_3\}$ so that $1\le i_1<j_1<i_2<j_2<i_3<j_3\le n$.

The following construction covers all indecomposable rank 2 modules in case when the category $\mathrm{CM}(B_{k,n})$ is tame and $(k,n)=(3,9)$.

We want to define a rank 2 module $\mathbb{L}(I,J)$ in $\mathrm{CM}(B_{k,n})$ in a similar way as rank 1 modules are defined in [5]. Let $V_i := \mathbb{C}[|t|]\oplus\mathbb{C}[|t|]$, $i=1,\dots,n$. The module $\mathbb{L}(I,J)$ has V_i at each vertex $1,2,\dots,n$ of Γ_n, where Γ_n is the quiver of the boundary algebra, that is, with vertices $1,2,\dots,n$ on a cycle and arrows $x_i : i-1\to i$, $y_i : i\to i-1$. Observe the following

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matrices:

$$A_1 := \begin{pmatrix} t & -2 \\ 0 & 1 \end{pmatrix}, \quad B_1 := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 := \begin{pmatrix} t & -1 \\ 0 & 1 \end{pmatrix}, \quad D_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$A_2 := \begin{pmatrix} 1 & 2 \\ 0 & t \end{pmatrix}, \quad B_2 := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \quad C_2 := \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix}, \quad D_2 := \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Note that these are all matrix factorisations of $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$: $A_1A_2 = B_1B_2 = C_1C_2 = D_1D_2 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$.

Definition 1.1. Let I,J be tightly 3-interlacing k-subsets of $\{1,2,\dots,n\}$. At the vertices of Γ_n, $\mathbb{L}(I,J)$ has the spaces $V_1,\dots,V_n$. We define the maps x_i, y_i as follows:

$$x_i : V_{i-1} \to V_i \text{ acts as } \begin{cases} A_1, & \text{if } i = i_1, \\ B_2, & \text{if } i = j_1, \\ B_1, & \text{if } i = i_2, \\ C_2, & \text{if } i = j_2, \\ C_1, & \text{if } i = i_3, \\ A_2, & \text{if } i = j_3, \\ D_1, & \text{if } i \in I \cap J, \\ D_2, & \text{if } i \in I^c \cap J^c. \end{cases} \qquad y_i : V_i \to V_{i-1} \text{ acts as } \begin{cases} A_2, & \text{if } i = i_1, \\ B_1, & \text{if } i = j_1, \\ B_2, & \text{if } i = i_2, \\ C_1, & \text{if } i = j_2, \\ C_2, & \text{if } i = i_3, \\ A_1, & \text{if } i = j_3, \\ D_2, & \text{if } i \in I \cap J, \\ D_1, & \text{if } i \in I^c \cap J^c. \end{cases}$$

One easily checks that $xy = yx$ and $x^k = y^{n-k}$ at all vertices and that $\mathbb{L}(I,J)$ is free over the center of the boundary algebra. Hence, the following proposition holds.

Proposition 1.2. *The module $\mathbb{L}(I,J)$ as constructed in Definition 1.1 is in $CM(B_{k,n})$.*

For the remainder of the paper, if $w = tv$, then $t^{-1}w$ stands for v.

Proposition 1.3. *Let I and J be tightly 3-interlacing, $n \geq 6$ arbitrary, $I \setminus J = \{i_1,i_2,i_3\}$ and $J \setminus I = \{j_1,j_2,j_3\}$ where $1 \leq i_1 < j_1 < i_2 < j_2 < i_3 < j_3 \leq n$. If $\varphi = (\varphi_i)_{i=1}^n \in Hom(\mathbb{L}(I,J),\mathbb{L}(I,J))$, then*

$$\varphi_{i_1} = \varphi_{i_2} = \varphi_{i_3} = \begin{pmatrix} a & bt \\ c & d \end{pmatrix},$$
$$\varphi_{j_1} = \begin{pmatrix} a & b \\ ct & d \end{pmatrix},$$
$$\varphi_{j_2} = \begin{pmatrix} a+c & b+t^{-1}(d-a-c) \\ ct & d-c \end{pmatrix},$$
$$\varphi_{j_3} = \begin{pmatrix} a+2c & b+2t^{-1}(d-a-2c) \\ ct & d-2c \end{pmatrix},$$
$$\varphi_i = \varphi_{i-1}, \text{ for } i \in (I^c \cap J^c) \cup (I \cap J),$$

with $a,b,c,d \in \mathbb{C}[|t|]$. Furthermore, $t \mid c$ and $t \mid (a-d)$.

Proof. First we prove the statement for $n = 6$, $I = \{1,3,5\}$, $J = \{2,4,6\}$, and $\varphi \in \text{End}(\mathbb{L}(I,J))$. Then $\varphi = (\varphi_1,\varphi_2,\dots,\varphi_6)$, where each φ_i is an element of $M_2(\mathbb{C}[[t]])$ (matrices over the center).

