7

The Standard Model

The data and facts about elementary particles introduced so far almost completely define the so-called Standard Model of elementary particles; the few missing pieces are:

1. a detailed description of the weak interactions as a gauge theory with the $SU(2)_L$ symmetry group, the gauge bosons of which interact only with fermions of left chirality,
2. a mechanism of providing mass to gauge bosons as well as other particles, and
3. a unification of the weak and the electromagnetic interaction.

Straightforwardly adding an $m^2\|\mathbb{A}_\mu\|^2 := m^2\,\mathrm{Tr}[\mathbb{A}_\mu\eta^{\mu\nu}\mathbb{A}_\nu]$ term to the Lagrangian density would certainly provide the 4-vector potential $\mathbb{A}_\mu$ with the mass m. However, that term is not invariant under the gauge transformation, and explicitly breaks precisely that symmetry because of which $\mathbb{A}_\mu$ was introduced. On the other hand, particles that mediate the weak interaction must have a mass [☞ discussion in the passage on the weak processes (2.56)]. Thus, finding a hopefully more skillful, and certainly gauge-invariant mechanism for providing gauge bosons with a mass is absolutely indispensable for consistency, and we first attend to that matter.

7.1 Boundary conditions and solutions of symmetric equations

Simply inserting an explicit mass term, $m^2\|\mathbb{A}_\mu\|^2$, into the Lagrangian density would destroy precisely that symmetry which is gauged by the 4-vector gauge potential $\mathbb{A}_\mu$ and would thus be self-contradicting. The equations of motion, and so also the Lagrangian density and the Hamilton action, therefore, must remain gauge invariant. Recall, however, that *concrete solutions* of a given system of equations need not have all the symmetries of the system that they solve [☞ Appendix A.1.3 and Comment A.2 on p. 458]. However, if a symmetry X of a system of equations is not a symmetry of a concrete solution f so $X(f) \neq f$, then $X(f)$ is nevertheless a (different) solution of the system, and X is the transformation that maps one solution into the other. Finally, recall that the solutions of a model are not determined only by the system of equations, but also by the boundary (initial, analyticity, etc.) conditions, so it must be that at least some of those conditions distinguish f from $X(f)$.

It is then – in principle – possible to find a solution of gauge-invariant equations of motion that represent massive gauge bosons, i.e., concrete solutions that break precisely the gauge symmetry of the system. This desired solution wherein gauge bosons have a mass breaks the gauge symmetry,

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and this "boundary" condition must be in the abstract space of gauge phases [☞ Comment 5.3 on p. 170], and this cannot be imposed "by hand" without destroying precisely the symmetry that we are trying to describe.

The mechanism in which a choice of such a condition in the space of gauge phases can be imposed does exist, and it is based on a concatenation of ideas:

1. In 1950, L. D. Landau and V. L. Ginzburg analyzed *phenomenologically* ferromagnet magnetization, following Landau's early work in 1937 [330, 207].[1]
2. P. Anderson's comment and Y. Nambu's research (1960), where the BCS (J. Bardeen, L. N. Cooper and J. R. Shiffer, 1957) model of superconductance is adapted to the description of vacuum in quantum field theory.
3. J. Goldstone's theorem (1961–2) about the Nambu–Goldstone modes (1961), the final proof of which within special relativistic theoretical systems was provided by J. Goldstone, A. Salam and S. Weinberg [214].
4. P. Anderson's work (1963 [15]) about *non-relativistic* plasmons, gauge symmetry and the emergence of effective mass.
5. Independent proposals (1964) by:
   (a) R. Brout and F. Englert,
   (b) P. Higgs,
   (c) G. Guralnik, C. R. Hagen and T. Kibble.
6. In 1971, G. 't Hooft (PhD work advised by M. J. G. Veltman) showed the renormalizability of models where a non-abelian gauge symmetry is broken *via* the Higgs mechanism.
7. 1973: S. Coleman and E. Weinberg analyzed the effect of quantum corrections.

Owing to this complex genesis of this group of ideas, I will not delve into the historical details, but will focus on the description of the effect and its technical details, leaving out the discussion of the individual contributions. Also, I will use the simple expressions such as the "Goldstone mode," the "Higgs mechanism" and the "Higgs particle," with no intention to downplay the relevance of others' contributions. Ever since the LHC at CERN started the experiment of which one of the aims is the detection of the Higgs particle, historical reminders have been (re-)emerging; see, for example, Ref. [252]. For more information, beyond the scope of this book, see Refs. [311, 499, 359, 368].

7.1.1 The Landau–Ginzburg phenomenological description of magnetization

To describe the magnetization of a magnet, introduce the vector function $\vec{M}(\vec{r},t)$, the direction and magnitude of which describe the state of magnetization in the object in an infinitesimally small volume (and which we regard as a tiny domain) at the point $\vec{r}$ at the time t. As the direction and magnitude of magnetization in nearby domains affects the magnetization in a given domain, one expects that the change in the magnetization spreads, at least in the first approximation, as a wave. One therefore expects that the magnetization satisfies an equation of the form

$$\Big[\vec{\nabla}^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\Big]\vec{M} = \cdots \tag{7.1}$$

where v is the propagation speed of the magnetization wave and where one must supply the missing terms on the right-hand side. If we temporarily define $\mathrm{x} := (vt,\vec{r})$, akin to the relativistic practice, this equation would follow from a Lagrangian density with the "kinetic" term

[1] With the benefit of hindsight, this analysis may today be viewed disparagingly as "fitting" the potential to describe the observed effect. However, this analysis is valuable as it indicates the essential result – precisely the effective potential – that every fundamental, so-called microscopic model *must* reproduce. This then presents an extraordinarily effective criterion to filter the possible, more fundamental models.

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$\frac{A}{2}\eta^{\mu\nu}(\partial_\mu\vec{M})\cdot(\partial_\nu\vec{M})$, where A is a constant with appropriate physical units. Adding a potential $V(\vec{M})$, one obtains the classical equations of motion

$$\delta_{ij}\Big[\vec{\nabla}^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\Big]M^j = -\frac{1}{A}\frac{\partial V}{\partial M^i}, \tag{7.2}$$

where M^i are the components of the magnetization vector in some arbitrary Cartesian coordinate system. In quantum theory, one must of course switch to operators and define an adequate Hamiltonian by integrating (in space) the Hamiltonian density obtained from the Lagrangian density. However, the essence of this procedure is that the ground state of the quantum model is defined by the global minimum of the potential $V(\vec{M})$. This phenomenological approach (based on Landau's theory of phase transitions [☞ for example, Ref. [340]]) then reduces to choosing an appropriate potential function $V(\vec{M})$.

In the familiar example of the harmonic oscillator, the potential $V(x) = \frac{1}{2}m\omega^2x^2$ has a unique minimum, $x = 0$. Correspondingly, the model has a unique ground state for all physically acceptable values of the parameters $\omega, m > 0$. Landau's essential insight, which provides the basis for the Landau–Ginzburg description of magnetization, is that a more complicated potential may well have several distinct minima, depending on the choice of its parameters. Thus, e.g., the anharmonic generalization of the linear harmonic potential, $V(x) = \frac{1}{2}\mu x^2 + \frac{1}{4}\lambda x^4$, has two phases:

1. **when** $\boldsymbol{\mu > 0}$: the minimum of the potential $V(x)$ is at $\check{x}_0 = 0$;
2. **when** $\boldsymbol{\mu < 0}$: the minima of the potential $V(x)$ are at $\check{x}_\pm = \pm\sqrt{-\mu/\lambda}$,

where $\check{x} := \min\big(V(x)\big)$. The quantum-mechanical expectation value of the observable x (the position of the oscillator) is the average value, $\langle x\rangle = 0$, but in the second case may be "localized" at $\check{x}_\pm$. For the Hilbert space to consist of normalizable bound states and so that the above local minima would in fact be global minima, one requires $\lambda > 0$. *[✎ Why?]* (The $\lambda < 0$ choice implies that $\lim_{x\to\infty} V(x) \to -\infty$, which is unphysical as it prevents the existence of a stable ground state.)

The application of this idea in the Landau–Ginzburg phenomenological model also uses the fact that the potential is a scalar function of the vector $\vec{M}(\vec{r},t)$, and so can depend only on the magnitude $|\vec{M}| = \sqrt{\delta_{ij}M^i(\vec{r},t)M^j(\vec{r},t)}$. If one also requires that the potential is an analytic function, it must be that

$$V(\vec{M}) = \tfrac{1}{2}\mu|\vec{M}|^2 + \tfrac{1}{4}\lambda\big(|\vec{M}|^2\big)^2 + \cdots \tag{7.3}$$

It then follows that the ground state of the quantum-mechanical description of magnetization is determined by minimizing the potential:

1. **if** $\boldsymbol{\mu > 0}$: the minimum of the potential $V(\vec{M})$ is at $\langle\vec{M}\rangle_0 = 0$;
2. **if** $\boldsymbol{\mu < 0}$: the minima of the potential $V(\vec{M})$ are at $\langle\vec{M}\rangle_> = \sqrt{-\mu/\lambda}\,\hat{M}$,

where $\hat{M}$ is one of a continuum of *arbitrary* unit vectors in the 3-dimensional space in which the magnetization $\vec{M}(\vec{r},t)$ is a 3-vector – and which *coincides* with the "actual," real space.

Before we return to the question: "*Which* arbitrary direction $\hat{M}$?," let us finish the parametrization of the Landau–Ginzburg model by noting that one of course knows that the magnet loses its magnetization when heated to a sufficiently high temperature. It then follows that μ must be a function of temperature, and so that $\mu < 0$ for $T < T_c$, whereas $\mu > 0$ for $T > T_c$. The concrete choice of the $\mu = \mu(T)$ dependence, as well as the presence of an additional $(|\vec{M}|^2)^3$ term in the expansion of the potential (7.3) in the original Landau–Ginzburg potential stems from additional requirements to also describe successfully physical characteristics such as the susceptibility – which may be ignored for the present purposes. We then simply adopt

$$V(\vec{M}) = \tfrac{1}{2}\mu_0(T^2 - T_c^2)|\vec{M}|^2 + \tfrac{1}{4}\lambda\big(|\vec{M}|^2\big)^2 + \cdots, \qquad \mu_0, \lambda > 0. \tag{7.4}$$

It then follows that the ground state of the quantum-mechanical description of the magnetization is determined by the minimum of the potential:

$$\min\big[V(\vec{M})\big] = \begin{cases} \langle\vec{M}\rangle_> = \sqrt{\mu_0(T_c^2-T^2)/\lambda}\,\hat{M} & \text{when } T < T_c; \\ \langle\vec{M}\rangle_0 = 0 & \text{when } T \geqslant T_c. \end{cases} \tag{7.5}$$

Notice that $\min[V(\vec{M})]$ is a continuous (but not smooth) function of the temperature.