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We check the relations which arise when we go from a peak of the rim of $\mathbb{L}(I,J)$ to a valley of the rim:

$$\begin{array}{llll} (i) & x_2\varphi_1 = \varphi_2 x_2, & (ii) & x_3\varphi_2 = \varphi_3 x_3, \\ (iii) & x_4\varphi_3 = \varphi_4 x_4, & (iv) & x_5\varphi_4 = \varphi_5 x_5, \\ (v) & x_6\varphi_5 = \varphi_6 x_6, & (vi) & x_1\varphi_6 = \varphi_1 x_1. \end{array}$$

Let $\varphi_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equalities $\varphi_1 = \varphi_3 = \varphi_5$ follow immediately from $t\varphi_3 = \varphi_3 B_1 B_2 = B_1 B_2 \varphi_1 = t\varphi_1$ and $t\varphi_5 = \varphi_5 C_1 C_2 = C_1 C_2 \varphi_3 = t\varphi_3$.

If we consider matrices x_i and y_i as elements of the ring $M_2(\mathbb{C}((t)))$, where all of them are units, then from $x_2\varphi_1 = \varphi_2 x_2$ follows that

$$\varphi_2 = x_2\varphi_1 x_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} a & t^{-1}b \\ ct & d \end{pmatrix}.$$

Thus, $t \mid b$, so if we replace b by bt, this yields

$$\varphi_1 = \begin{pmatrix} a & bt \\ c & d \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} a & b \\ ct & d \end{pmatrix}.$$

Similarly, from $x_4\varphi_3 = \varphi_4 x_4$, we have

$$\varphi_4 = x_4\varphi_3 x_4^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix}\begin{pmatrix} a & bt \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -t^{-1} \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} a+c & b+t^{-1}(d-a-c) \\ ct & d-c \end{pmatrix},$$

and from $x_6\varphi_5 = \varphi_6 x_6$, we have

$$\varphi_6 = x_6\varphi_5 x_6^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & t \end{pmatrix}\begin{pmatrix} a & bt \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -2t^{-1} \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} a+2c & b+2t^{-1}(d-a-2c) \\ ct & d-2c \end{pmatrix}.$$

The statement about the divisibility follows since we have the two properties $t \mid (d-c-a)$ and $t \mid (d-a-2c)$. Combined, they imply $t \mid c$ and $t \mid d-a$ as claimed.

In the general case, the proof is almost the same as in the case $n=6$. The only thing left to note is that if $i \in (I^c \cap J^c) \cup (I \cap J)$, then x_i is a scalar matrix (either identity or t times identity), so the equality $x_i\varphi_{i-1} = \varphi_i x_i$ yields $\varphi_{i-1} = \varphi_i$. ∎

PROPOSITION 1.4. *Let I,J be tightly 3-interlacing, $n \geq 6$ arbitrary. Then the module $\mathbb{L}(I,J)$ is indecomposable.*

Proof. We first consider $n=6$. In this case, we can assume $I = \{1,3,5\}$ and $J = \{2,4,6\}$. Take $\varphi = (\varphi_i)_i \in \operatorname{End}(\mathbb{L}(I,J))$ as in the previous proposition.

To show the indecomposability, we assume that φ is an idempotent endomorphism of $\mathbb{L}(I,J)$ and show that φ is trivial (the identity or the zero endomorphism).

Assume that $\varphi_2^2 = \varphi_2$, that is

$$\varphi_2^2 = \begin{pmatrix} a^2+bct & (a+d)b \\ (a+d)ct & d^2+bct \end{pmatrix} = \begin{pmatrix} a & b \\ ct & d \end{pmatrix}.$$

The equations $a^2+bct = a$ and $d^2+bct = d$ on the diagonal entries give $a-a^2 = d-d^2$, that is, $a-d = a^2-d^2 = (a-d)(a+d)$ and hence $a=d$ or $a+d=1$. The equations also show that $t \mid a(1-a)$ and that $t \mid d(1-d)$.

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Assume first $a = d$. If $b \neq 0$, we get $a = \frac{1}{2}$, which contradicts to $t \mid a - a^2$. Analogously for $c \neq 0$. Thus $b = c = 0$ and $a = d = 0$ or $a = d = 1$, the two trivial cases (note that if φ_2 is trivial, then $x_i \varphi_{i-1} = \varphi_i x_i$ yields $\varphi_2 = \varphi_i$, for all i).

So assume that $a \neq d$ and $d = 1 - a$. Combining $t \mid a(1-a)$ with the fact that t divides $a - d = 2a - 1$ implies that $t \mid 1$, which is a contradiction.

For a general n, since $\varphi_i = \varphi_{i+1}$ for $i + 1 \in (I^c \cap J^c) \cup (I \cap J)$, the proof follows as for $n = 6$. □

The question of uniqueness of such a rank 2 indecomposable module is studied in [2]. For given tightly 3-interlacing I and J, there is a unique indecomposable rank 2 module with filtration $L_I \mid L_J$. This statement is clear in case when the category $\mathrm{CM}(B_{k,n})$ is of finite representation type and in case when $\mathrm{CM}(B_{k,n})$ is tame, with $(k,n) \in \{(3,9), (4,8)\}$. Consequently, we have the following theorem.

THEOREM 1.5 [2, Th. 1.2]. *Let $M \in CM(B_{k,n})$ be an indecomposable module with profile $I \mid J$. Then, up to isomorphism, M is the unique indecomposable rank 2 module with filtration $I \mid J$ if and only if its poset is $1^3 \mid 2$ and I and J are almost tightly 3-interlacing.*

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