Thus, at a sufficiently high temperature, the object has no magnetization, $\langle\vec{M}\rangle_0 = 0$, whereas lowering of the temperature below T_c causes the object to *spontaneously magnetize*. That is, we have that $\langle\vec{M}\rangle_> = \sqrt{\mu_0(T_c^2-T^2)/\lambda}\,\hat{M}$, in the direction $\hat{M}$ – which remains undetermined by the dynamics of the model.

In an actual, real situation, there *always* exists some small *external* magnetic field, which "chooses" a preferred direction: The interactions of the tiny domains with this external magnetic field then direct them opposite to this external magnetic field, which removes the arbitrariness of the choice of $\hat{M}$.

Comment 7.1 *In the Landau–Ginzburg description of magnetization, the 3-dimensional space in which the magnetization vector $\vec{M}(\vec{r},t)$ has magnitude and direction is in fact the "actual," real space in which we ourselves live and move. In the other applications of this idea, this need not be so.*

Classical analysis straightforwardly shows the following properties:

1. As temperature decreases through the critical value T_c, the character of the potential $V(\vec{M})$ changes suddenly. However, the gradient of the potential (the generalized "force" that moves the magnetization of the object) in fact always vanishes at the point $\vec{M} = 0$; that is a consequence of the fact that an analytic potential function must depend on $|\vec{M}|^2$ and not on $|\vec{M}|$. This necessitates an influence to "move" the system from $\vec{M} = 0$, and this external influence then also fixes the ultimate direction of the magnetization $\hat{M}$. This may literally be an external influence (a small external magnetic field), or also a simply random (quantum) fluctuation within the object/system itself.
2. Immediately after the transition, when $T \lesssim T_c$, the potential has a very mild "slope" near $\vec{M} = 0$, the "inclination" of which grows with the distance from the $\vec{M} = 0$ point. The global minimum of the potential function moves from $\vec{M} = 0$ to a circle of "radius" $|\vec{M}(T)| = \sqrt{\mu_0(T_c^2-T^2)/\lambda}$, which grows as the temperature decreases: $T < T_c$ and $T \to 0$.
3. Even if moved by an external influence, the actual value $\langle\vec{M}\rangle$ will lag behind the growing value of the "radius" $M(T)$. The change in the magnetization from $\langle\vec{M}\rangle_0$ towards $\langle\vec{M}\rangle_>$ will be accelerated, just as with rolling down a steepening slope.
4. When the system reaches $\langle\vec{M}\rangle_>$, the motion regime turns oscillatory around $\langle\vec{M}\rangle_>$, where the loss of energy through interaction with the environment leads to a stabilization of the value $\langle\vec{M}\rangle \to \langle\vec{M}\rangle_>$.
5. In this entire process, the system has (through interaction with the environment) lost the energy
$$\triangle V := V\Big(|\vec{M}|{=}0\Big) - V\Big(|\vec{M}|{=}\sqrt{\mu_0(T_c^2-T^2)/\lambda}\Big), \tag{7.6}$$
which somewhat akin to the latent heat of a first-order (discontinuous) phase transition such as freezing of water. To be precise, magnetization is however a second-order (continuous) transition, where $\langle\vec{M}\rangle$ continuously changes between its values.

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7.1.2 The Goldstone theorem

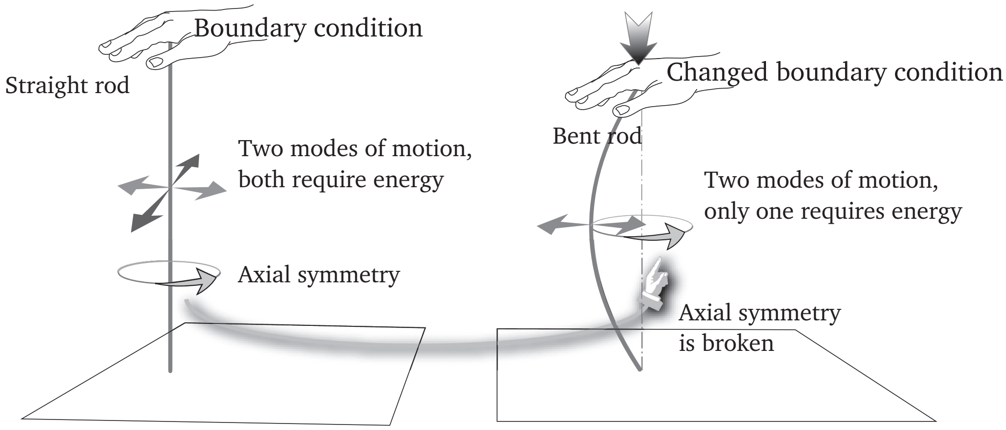

Figure 7.1 The example of a straight and a bent rod. Notice that the mode of motion of the bent rod that (ignoring friction) requires no energy is identical with the symmetry that is broken by bending. The difference is induced by changing the boundary conditions.

Before we apply the ideas from the previous section to a scalar field in a relativistic theory, consider a simple model, shown in Figure 7.1. This model illustrates several of the characteristics of symmetry breaking, with a faithful analogy in the case of spontaneous magnetization as the temperature drops.

The straight rod has a manifest axial symmetry. Analogously, at temperatures above T_c, a magnetic material has $\langle \vec{M} \rangle = 0$, i.e., the magnetic domain orientation distribution is spherically symmetric. The bent rod does not have the axial symmetry, but its horizontal rotation requires no energy if we neglect friction. Analogously, at temperatures below T_c, the magnetic material has $\langle \vec{M} \rangle \neq 0$, i.e., the magnetic domain orientation distribution is no longer spherically symmetric. However, fluctuations in the magnetization orientation form a wave (dubbed a *magnon*), the propagation of which in the magnet requires very little energy, which fails to vanish only because of imperfections and finiteness of a real, physical magnet.

Similarly, the molecular velocity distribution in any fluid is spherically symmetric. When the temperature of the fluid drops below the freezing point, the material can form a crystal, in which molecules move only in modes permitted by the crystalline geometry; this breaks the continuous spherical symmetry into the discrete crystalline symmetry. Correspondingly, there appear *phonons* in the crystal, the propagation of which in the crystal requires very little energy, which fails to vanish because of the imperfection and finiteness of the real, physical crystal.

These examples exhibit the essence of the Goldstone theorem, a technically simplified form of which is:

> ***Theorem 7.1*** *For every continuous (and local) symmetry of a system (and for which there then exists a current that satisfies the continuity equation and a conserved charge) that is not a symmetry of the vacuum (ground state), there exists an excitation (a motion/fluctuation mode) of the system that requires no energy.*

The idea of the proof is very simple: the ground state that breaks the continuous symmetry is only one of continuously many possible such states. In the example of a bent rod, the direction of bending is one of continuously many arbitrary directions; similarly, the ultimate direction of magnetization $\langle \vec{M} \rangle$ and of the crystalline lattice represent arbitrary choices from among continuously many possibilities. Thus, a system with a broken symmetry has a continuum of possible ground states – which are of course degenerate and which the broken symmetry transforms one into another. The motion/change of the system from one of these continuously many possible choices into another then requires no energy.

As the analysis in Appendix A.1.3 shows, the symmetry of a system of equations is always a symmetry of the complete space of solutions. If some particular – e.g., ground – state of the system is not itself symmetric, then this symmetry transforms this particular (ground) state into another (also ground) state. As the symmetry of a system is by definition a transformation that commutes with the Hamiltonian, then the mode of motion/change of the system from one state into another cannot require any energy. Symbolically (see Section A.1.3):

$$\begin{array}{rcl} X\ (=(X^\dagger)^{-1}) \text{ is a symmetry of } \boldsymbol{M}. & \leftrightarrow & [H,X]=0. \qquad (7.7a)\\ X \text{ is a symmetry of the } \textit{complete} \text{ solution space, } \mathscr{X}(\boldsymbol{M}). & \leftrightarrow & X|\Psi\rangle\in\mathscr{X}(\boldsymbol{M}) \Leftrightarrow |\Psi\rangle\in\mathscr{X}(\boldsymbol{M}). \qquad (7.7b)\\ |\Psi\rangle\in\mathscr{X}(\boldsymbol{M}) \text{ breaks } X. & \leftrightarrow & \big(X|\Psi\rangle=|\Psi'\rangle\big)\neq|\Psi\rangle. \qquad (7.7c)\\ \hline \Rightarrow \text{ The ``motion'' } |\Psi\rangle\to|\Psi'\rangle \text{ in } \mathscr{X}(\boldsymbol{M}) \text{ requires no energy.} & \leftrightarrow & \langle\Psi|H|\Psi\rangle-\langle\Psi'|H|\Psi'\rangle=0. \qquad (7.7d) \end{array}$$

The final result follows since

$$\langle\Psi'|H|\Psi'\rangle=\langle\Psi|X^\dagger HX|\Psi\rangle \overset{(7.7a)}{=} \langle\Psi\big|X^{-1}XH\big|\Psi\rangle=\langle\Psi|H|\Psi\rangle. \tag{7.8}$$

Since $\langle\Psi'|H|\Psi'\rangle=\langle\Psi|H|\Psi\rangle$, the transformation/motion $|\Psi\rangle\to|\Psi'\rangle$ cannot possibly require any energy. ☑

The careful Reader must have noticed the minor differences in the implied definitions and concepts in the above several paragraphs, and a technically complete treatment of the Goldstone theorem requires a consistent and technically precise connection between these ideas. Besides, one must keep in mind the finiteness of the resolution of any concrete measurement, which then implies limitations in the definition of physical quantities. For example, the "bare" electric charge is not distinguishable from a system consisting of that same electric charge but together with the electromagnetic field created by that charge, the intensity of which is below the threshold of observability. That is, the "bare" electric charge is indistinguishable from the electric charge surrounded by a sea of photons that are either sufficiently "soft" (of small frequency) or are reabsorbed too fast to permit detection.

7.1.3 The Higgs effect for gauge symmetry

As we begin analyzing the gradual development of a model based on the ideas from the previous Sections 7.1.1–7.1.2, note that in field theory the quadratic term provides a field with a mass, as was mentioned in the beginning of this section.

Field shift

A simple relativistic Lagrangian density (constructed in the spirit of the discussion in Section 7.1.1) for one, real, scalar field, $\phi(\mathrm{x})$ is

$$\mathscr{L}_0=\tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)-\tfrac{\varkappa}{2}\big(\tfrac{mc}{\hbar}\big)^2\phi^2-\tfrac{1}{4}\lambda\phi^4, \tag{7.9}$$

so that the classical, Euler–Lagrange equation of motion is

$$0=\partial_\mu\frac{\partial\mathscr{L}_0}{\partial(\partial_\mu\phi)}-\frac{\partial\mathscr{L}_0}{\partial\phi} = \partial_\mu\big(\eta^{\mu\nu}\partial_\nu\phi\big)+\varkappa\big(\tfrac{mc}{\hbar}\big)^2\phi+\lambda\phi^3, \tag{7.10}$$

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that is,[2]

$$\big[\tfrac{1}{c^2}\partial_t^2 - \vec{\nabla}^2 + \varkappa\big(\tfrac{mc}{\hbar}\big)^2\big]\phi = -\lambda\phi^3 \quad \Leftrightarrow \quad \big[-E^2 + \vec{p}^2c^2 + \varkappa m^2c^4\big]\phi = (\hbar^2c^2\lambda)\phi^3, \tag{7.11}$$

identifying $\sqrt{\varkappa}\,m$ as the mass of the ϕ field, so we fix $\varkappa \to 1$ for now.

> ***Comment 7.2*** *Since $[\int \mathrm{d}^4\mathrm{x}\,\mathscr{L}_0] = [\hbar] = ML^2/T$, then $[\mathscr{L}_0] = M/L^2T$. As the metric tensor $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ are dimensionless and $[\partial_\mu] = L^{-1}$, it follows that the units of the so-defined scalar field $[\phi] = \sqrt{M/T}$ and the units of λ are $[\lambda] = T/ML^2$. In turn, comparing the ϕ^2-terms in the Lagrangian density (7.9), $[\partial_\mu] = [\frac{mc}{\hbar}]$ and m is really a mass, $[m] = M$.*

The potential energy density in the Lagrangian density (7.9) is $\mathscr{V}_0 = \frac{\varkappa}{2}\big(\frac{mc}{\hbar}\big)^2\phi^2 + \frac{1}{4}\lambda\phi^4$, and the Hamiltonian density is

$$\mathscr{H}_0 := (\dot{\phi})\frac{\partial\mathscr{L}_0}{\partial(\dot{\phi})} - \mathscr{L}_0 = \tfrac{1}{2}\big[\tfrac{1}{c^2}\dot{\phi}^2 + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi)\big] + \tfrac{\varkappa}{2}\big(\tfrac{mc}{\hbar}\big)^2\phi^2 + \tfrac{1}{4}\lambda\phi^4. \tag{7.12}$$

The expressions (7.11) and (7.12) indicate that changing $\varkappa = 1 \to -1$, aiming to describe a symmetry breaking as in Section 7.1.1, implies that the mass of the ϕ field has become imaginary ($\sqrt{\varkappa}m = m \to im$) – which is nonsensical in classical physics.

However, recall that the parameters in the classical Lagrangian are only auxiliary, helping parameters, and that the true, physically measurable values are obtained only after an adequate redefinition of those parameters, i.e., after renormalization [☞ Sections 5.3.3 and 6.2.4]. With that idea, in 1973 Sydney Coleman and Erick Weinberg analyzed the effect of the electromagnetic field on the mass of an electrically charged scalar particle [112] and found that there exists a regime (phase) of the parameter m, λ choices where the effective mass of the field (owing to renormalization effects) really does become imaginary and so induces the breaking of a symmetry, i.e., indicates an instability of the state with the unbroken symmetry. With this in mind, we now simply change $m^2 \to -m^2$ without delving into the detailed reasons and dynamics of this change.

With the potential energy density $\widetilde{\mathscr{V}}_0 = -\frac{1}{2}\big(\frac{mc}{\hbar}\big)^2\phi^2 + \frac{1}{4}\lambda\phi^4$, the system is unstable at $\phi_0 = 0$, and the global minima appear at the values $\phi \to \pm\frac{mc}{\hbar\sqrt{\lambda}}$. One thus expects that, after enough time, the system settles at either $\langle\phi\rangle = +\frac{mc}{\hbar\sqrt{\lambda}}$ or $\langle\phi\rangle = -\frac{mc}{\hbar\sqrt{\lambda}}$. Feynman's perturbative computation would then have to be adapted so that all fields vanish at the chosen classical minimum, i.e., that the fields represent fluctuations around that minimum. It is thus convenient to introduce one of the two substitutions:

$$\text{either} \quad \varphi_+ := \phi - \frac{mc}{\hbar\sqrt{\lambda}}, \qquad \text{when } \langle\phi\rangle = +\frac{mc}{\hbar\sqrt{\lambda}}, \qquad \text{so } \langle\varphi_+\rangle = 0, \tag{7.13a}$$

$$\text{or} \quad \varphi_- := \phi + \frac{mc}{\hbar\sqrt{\lambda}}, \qquad \text{when } \langle\phi\rangle = -\frac{mc}{\hbar\sqrt{\lambda}}, \qquad \text{so } \langle\varphi_-\rangle = 0, \tag{7.13b}$$

whereby the Lagrangian density (7.9), with the sign in the mass-term changed by hand,

$$\widetilde{\mathscr{L}}_0 = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2\phi^2 - \tfrac{1}{4}\lambda\phi^4, \tag{7.14}$$

becomes – corresponding to the choice (7.13) – one of the two Lagrangian densities:

$$\text{either} \quad \mathscr{L}_+ = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\varphi_+)(\partial_\nu\varphi_+) - \big(\tfrac{mc}{\hbar}\big)^2\varphi_+^2 - \tfrac{mc\sqrt{\lambda}}{\hbar}\varphi_+^3 - \tfrac{1}{4}\lambda\varphi_+^4 + \tfrac{m^4c^4}{4\lambda\hbar^4}, \tag{7.15a}$$

[2] Identification of the operator ∂_μ with the components of the 4-momentum is obtained fastest by using the quantum-mechanical relations in the coordinate representation, $H = i\hbar\partial_t = i\frac{\hbar}{c}\partial_0$ and $\vec{p} = -i\hbar\vec{\nabla}$, so that substituting the eigenvalues, $\hbar^2\partial_t^2 \mapsto -E^2$ and $\hbar^2\vec{\nabla}^2 \mapsto -\vec{p}^2$ [☞ Digression 3.6 on p. 93, and the relation (3.37) that holds when $\lambda \to 0$].

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$$\text{or}\quad \mathscr{L}_- = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\varphi_-)(\partial_\nu\varphi_-) - \left(\tfrac{mc}{\hbar}\right)^2\varphi_-^2 + \tfrac{mc\sqrt{\lambda}}{\hbar}\varphi_-^3 - \tfrac{1}{4}\lambda\varphi_-^4 + \tfrac{m^4c^4}{4\lambda\hbar^4}. \tag{7.15b}$$

For these "shifted" fields (7.13), the mass is real, $m_\pm = \sqrt{2}m$, as the sign of the quadratic term turned negative, and other than the anharmonic term $\varphi_\pm^4$, now there is also a cubic term, $\varphi_\pm^3$. Finally, the total value of the Lagrangian density shifted by the constant $+\frac{m^4c^4}{4\lambda\hbar^4}$, which means that the value of the total energy density (Hamiltonian density) of the system *decreased* by $-\frac{m^4c^4}{4\lambda\hbar^4}$; recall, $\mathscr{H} = p_i\dot{q}^i - \mathscr{L}$. This contribution to the energy of the system is the excess energy density of the phase transition from the phase where the effective mass is real and $\langle\phi\rangle = 0$ into the phase where the effective mass is imaginary and $\langle\phi\rangle = \pm\frac{mc}{\hbar\sqrt{\lambda}}$.[3]

As the minimum of the total energy in the phase with $\langle\phi\rangle = \pm\frac{mc}{\hbar\sqrt{\lambda}}$ is lower than that in the phase with $\langle\phi\rangle = 0$, it follows that the ground state of the system after the sign change of the quadratic term must have one of the two possible values: $\langle\phi\rangle = \pm\frac{mc}{\hbar\sqrt{\lambda}}$, and the choice between these two values is arbitrary.

Conclusion 7.1 *The Lagrangian density (7.9) describes two phases of the system:*

1. *the symmetric phase, where $\varkappa > 0$ and $\langle\phi\rangle = 0$,*
2. *the broken symmetry phase, where $\varkappa < 0$ and $\langle\phi\rangle = \pm\frac{mc}{\hbar\sqrt{\lambda}} \neq 0$.*

Typically, the parameter $\varkappa$ is a function of temperature and turns negative when the temperature drops below some critical value. The change $\varkappa > 0 \leftrightarrow \varkappa < 0$ is, of course, a ***phase transition****, for which the excess energy density equals $\frac{m^4c^4}{4\lambda\hbar^4}$, as seen in the expressions (7.15).*

The Lagrangian density (7.9), and then also the equations of motion (7.11), have the symmetry $\varpi : \phi \to -\phi$. However, when the parameter m^2 turns into $-m^2$ and the mass becomes unphysically imaginary, as in the Lagrangian density (7.14), the state where $\langle\phi\rangle = 0$ becomes unstable. Instead, one chooses one of the two states where $\langle\phi\rangle = \pm\frac{mc}{\hbar\sqrt{\lambda}}$ and, corresponding to the change in the notation (7.13), one uses one of the two Lagrangian densities (7.15). The transformation ϖ is still a symmetry of the system:

$$\varpi : \phi \to -\phi \quad\Rightarrow\quad \varphi_\pm \to -\varphi_\pm \mp 2\tfrac{mc}{\hbar\sqrt{\lambda}} \quad\Rightarrow\quad \mathscr{L}_\pm \to \mathscr{L}_\pm. \tag{7.16}$$

As this is a discrete transformation, the Goldstone theorem does not apply. However, there evidently exists a mapping $\varphi_\pm \to -\varphi_\mp$ that turns $\mathscr{L}_\pm(\varphi_\pm) \to \mathscr{L}_\mp(\varphi_\mp)$; i.e., that connects the two existing and degenerate vacua.[4] By breaking discrete symmetries, the existence of such a discrete mapping is a property that is closest to the existence of a Goldstone mode. Although this "goldstonesque" transformation $\varphi_\pm \to -\varphi_\mp$ is not identically equal to the initial symmetry (7.16), the two transformations are isomorphic: both are reflections, albeit across different points in the field space.

Basic building blocks of Feynman diagrams correspond to the terms in the Lagrangian density (7.15). Terms that are quadratic in φ define the "propagator," i.e., the Green function. Its physical meaning is that the change in the φ field in one spacetime point correlates with a change in a neighboring point. For a scalar field, this function is represented in Feynman diagrams by a

[3] The contribution to the total energy is, of course, $-\int d^3\vec{r}\,\frac{m^4c^4}{4\lambda\hbar^4}$, which diverges because of the integral over the infinitely large space. However, this is but one example of the need to renormalize the reference energy level of the "empty spacetime."

[4] When the number of degenerate states is finite, as here, it makes sense to construct (anti)symmetrized linear combinations $\mathscr{L}_+$ and $\mathscr{L}_-$. However, we will be interested in the breaking of *continuous* symmetries, where this is not possible – or at least does not have the same physical meaning.

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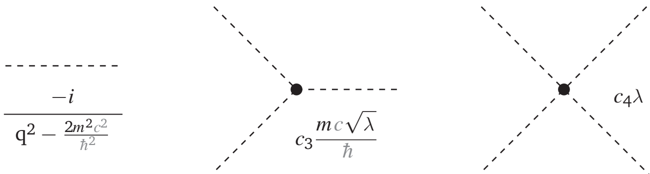
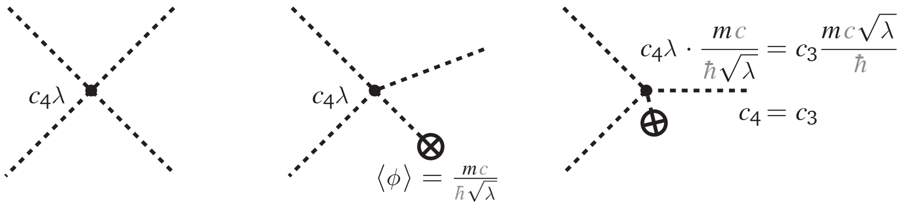
dashed line. Cubic and quartic terms, respectively, represent correlated changes in three and four spacetime points, and are represented by vertices where, respectively, three and four dashed lines meet:



(7.17)

where the concrete choice of (combinatorial and normalizing) constants c_3, c_4 is not relevant here. Note, however, that the triple vertex may be obtained from the quadruple one by formally "ending" one of its four edges – as if the field ϕ here *sinks* into the vacuum:



(7.18)

or *wells* up from it [☞ discussion about the diagram (3.82)]. The nonzero value $\langle\phi\rangle$ indicates that the number of ϕ-quanta is not conserved in the vacuum with the broken symmetry. In contrast, the number of $\varphi_\pm$-quanta is conserved as $\langle\varphi_\pm\rangle = 0$, and this is the normal mode for describing the system in vacuum with the broken symmetry. After the substitution $\phi \to \varphi_\pm$, the system has only elementary diagrams of the type (7.17), from which one can, of course, construct much more complex Feynman diagrams, and so also much more complex processes. However, in the φ_+- or φ_--description (depending on the choice of the vacuum), there are no diagrams with "sinks" or "sources" such as in the ϕ-description (7.18).

Finally, the Feynman diagrams represent corresponding perturbative contributions, understanding that the fields fluctuate about their classical solutions. Thereby, the choice of the Lagrangian density $\mathscr{L}_\pm$ implies that the ϕ field fluctuates about the expectation value $\langle\phi\rangle = \pm\frac{mc}{\hbar\sqrt{\lambda}}$, so $\langle\varphi_\mp\rangle = 0$. Similarly, just as the ground state of the linear harmonic oscillator is centered at $x = 0$, so is the ground state of the model with the Lagrangian density $\mathscr{L}_+$ centered at $\varphi_+ = 0$, and for $\mathscr{L}_-$ at $\varphi_- = 0$. These then are two distinct models of the system, which the "goldstonesque" transformation $\varphi_\pm \to -\varphi_\mp$ maps one into the other, and proves them to be physically equivalent descriptions of the same system.

> ***Conclusion 7.2*** *In the **symmetric** phase, one uses the Lagrangian density (7.9) with $\varkappa > 0$ and the ϕ field, so that the Feynman diagrams (7.17) – without the triple vertex – correspond to the so-described processes. When $\varkappa < 0$, for the description of this **non-symmetric** phase one picks either the Lagrangian density (7.15a) or (7.15b) and, correspondingly, either φ_+ or φ_-; correspondingly, the Feynman diagrams (7.17) change their meaning although the mechanics of the computations remains the same.*

Breaking of continuous symmetry

One of the simplest generalizations of the above results to the case where a continuous symmetry is broken by the choice of the ground state uses two real scalar fields in place of one: $\phi(\mathrm{x}) \to (\phi_1(\mathrm{x}), \phi_2(\mathrm{x}))$. The Lagrangian density is chosen akin to (7.9)

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$$\mathscr{L}_{2d} = \tfrac{1}{2}\eta^{\mu\nu}\delta^{ij}(\partial_\mu\phi_i)(\partial_\nu\phi_j) - \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2(\delta^{ij}\phi_i\phi_j) - \tfrac{1}{4}\lambda(\delta^{ij}\phi_i\phi_j)^2. \tag{7.19}$$

Owing to the specific choice of the potential function, the Lagrangian density (7.19) is invariant under the action of an arbitrary rotation

$$\varpi_\vartheta : \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \to \begin{bmatrix}\phi_1'\\ \phi_2'\end{bmatrix} := \begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{7.20}$$

in the abstract (ϕ_1, ϕ_2)-plane. Flipping the sign of the quadratic term, we obtain

$$\widetilde{\mathscr{L}}_{2d} = \tfrac{1}{2}\eta^{\mu\nu}\delta^{ij}(\partial_\mu\phi_i)(\partial_\nu\phi_j) + \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2(\delta^{ij}\phi_i\phi_j) - \tfrac{1}{4}\lambda(\delta^{ij}\phi_i\phi_j)^2, \tag{7.21}$$

where the potential energy density is easily found to have continuously many minima, forming the *circle*

$$({\phi_1}^2 + {\phi_2}^2)\big|_{\min} = \tfrac{m^2c^2}{\hbar^2\lambda}, \qquad \text{i.e.,} \qquad (\phi_1, \phi_2)_{\min} = \Big(\tfrac{mc}{\hbar\sqrt{\lambda}}\cos\theta, \tfrac{mc}{\hbar\sqrt{\lambda}}\sin\theta\Big), \tag{7.22}$$

where the angle θ is arbitrary. Clearly, the transformation (7.20) maps the arbitrary choice of minima at the angle θ into the choice with the angle $\theta + \vartheta$.

Consider, e.g., the minimum $(\phi_1, \phi_2) \to (\tfrac{mc}{\hbar\sqrt{\lambda}}, 0)$ and the correspondingly shifted fields:

$$\varphi_1 = \phi_1 - \tfrac{mc}{\hbar\sqrt{\lambda}}, \qquad \varphi_2 = \phi_2. \tag{7.23}$$

With these, the Lagrangian density (7.19) becomes

$$\begin{aligned}\widetilde{\mathscr{L}}_{2d} &= \tfrac{1}{2}\eta^{\mu\nu}\delta^{ij}(\partial_\mu\varphi_i)(\partial_\nu\varphi_j) - \big(\tfrac{mc}{\hbar}\big)^2{\varphi_1}^2\\ &\quad - \tfrac{mc\sqrt{\lambda}}{\hbar}\varphi_1({\varphi_1}^2 + {\varphi_2}^2) - \tfrac{1}{4}\lambda({\varphi_1}^2 + {\varphi_2}^2)^2 + \tfrac{m^4c^4}{4\lambda\hbar^4}\end{aligned} \tag{7.24a}$$

$$\begin{aligned}&= \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\varphi_1)(\partial_\nu\varphi_1) - \big(\tfrac{mc}{\hbar}\big)^2{\varphi_1}^2 - \tfrac{mc\sqrt{\lambda}}{\hbar}{\varphi_1}^3 - \tfrac{1}{4}\lambda\varphi_1^4\\ &\quad + \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\varphi_2)(\partial_\nu\varphi_2) - \tfrac{1}{4}\lambda{\varphi_2}^4\\ &\quad - \tfrac{mc\sqrt{\lambda}}{\hbar}\varphi_1{\varphi_2}^2 - \tfrac{1}{2}\lambda{\varphi_1}^2{\varphi_2}^2 + \tfrac{m^4c^4}{4\lambda\hbar^4},\end{aligned} \tag{7.24b}$$

where we separated the terms that produce the dynamics of the φ_1 and the φ_2 fields into two separate rows, and left the coupling terms and the excess energy density for the last row.

Just as in the one-dimensional example (7.9)–(7.16), the φ_1 field has acquired a real mass $m_1 = \sqrt{2}|m|$, as well as an additional cubic term, besides the φ_1^4 term. However, the $\varphi_2 = \phi_2$ field has *lost* its mass, and only has a ${\varphi_2}^4$ term in the potential! Finally, the φ_1 and the φ_2 fields are coupled via the $\varphi_1{\varphi_2}^2$ and the ${\varphi_1}^2{\varphi_2}^2$ terms, in the sense that the Euler–Lagrange equations of motion form a coupled system. The transformation

$$\varpi_\vartheta : \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} \to \begin{bmatrix}\varphi_1'\\ \varphi_2'\end{bmatrix} := \begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} + \begin{bmatrix}\tfrac{mc}{\hbar\sqrt{\lambda}}(\cos\vartheta - 1)\\ \tfrac{mc}{\hbar\sqrt{\lambda}}\sin\vartheta\end{bmatrix} \tag{7.25}$$

is still a symmetry of the system – and is merely rewritten into the new coordinates, making it evident that this is not a rotation about the coordinate origin in the (φ_1, φ_2)-plane. Since the rotations in the (φ_1, φ_2)-plane about the point $(0, 0)$ are not symmetries, the fact that the φ_2 field has lost its mass indicates that (in this choice of the parametrization of the system) φ_2 represents the Goldstone boson.

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Conclusion 7.3 *Following Conclusion 7.2 on p. 259, one may use the Lagrangian densities (7.19) and (7.24), respectively, to describe the symmetric ($\varkappa > 0$) and the non-symmetric ($\varkappa < 0$) phases of the system. Unlike in the situation in Conclusion 7.2, the non-symmetric phase now contains a* **continuous degeneracy***: any one of continuously many scalar fields that satisfy the relations (7.22) represents a minimum of the potential in the non-symmetric phase. Any one concrete choice, such as (7.23), then represents one concrete spontaneous breaking of the original symmetry, from among continuously many such choices.*

Digression 7.1 Note that after ad hoc changing the sign from the Lagrangian density (7.19) into the Lagrangian density (7.21), varying the ϕ field produces the change in the equation of motion:

$$\big[\Box + \big(\tfrac{mc}{\hbar}\big)^2\big]\phi_j = -\lambda\phi_j\|\phi\|^2 \qquad \to \qquad \big[\Box - \big(\tfrac{mc}{\hbar}\big)^2\big]\phi_j = -\lambda\phi_j\|\phi\|^2, \tag{7.26a}$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ is the wave operator, a.k.a., the d'Alembertian. In the absence of interactions ($\lambda \to 0$), the Klein–Gordon operator $[\Box + (\frac{mc}{\hbar})^2]$ thus changes into $[\Box - (\frac{mc}{\hbar})^2]$. Since the standard Klein–Gordon operator corresponds to the relation (3.36), we have

$$\big[\Box+\big(\tfrac{mc}{\hbar}\big)^2\big]\phi_j = 0 \;\;\Leftrightarrow\;\; E^2 = \vec{p}^2c^2+m^2c^4 \;\;\Leftrightarrow\;\; \frac{\vec{p}^2}{E^2/c^4} = v^2 = c^2\Big(1-\frac{m^2c^4}{E^2}\Big) < c^2. \tag{7.26b}$$

However, flipping the sign of the $m^2\phi^2$ term, by hand, would produce

$$\big[\Box-\big(\tfrac{mc}{\hbar}\big)^2\big]\phi_j = 0 \;\;\Leftrightarrow\;\; E^2 = \vec{p}^2c^2-m^2c^4 \;\;\Leftrightarrow\;\; \frac{\vec{p}^2}{E^2/c^4} = v^2 = c^2\Big(1+\frac{m^2c^4}{E^2}\Big) > c^2. \tag{7.26c}$$

Thus, simply flipping the sign of the $m^2\phi^2$ term in the Lagrangian density would have two correlated consequences:

1. The vacuum where $\langle\phi\rangle = 0$ would become unstable, as a local *maximum* of the potential energy density, which indicates the tendency of the system to transition into a state where $\langle\phi\rangle = \frac{mc}{\hbar\sqrt{\lambda}} \neq 0$.
2. The ϕ field would become tachyonic (superluminal): it would propagate faster than the speed of light in the "false" vacuum where $\langle\phi\rangle = 0$; by transitioning into the "true" vacuum where $\langle\phi\rangle = \frac{mc}{\hbar\sqrt{\lambda}} \neq 0$, ϕ (i.e., now φ) becomes again a physical, tardionic (subluminal) field.

However, the sign of the (quadratic) mass term is in reality a continuously variable parameter, and the evolution of the system is considerably more involved than could be shown here; see for example [81, 20]. Nevertheless, the appearance of a tachyonic particle/state in a simple analysis as shown here does signal vacuum instability.

The correspondence between the broken symmetry and the Goldstone boson is not perfectly evident in this parametrization, since φ_2 does not represent rotations. This correspondence becomes clearer by using, instead of (7.23), the nonlinear transformation

$$\phi_1 = \rho\cos\theta, \qquad \phi_2 = \rho\sin\theta, \tag{7.27}$$

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with which the Lagrangian density with the flipped sign of the quadratic term becomes

$$\widetilde{\mathscr{L}}_{2d} = \tfrac{1}{2}\eta^{\mu\nu}\big[(\partial_\mu\rho)(\partial_\nu\rho) + \rho^2(\partial_\mu\theta)(\partial_\nu\theta)\big] + \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2\rho^2 - \tfrac{1}{4}\lambda\rho^4. \tag{7.28a}$$

Finding the minimum on the circle of radius $\rho = \frac{mc}{\hbar\sqrt{\lambda}}$ and after the substitution

$$\varrho := \rho - \tfrac{mc}{\hbar\sqrt{\lambda}}, \tag{7.28b}$$

we obtain

$$\begin{aligned}\widetilde{\mathscr{L}}_{2d} = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\varrho)(\partial_\nu\varrho) - \big(\tfrac{mc}{\hbar}\big)^2\varrho^2 - \tfrac{mc\sqrt{\lambda}}{\hbar}\varrho^3 - \tfrac{1}{4}\lambda\varrho^4 \ + \ \tfrac{m^4c^4}{4\lambda\hbar^4}\\ + \tfrac{1}{2}\big(\varrho + \tfrac{mc}{\hbar\sqrt{\lambda}}\big)^2\eta^{\mu\nu}(\partial_\mu\theta)(\partial_\nu\theta), \qquad \varpi_\vartheta : \theta \xrightarrow{(7.20)} \theta + \vartheta.\end{aligned} \tag{7.28c}$$

This makes it evident that the rotations (7.20) map the system from a parametrization where the Feynman calculus is defined about the ground state with $(\varrho,\theta) = (0,0)$ into a parametrization centered at $(\varrho,\theta) = (0,\theta_*)$, and where the θ field has no mass – nor in fact any potential – and so represents the Goldstone mode.

In turn, by shifting the fields in a ϑ-dependent fashion:

$$\varphi_1 = \phi_1 - \tfrac{mc}{\hbar\sqrt{\lambda}}\cos(\vartheta), \qquad \varphi_2 = \phi_2 - \tfrac{mc}{\hbar\sqrt{\lambda}}\sin(\vartheta), \tag{7.29}$$

we obtain

$$\begin{aligned}\widetilde{\widetilde{\mathscr{L}}}_{2d} = \tfrac{1}{2}\eta^{\mu\nu}\delta^{ij}(\partial_\mu\phi_i)(\partial_\nu\phi_j) - \big(\tfrac{mc}{\hbar}\big)^2\Big(\cos(\vartheta)\,\phi_1 + \sin(\vartheta)\,\phi_2\Big)^2 + \tfrac{m^4c^4}{4\lambda\hbar^4}\\ + \sqrt{\lambda}\big(\tfrac{mc}{\hbar}\big)\Big(\cos(\vartheta)\,\phi_1 + \sin(\vartheta)\,\phi_2\Big)(\phi_1^2+\phi_2^2) - \tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2,\end{aligned} \tag{7.30}$$

which evidently interpolates between (7.23)–(7.24) and a continuum of equivalently shifted Lagrangian densities.

Notice the extraordinary similarity between the descriptions (7.27)–(7.30) and the illustration in Figure 7.1 on p. 255, whereby it is possible to identify the pair of fields (ϕ_1,ϕ_2) with the motion denoted by the dark/light arrows on the left-hand side, and ϱ with the radial motion (dark arrows) on the left-hand side, and where the ϖ_ϑ rotation evidently perfectly corresponds to the rotational motion denoted by the light and outlined arrow. Unfortunately, the nonlinear coupling in the kinetic term, between $(\partial_\mu\theta)$ and ϱ, is the "price" of making this relationship between the Goldstone mode and the broken symmetry evident. This "polar" parametrization of the model is therefore rather cumbersome for defining Feynman diagrams and the perturbative computations, and is not used except to identify symmetries.

The Higgs effect for gauge $U(1)$ symmetry

The 2-dimensional model from the previous section may be combined with gauge symmetry. One only needs to reinterpret the pair of real scalar fields, ϕ_1,ϕ_2, as one complex scalar field, $\boldsymbol{\phi} = \phi_1 + i\phi_2$. This complex field then has a phase, and the description from Sections 5.1 and 5.3 may be adapted. Start therefore with the Lagrangian density

$$\mathscr{L}_{CW} = \tfrac{1}{2}\eta^{\mu\nu}(D_\mu\boldsymbol{\phi})^*(D_\nu\boldsymbol{\phi}) - \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2|\boldsymbol{\phi}|^2 - \tfrac{1}{4}\lambda\big(|\boldsymbol{\phi}|^2\big)^2 - \tfrac{4\pi\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu}, \tag{7.31}$$

where

$$D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + \tfrac{iq_\phi}{\hbar c}A_\mu\,\boldsymbol{\phi}, \tag{7.32}$$

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is the (electromagnetically) $U(1)$-covariant derivative, and $q_{\boldsymbol{\phi}}$ is the electric charge of the complex field $\boldsymbol{\phi}$. Varying by $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^*$, we obtain the Euler–Lagrange equations of motion, varying by $\dot{\boldsymbol{\phi}}$ and $\dot{\boldsymbol{\phi}}^*$ produces the canonical momenta, etc. However, we are here concerned with the breaking of the gauge symmetry

$$\boldsymbol{\phi}(\mathrm{x}) \to \exp\left\{\tfrac{i}{\hbar} q_{\boldsymbol{\phi}}\, \chi(\mathrm{x})\right\} \boldsymbol{\phi}(\mathrm{x}), \qquad A_\mu(\mathrm{x}) \to A_\mu(\mathrm{x}) - c\,\partial_\mu \chi(\mathrm{x}), \tag{7.33}$$

and will, as before, consider the Lagrangian density (7.31), the one with the "wrong" sign of the quadratic term:

$$\widetilde{\mathscr{L}}_{\mathrm{CW}} = \tfrac{1}{2}\eta^{\mu\nu}(D_\mu\boldsymbol{\phi})^*(D_\nu\boldsymbol{\phi}) + \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2|\boldsymbol{\phi}|^2 - \tfrac{1}{4}\lambda\big(|\boldsymbol{\phi}|^2\big)^2 - \tfrac{4\pi\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu}. \tag{7.34}$$

It is not hard to show, e.g., by parametrizing $\boldsymbol{\phi} = R\,e^{i\Theta}$, *[✐ do it]* that the potential energy density

$$\widetilde{\mathscr{V}}_{\mathrm{CW}} = -\tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2|\boldsymbol{\phi}|^2 + \tfrac{1}{4}\lambda\big(|\boldsymbol{\phi}|^2\big)^2 \;=\; -\tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2 R^2 + \tfrac{1}{4}\lambda R^4 \tag{7.35}$$

has a minimum when $R := |\boldsymbol{\phi}| = \frac{mc}{\hbar\sqrt{\lambda}}$ and the "angle" $\Theta \in [0, 2\pi]$ is arbitrary, which parametrizes a circle of radius $\frac{mc}{\hbar\sqrt{\lambda}}$ – in the complex field plane of $\boldsymbol{\phi} = \phi_1 + i\phi_2$. The classical solutions, i.e., the quantum-expectation values $|\langle\boldsymbol{\phi}\rangle| = \frac{mc}{\hbar\sqrt{\lambda}}$, are equally probable for every choice of the "angle" Θ, and the ultimate value $\langle\Theta\rangle$ is determined by the initial conditions and external influences. (As per Conclusion 1.1, perfect initial conditions do not exist.)

Choosing, e.g., $\Theta = 0$ for the ground state and in the Feynman diagrammatic calculus,[5] we must redefine the fields so that they describe fluctuations about the chosen classical solution. We thus define $\boldsymbol{\varphi} = \boldsymbol{\phi} - \frac{mc}{\hbar\sqrt{\lambda}}$, but are free to return to the Cartesian basis, with $\varphi_1 := \Re e(\boldsymbol{\phi}) - \frac{mc}{\hbar\sqrt{\lambda}}$ and $\varphi_2 := \Im m(\boldsymbol{\phi})$. This substitution yields

$$\begin{aligned}
\widetilde{\mathscr{L}}_{\mathrm{CW}} &= \tfrac{1}{2}\eta^{\mu\nu}\big[D_\mu\big((\varphi_1 + \tfrac{mc}{\hbar\sqrt{\lambda}}) + i\varphi_2\big)\big]^*\big[D_\nu\big((\varphi_1 + \tfrac{mc}{\hbar\sqrt{\lambda}}) + i\varphi_2\big)\big] - \tfrac{4\pi\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu}\\
&\quad + \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2\big|(\varphi_1 + \tfrac{mc}{\hbar\sqrt{\lambda}}) + i\varphi_2\big|^2 - \tfrac{1}{4}\lambda\big(\big|(\varphi_1 + \tfrac{mc}{\hbar\sqrt{\lambda}}) + i\varphi_2\big|^2\big)^2\\
&= \Big[\tfrac{1}{2}(\partial_\mu\varphi_1)(\partial^\mu\varphi_1) - \tfrac{m^2c^2}{\hbar^2}{\varphi_1}^2\Big] + \Big[\tfrac{1}{2}(\partial_\mu\varphi_2)(\partial^\mu\varphi_2)\Big] - \Big[\tfrac{4\pi\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\tfrac{q_\varphi^2 m^2}{\hbar^4\lambda}A_\mu A^\mu\Big]\\
&\quad + \tfrac{q_\varphi\, m}{\hbar^2\sqrt{\lambda}}\underline{A_\mu(\partial^\mu\varphi_2)} + \tfrac{q_\varphi}{c\hbar}A_\mu[\varphi_1(\partial^\mu\varphi_2) - (\partial^\mu\varphi_1)\varphi_2]\\
&\quad + \tfrac{q_\varphi^2 m}{c\hbar^3\sqrt{\lambda}}\varphi_1 A_\mu A^\mu - \tfrac{mc\sqrt{\lambda}}{\hbar}\varphi_1({\varphi_1}^2 + {\varphi_2}^2)\\
&\quad + \tfrac{1}{2}\tfrac{q_\varphi^2}{c^2\hbar^2}A_\mu A^\mu({\varphi_1}^2 + {\varphi_2}^2) - \tfrac{1}{4}\lambda({\varphi_1}^2 + {\varphi_2}^2)^2 + \tfrac{m^4c^4}{4\lambda\hbar^4}.
\end{aligned} \tag{7.36}$$

The appearance of the underlined "mixed" quadratic term $\frac{q_\varphi\, m}{\hbar^2\sqrt{\lambda}}A_\mu(\partial^\mu\varphi_2)$ indicates that the functions φ_1, φ_2, A_0, A_1, A_2 and A_3 are not the normal modes of this system. *[✐ Why?]* However, instead of pursuing the diagonalization procedure, we may use the gauge transformation

$$\begin{aligned}
\boldsymbol{\phi} \to e^{i\vartheta}\boldsymbol{\phi} &= (\cos\vartheta + i\sin\vartheta)(\phi_1 + i\phi_2)\\
&= (\phi_1\cos\vartheta - \phi_2\sin\vartheta) + i(\phi_1\sin\vartheta + \phi_2\cos\vartheta)
\end{aligned} \tag{7.37}$$

where we select [☞ definition (5.104a)]

$$\vartheta = -\operatorname{ATan}(\phi_1, \phi_2) = -\operatorname{ATan}\Big(\varphi_1 + \frac{mc}{\hbar\sqrt{\lambda}}, \varphi_2\Big), \tag{7.38}$$

[5] In classical physics, where $\phi_1 = \Re e(\boldsymbol{\phi})$ and $\phi_2 = \Im m(\boldsymbol{\phi})$ would be functions of (only) time, a similar choice would be convenient for describing small oscillations. Feynman's diagrammatic calculus is indeed a generalization of small oscillations in field theory, in a quite general sense.

so that $\varphi'_2 := \Im m(e^{i\vartheta}\boldsymbol{\phi}) = 0$. Also, $\varphi'_1 := \Re e\left(e^{i\vartheta}(\boldsymbol{\phi} - \frac{mc}{\hbar\sqrt{\lambda}})\right)$ and, of course, $A'_\mu := A_\mu + (\hbar c\,\partial_\mu\vartheta)$. The Lagrangian density (7.36) being invariant with respect to gauge transformations, it follows that *the same* Lagrangian density may also be expressed in terms of these new, gauge-transformed fields:

$$
\begin{aligned}
\widetilde{\mathscr{L}}_{\rm CW} = \Big[\tfrac{1}{2}(\partial_\mu\varphi'_1)(\partial^\mu\varphi'_1) - \tfrac{m^2c^2}{\hbar^2}\varphi_1'^2\Big] - \Big[\tfrac{4\pi\epsilon_0}{4}F'_{\mu\nu}F'^{\mu\nu} - \tfrac{1}{2}\tfrac{q_\varphi^2 m^2}{\hbar^4\lambda}A'_\mu A'^\mu\Big] \\
+ \tfrac{q_\varphi^2 m}{c\hbar^3\sqrt{\lambda}}\varphi'_1 A'_\mu A'^\mu - \tfrac{mc\sqrt{\lambda}}{\hbar}\varphi_1'^3 + \tfrac{1}{2}\tfrac{q_\varphi^2}{c^2\hbar^2}A'_\mu A'^\mu\varphi_1'^2 - \tfrac{1}{4}\lambda\varphi_1'^4 + \tfrac{m^4c^4}{4\lambda\hbar^4},
\end{aligned}
\tag{7.39}
$$

where we note that φ'_2 no longer appears. The same result could, of course, have been obtained by the standard diagonalization procedure.

It must be kept in mind that the three Lagrangian densities (7.34), (7.36) and (7.39) all describe the same system, only in slightly different parametrization, and where the ultimate version (7.39) achieves the most concise description. Varying the Lagrangian density (7.39) by A_μ produces the Euler–Lagrange equations of motion:

$$
\Box A'^\nu - \partial^\nu(\partial_\mu A'^\mu) + \frac{q_\varphi^2 m^2}{4\pi\epsilon_0\hbar^4\lambda}A'^\nu = -\frac{q_\varphi^2}{4\pi\epsilon_0 c^2\hbar^2}\big(\varphi'_1 + \tfrac{mc}{2\hbar\sqrt{\lambda}}\big)\varphi'_1 A'^\nu. \tag{7.40}
$$

This proves that the gauge field A'_μ acquired the mass

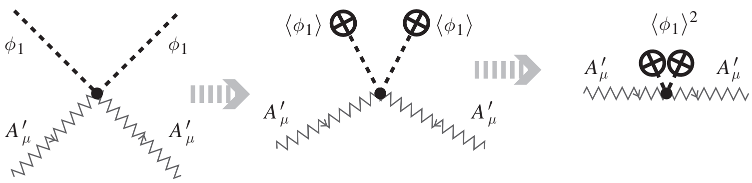
$$
m_A = \frac{q_\varphi\, m}{\sqrt{4\pi\epsilon_0}\hbar c\sqrt{\lambda}} = \frac{q_\varphi}{\sqrt{4\pi\epsilon_0}}\frac{1}{c^2}\langle\phi_1\rangle, \qquad \langle\phi_1\rangle = \frac{mc}{\hbar\sqrt{\lambda}}, \tag{7.41}
$$

since by using the Lorenz gauge, $\partial_\mu A'^\mu = 0$, the equation of motion (7.40) becomes

$$
\Big[\Box + \big(\tfrac{q_\varphi\, m}{\sqrt{4\pi\epsilon_0}\hbar^2\sqrt{\lambda}}\big)^2\Big]A'^\nu = -\frac{q_\varphi^2}{4\pi\epsilon_0 c^2\hbar^2}\big(\varphi'_1 + \tfrac{mc}{2\hbar\sqrt{\lambda}}\big)\varphi'_1 A'^\nu, \tag{7.42}
$$

where the operator in the square brackets is the same as in the Klein–Gordon equation (5.26).

The algebraic substitutions and operations that turn the Lagrangian densities (7.34)–(7.36) into (7.39) may also be represented graphically, since the various homogeneous terms[6] unambiguously correspond to the Feynman diagrams. So, e.g., the gauge boson mass stems from the interaction of these bosons with the Higgs field, where both "scalar" legs of this 2+2-leg vertex sink into the vacuum, or well from it:



(7.43)

The incessant sinking into the vacuum and welling from it of the ϕ_1-field acts as an effective "viscosity" for all the fields interacting with ϕ_1. This is what impedes the propagation of gauge fields A'_μ, so the quanta of this field acquire an (increased) inertia, i.e., mass. It is not hard to show that, in the (ϕ_1, ϕ_2)-picture, the Feynman diagrams of all "additional" terms in the Lagrangian density (7.39) have dashed lines that sink into the vacuum or well from it, as shown in diagrams (7.18) and (7.43). After the substitution $\phi_1 \to \varphi_1 + \frac{mc}{\hbar\sqrt{\lambda}}$, all diagrams that contain sinks/sources $\langle\phi_1\rangle$ are simply drawn as new, independent diagrams.

[6] By "homogeneous terms" one understands all the terms that have the same power of the various fields of the model. For example, $(\partial_\mu\phi_1)(\partial^\mu\phi_1)$ and $\frac{m^2c^2}{\hbar^2}\phi_1^2$ are homogeneous and together contribute to the propagator, i.e., the first Feynman diagram (7.17). Similarly, the Lagrangian density (7.39) has only one cubic term, $-\frac{mc\sqrt{\lambda}}{\hbar}\varphi_1'^3$, and this is the only term that contributes to the triple vertex Feynman diagram, shown as the middle diagram (7.17).

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Conclusion 7.4 *In a diagram such as (7.18) or (7.43), the crucial role is played by the property of Higgs bosons that they have a non-vanishing vacuum expectation value. The direct interpretation of these diagrams is that the Higgs bosons mediate the interaction of other particles with the* ***true*** *vacuum, so that the Higgs bosons in fact also* ***mediate*** *a type of interaction.*

Supporting the claim that these are but different descriptions of the same system, let us count the degrees of freedom in the Lagrangian density:

Equation (7.34) The complex scalar field $\boldsymbol{\phi}$ has two real functions, $\phi_1(\mathrm{x})$ and $\phi_2(\mathrm{x})$. The $U(1)$-gauge potential $A_\mu(\mathrm{x})$ has four real components, but only two are physical, as the gauge symmetry permits the imposition of the Lorenz and the Coulomb gauge, which leave only the two components (those orthogonal to the photon's direction of motion) having a physical meaning. Jointly, these count as four real functions.

Equation (7.39) The real scalar (Higgs) field $\varphi_1'(\mathrm{x})$ is of course just one real function. The vector potential $A_\mu(\mathrm{x})$ here has a mass, and so also has, besides the two components that are orthogonal to the direction of motion, the longitudinal component.[7] Jointly, these again count as four real functions.

By rewriting the Lagrangian density from its form (7.34) into the form (7.39), the imaginary part of the scalar field $\boldsymbol{\phi}$ became the *physical, longitudinal component* of the 4-vector gauge potential A_μ, whereby that gauge boson acquired the mass (7.41), proportional to the charge and the vacuum expectation value of the Higgs field $\boldsymbol{\phi}$. One says that the gauge boson "ate" the imaginary part of the Higgs field, φ_2, which had no mass in the Lagrangian density (7.36) and so represented the Goldstone boson. Suffice it here then to state, without a detailed proof [257, 307, 159, 422, 423, 538, 250, 389, 243]:

Conclusion 7.5 *In the general case of non-abelian (non-commutative) gauge symmetry breaking via the Higgs effect, there exists a symmetric ($\varkappa > 0$) phase, where the complete gauge symmetry is exact, and all Higgs fields are "accounted for" and have the same, real mass.*

There also exists a non-symmetric ($\varkappa < 0$), i.e., Higgs phase, where the gauge symmetry is broken so that from the original group of symmetries G only a subgroup H of symmetries is exact. For each generator of the so-called coset G/H [☞ the lexicon entry, in Appendix B.1] and corresponding to each broken symmetry:

1. *one Higgs scalar field turns into*
2. *the longitudinal component of one gauge 4-vector potential,*
3. *and the particle represented by that 4-vector potential becomes massive.*

The choice between the symmetric or non-symmetric phase is made by the sign $\varkappa$, which is a function of the ***order parameter*** *(typically, the temperature T), so that*

$$\varkappa(T) = \begin{cases} \varkappa > 0 & \text{for} \quad T > T_c \quad \text{symmetric phase,} \\ \varkappa < 0 & \text{for} \quad T < T_c \quad \text{non-symmetric phase.} \end{cases} \tag{7.44}$$

Comment 7.3 *All the Lagrangian densities involving a Higgs field such as (7.39) exhibit an excess energy density, $\frac{m^4c^4}{4\lambda\hbar^4}$. This quantity must contribute to the vacuum energy density of our universe (there is no external reservoir to siphon it away), the $8\pi G_N/c^4$-multiple of*

[7] See the discussion on p. 186, as well as the explanation in Footnote *29* on p. 67.

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which is the cosmological constant, and which is known to be some 55 orders of magnitude smaller than $\frac{m^4c^4}{4\lambda\hbar^4}$*; whence the term "excess energy density." This discrepancy only becomes worse with the grand-unifying attempts that we will explore in the next chapter. Ultimately, a theory also including gravity would – based on dimensional arguments alone – predict a vacuum energy density that is some 122 orders of magnitude larger than what is observed. This is often cited as the "vacuum catastrophe" and the "worst theoretical prediction in the history of physics" [272]. However, this is not the first time dimensional analysis alone presented a manifestly wrong answer; see Section 1.2.5.*

Comment 7.4 *In processes where the energies of the involved particles are bigger (smaller) than* $k_B T_c$*, one expects the system to be in the symmetric (non-symmetric) phase. In practice therefore, the energy available to the particles in observed processes is identified with the order parameter, i.e., temperature. Finally, the critical energy then must be proportional to the value* $\langle\boldsymbol{\phi}\rangle$*, and dimensional analysis dictates that*

$$E_c = \hbar c\sqrt{\lambda}\langle\boldsymbol{\phi}\rangle \;= k_B T_c. \tag{7.45}$$

7.1.4 Exercises for Section 7.1

✎ ***7.1.1*** *Confirm the results (7.15) by explicit computation.*

✎ ***7.1.2*** *Confirm the results (7.22) by explicit computation.*

✎ ***7.1.3*** *Expanding the Lagrangian density (7.21) about* $(\varphi_1, \varphi_2) = (\phi_1, \phi_2 - \frac{mc}{\hbar\sqrt{\lambda}})$*, verify that now* φ_1 *plays the role of the Goldstone boson.*

✎ ***7.1.4*** *Confirm the results (7.24) by explicit computation.*

✎ ***7.1.5*** *Confirm the results (7.36) by explicit computation.*

7.2 The weak nuclear interaction and its consequences

Interactions of gauge 4-vector potentials and spin-$\frac{1}{2}$ fermions studied in Chapter 5 faithfully describe the interactions of electromagnetic and strong interactions, the gauge bosons of which are massless. The Higgs effect, described in the previous section, provides a correct description of massive $W^{\pm}$- and Z^0-bosons. However, for the description of the interaction of these bosons with spin-$\frac{1}{2}$ fermions, we need one additional detail, to which we now turn.

7.2.1 The asymmetry in weak interactions

Chapter 5 describes interactions of gauge bosons with 4-component Dirac fermions, which were shown in Section 5.2.1 on p. 172 to decompose in a Lorentz-invariant way into the eigenstates of $\boldsymbol{\gamma}_{\pm}$ [☞ Conclusion 5.2 on p. 179], the so-called Weyl spinors:

$$\Psi = \Psi_+ + \Psi_-, \qquad \Psi_{\pm} := \big(\boldsymbol{\gamma}_{\pm}\Psi\big), \qquad \boldsymbol{\gamma}_{\pm} = \tfrac{1}{2}[\mathbb{1} \pm \hat{\boldsymbol{\gamma}}]. \tag{7.46}$$

Using the relations (A.121a)–(A.121b) and (A.130), we obtain that

$$\begin{aligned}&\overline{\Psi}\big[i\hbar c\boldsymbol{\gamma}^{\mu}D_{\mu} - \tfrac{mc}{\hbar}\mathbb{1}\big]\Psi \\ &\quad= \overline{\Psi_+}\big[i\hbar c\boldsymbol{\gamma}^{\mu}D_{\mu}\big]\Psi_+ + \overline{\Psi_-}\big[i\hbar c\boldsymbol{\gamma}^{\mu}D_{\mu}\big]\Psi_- - \tfrac{mc}{\hbar}\big[\overline{\Psi_-}\Psi_+ + \overline{\Psi_+}\Psi_-\big].\end{aligned} \tag{7.47}$$

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That is, the interaction of a spin-$\frac{1}{2}$ fermion with the gauge field as described in Chapter 5 includes both "left-handed" ($\Psi_- \equiv \Psi_L$) and "right-handed" ($\Psi_+ \equiv \Psi_R$) fermions.[8]

Note that the Lagrangian term that defines the mass, $-\frac{mc}{\hbar}\overline{\Psi_-}\Psi_+$, couples Ψ_+ and Ψ_-. This is the so-called Dirac mass. By contrast, the previous two terms in the expression (7.47) "link" fermions of *the same* chirality. This property permits massless spin-$\frac{1}{2}$ particles to satisfy the simpler, Weyl equation (5.62) instead of the more complicated Dirac equation (5.34).

As was discussed in Section 4.2.1, the weak interactions *maximally* break the parity symmetry as the interaction of the $W^\pm$ boson with a charged lepton and a neutrino exclusively couples the "left-handed" fermions. Thus, e.g., the interaction $e^- \to W^- + \nu_e$ in the Lagrangian density must correspond to the term

$$\begin{aligned}
&\overline{\Psi_-^{(\nu,e)}}\big[i\hbar c\boldsymbol{\gamma}^\mu\big(\mathbb{1}\partial_\mu + \tfrac{i g_w}{\hbar c}\mathbb{W}_\mu\big)\big]\Psi_-^{(\nu,e)}, \qquad \mathbb{W}_\mu := \tfrac{1}{2}\boldsymbol{\sigma}_a W^a_\mu,\\
&\quad= \overline{\Psi^{(\nu,e)}\boldsymbol{\gamma}_-}\big[i\hbar c\boldsymbol{\gamma}^\mu\big(\mathbb{1}\partial_\mu + \tfrac{i g_w}{\hbar c}\mathbb{W}_\mu\big)\big]\big(\boldsymbol{\gamma}_-\Psi^{(\nu,e)}\big)\\
&\quad\overset{(A.130)}{=} \overline{\Psi^{(\nu,e)}}\boldsymbol{\gamma}_+\big[i\hbar c\boldsymbol{\gamma}^\mu\big(\mathbb{1}\partial_\mu + \tfrac{i g_w}{\hbar c}\mathbb{W}_\mu\big)\big]\boldsymbol{\gamma}_-\Psi^{(\nu,e)}\\
&\quad= \overline{\Psi^{(\nu,e)}}\big[i\hbar c\boldsymbol{\gamma}_+\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}_-\big(\mathbb{1}\partial_\mu + \tfrac{i g_w}{\hbar c}\mathbb{W}_\mu\big)\big]\Psi^{(\nu,e)}\\
&\quad= \overline{\Psi^{(\nu,e)}}\big[i\hbar c\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}_-^2\big(\mathbb{1}\partial_\mu + \tfrac{i g_w}{\hbar c}\mathbb{W}_\mu\big)\big]\Psi^{(\nu,e)}\\
&\quad\overset{(A.121b)}{=} \overline{\Psi^{(\nu,e)}}\big[i\hbar c\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}_-\big(\mathbb{1}\partial_\mu + \tfrac{i g_w}{\hbar c}\mathbb{W}_\mu\big)\big]\Psi^{(\nu,e)}.
\end{aligned} \tag{7.48}$$

That is, the first term in the left–right symmetric expression (7.47) must not appear in the Lagrangian density for weak interactions. As the key terms in the Lagrangian density must include factors of the type

$$\begin{aligned}
\overline{\Psi_-^{(\nu,e)}}\boldsymbol{\gamma}^\mu\mathbb{W}_\mu\Psi_-^{(\nu,e)} &= \overline{\Psi^{(\nu,e)}}\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}_-\mathbb{W}_\mu\Psi^{(\nu,e)} = \tfrac{1}{2}\overline{\Psi^{(\nu,e)}}\boldsymbol{\gamma}^\mu[\mathbb{1}-\widehat{\boldsymbol{\gamma}}]\mathbb{W}_\mu\Psi^{(\nu,e)}\\
&= \tfrac{1}{2}\Big[\underbrace{\overline{\Psi^{(\nu,e)}}\boldsymbol{\gamma}^\mu\tfrac{1}{2}\boldsymbol{\sigma}_a\Psi^{(\nu,e)}}_{\text{vector}} - \underbrace{\overline{\Psi^{(\nu,e)}}\boldsymbol{\gamma}^\mu\widehat{\boldsymbol{\gamma}}\tfrac{1}{2}\boldsymbol{\sigma}_a\Psi^{(\nu,e)}}_{\text{axial vector}}\Big]W^a_\mu,
\end{aligned} \tag{7.49}$$

one says that weak interactions are of the "$V-A$" type – contrary to the electrodynamics and chromodynamics interactions that are of purely "V" (vector) type.

Thus, the Lagrangian density describing the interactions of gauge bosons $W^\pm$ may be written with the projectors $\boldsymbol{\gamma}_-$ consistently inserted for all fermions; interactions with the Z^0-boson are still more complicated [☞ Sections 7.2.4 and 7.2.5].

7.2.2 The GIM mechanism

Section 2.3.14 showed that the quark states that interact by weak interaction are not the eigenstates of the "free" Hamiltonian that defines the mass: The quark states that are detected as d-, s- and b-quarks primarily differ in mass [☞ Figure 2.1 on p. 76, and Table 4.1 on p. 152]. However, the eigenstates of the Hamiltonian term describing the interaction with the $W^\pm$- and Z^0-bosons are nontrivial linear combinations (2.53) of these mass-identified states.

The first-order effect

When Nicola Cabibbo suggested the first variation of this phenomenon in 1963, only the u-, d- and s-quarks were known. Proposing that the states that interact with the $W^\pm$- and Z^0-bosons are in fact

$$|u\rangle, \quad |d_w\rangle := \cos\theta_c|d\rangle + \sin\theta_c|s\rangle, \qquad |s_w\rangle := \cos\theta_c|s\rangle - \sin\theta_c|d\rangle, \tag{7.50}$$

$$\text{so} \quad |d\rangle = \cos\theta_c|d_w\rangle - \sin\theta_c|s_w\rangle, \qquad |s\rangle = \cos\theta_c|s_w\rangle + \sin\theta_c|d_w\rangle, \tag{7.51}$$

[8] It is standard to use the adjectives "left/right-handed" regarding both the chirality eigenstates and the helicity eigenstates of spin-$\frac{1}{2}$ fermions – although these coincide only for massless particles. The context usually makes it clear which of these two characteristics is meant; herein, we have in mind only chirality.

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Cabibbo explained the existence of processes of the type

$$d \to W^- + u \qquad \text{and} \qquad s \to W^- + u. \tag{7.52}$$

Since the s-quark carries strangeness, and u- and d-quarks do not, the first process is assigned $\triangle S = 0$ and the second one $\triangle S = 1$. In these processes the W^--boson is said to interact with the quark "current" $d \to u$ (which preserves strangeness), and respectively $s \to u$ (where strangeness is broken). Using the principle of detailed balance [☞ Section 2.14], we also have the processes $W^- \leftrightarrow d_w + \overline{u}$, and akin to the expression (7.49), we define the quark 4-current density that interacts with the weak gauge bosons:

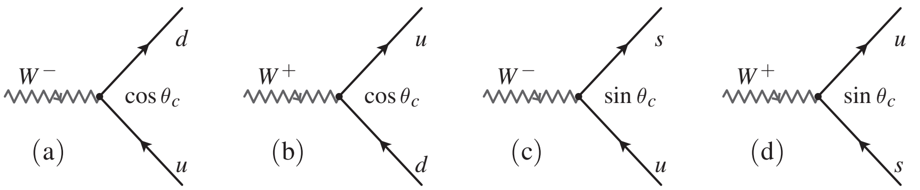
$$W^\mu_+ : \quad \mathfrak{J}^\mu_+ = \overline{d}_{w\,L}\boldsymbol{\gamma}^\mu u_L \qquad \to \cos\theta_c\,\overline{d}u + \sin\theta_c\,\overline{s}u, \tag{7.53a}$$

$$W^\mu_- : \quad \mathfrak{J}^\mu_- = \overline{u}_L\boldsymbol{\gamma}^\mu d_{w\,L} \qquad \to \cos\theta_c\,\overline{u}d + \sin\theta_c\,\overline{u}s, \tag{7.53b}$$

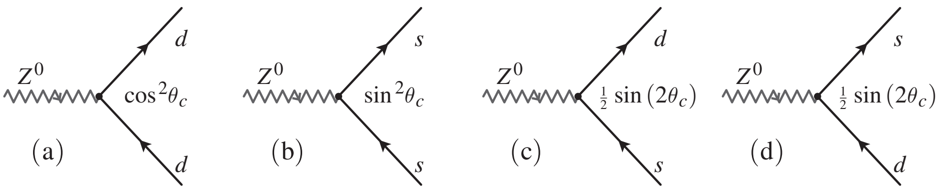
whereby it follows that

$$Z^0 : \quad \mathfrak{J}^\mu_0 = \overline{u}_L\boldsymbol{\gamma}^\mu u_L - \overline{d}_{w\,L}\boldsymbol{\gamma}^\mu d_{w\,L} \to \overline{u}u - \cos^2\theta_c\,\overline{d}d - \tfrac{1}{2}\sin 2\theta_c(\overline{d}s + \overline{s}d) - \sin^2\theta_c\,\overline{s}s. \tag{7.53c}$$

This implies the existence and relative strength of the following processes:



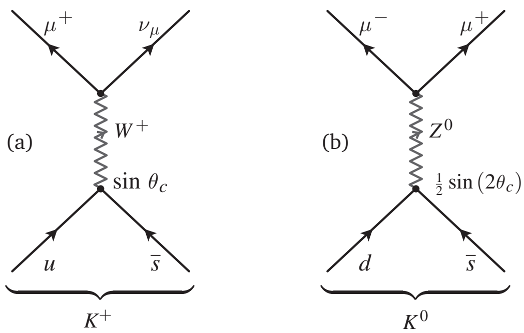
(7.54)



(7.55)

as well as their variations obtained through the principle of detailed balance, and where the relative θ_c-dependent factors for the amplitudes of these processes are written next to the vertices. The processes (7.54a,b) and (7.55a,b) have $\triangle S = 0$, and the processes (7.54c,d) and (7.55c,d) have $\triangle S = \pm 1$.

Combining the processes (7.54d) and (7.55d) with similar processes where the $W^\pm$- and Z^0-bosons create a lepton–antilepton pair, we obtain the Feynman diagrams



(7.56)

Except for the θ_c-dependent factor and the dependence on the particle masses, the amplitude of these processes would have to be approximately the same since

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$$\left|\frac{\frac{1}{2}\sin(2\theta_c)}{\sin(\theta_c)}\right|^2 = \cos^2(\theta_c) \sim O(\tfrac{1}{2})\text{–}O(1). \tag{7.57}$$

However, experiments confirm that the first of these two processes really happens and with the expected probability, but the second of these two processes practically does not occur [293]:[9]

$$\frac{\Gamma(K^+ \to \mu^+ + \nu_\mu)}{\Gamma(K^+ \to \text{all})} \approx 64\%, \qquad \frac{\Gamma(K^0 \to \mu^- + \mu^+)}{\Gamma(K^0 \to \text{all})} < 9\times 10^{-9}. \tag{7.58}$$

In the general case, it is experimentally verified that the processes with $\triangle S = \pm 1$ mediated by the Z^0-boson occur many orders of magnitude less frequently than other weak processes that can be described using the diagrams (7.54)–(7.55) and their equivalents with leptons instead of quarks. Cabibbo's original parametrization (7.50) thus implies the result (7.53c), which – besides the experimentally confirmed processes of the type (7.56a) – also predicts the flavor-changing neutral current processes, such as (7.56b), which do not occur. According to the discussion that led to Conclusion 1.1 on p. 6, Cabibbo's then model *must* be corrected.

To explain the tremendous difference (7.58), Glashow, Iliopoulos and Maiani (GIM) proposed in 1970 that there exists a fourth quark, c, so that the quark current densities that interact with the $W^\pm$- and Z^0 bosons are

$$W^+_\mu: \quad \mathfrak{J}^\mu_+ = \overline{d}_{w\,L}\boldsymbol{\gamma}^\mu u_L + \overline{s}_{w\,L}\boldsymbol{\gamma}^\mu c_L \to \cos\theta_c\,\overline{d}u + \sin\theta_c\,\overline{s}u - \sin\theta_c\,\overline{d}c + \cos\theta_c\,\overline{s}c, \tag{7.59a}$$

$$W^-_\mu: \quad \mathfrak{J}^\mu_- = \overline{u}_L\boldsymbol{\gamma}^\mu d_{w\,L} + \overline{c}_L\boldsymbol{\gamma}^\mu s_{w\,L} \to \cos\theta_c\,\overline{u}d + \sin\theta_c\,\overline{u}s - \sin\theta_c\,\overline{c}d + \cos\theta_c\,\overline{c}s, \tag{7.59b}$$

$$\begin{aligned} Z^0: \quad \mathfrak{J}^\mu_0 &= \overline{u}_L\boldsymbol{\gamma}^\mu u_L - \overline{d}_{w\,L}\boldsymbol{\gamma}^\mu d_{w\,L} + \overline{c}_L\boldsymbol{\gamma}^\mu c_L - \overline{s}_{w\,L}\boldsymbol{\gamma}^\mu d_{s\,L} \\ &\to \overline{u}u + \overline{c}c - \overline{d}d - \overline{s}s. \end{aligned} \tag{7.59c}$$

This proposal corrects Cabibbo's model in that it does not alter the results for the processes of the type (7.56a), but – in agreement with the experimental non-observation – prohibits processes of the type (7.56b). That is, in contrast to the quark current density (7.53c) that contains mixing terms $\overline{d}s$ and $\overline{s}d$, the quark current density (7.59c) contains no mixing term. The "price" for so diagonalizing the Z^0-boson interaction in the flavor space was the postulate of the existence of the fourth quark, and that proposal and its consequences are usually called the GIM mechanism.

> ***Comment 7.5*** *The Reader should notice the conceptual parallel between Glashow, Iliopoulos and Maiani's postulation of a new quark so as to preserve the logical consistency of the model and Pauli's postulation of the neutrino so as to preserve the energy conservation law [☞ Section 2.3.9].*

The second-order effect

Now, even if the decay $K^0 \to \mu^+ + \mu^-$ by way of a simple $O(g_w^2)$ process (7.56b) is forbidden, it does not follow that this physical process cannot happen by way of a more complex interaction, i.e., by way of a more complex Feynman diagram. Indeed, one straightforwardly constructs the $O(g_w^4)$ diagrams:

[9] Processes mediated by Z^0-bosons are usually labeled by the FCNC acronym, standing for *flavor-changing neutral current*.

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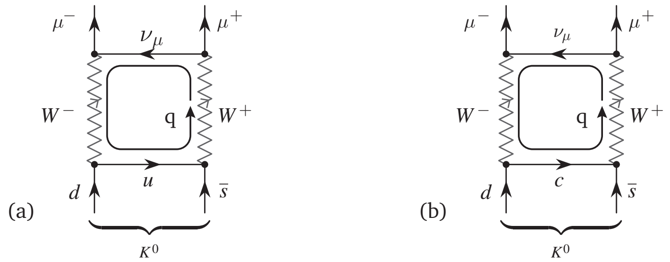

(7.60)

The sub-processes described by these two diagrams are identical, except that the u-quark in the left-hand diagram is replaced by a c-quark in the right-hand one. As these quarks are virtual in these diagrams, according to Conclusion 2.3 on p. 56, both sub-processes contribute to the decay $K^0 \to \mu^+\mu^-$. However, the diagram (7.54) implies that the amplitude of the diagram (7.60a) is proportional to $(\cos\theta_c)(-\sin\theta_c)$, while the amplitude for the diagram (7.60b) is proportional to $(\sin\theta_c)(\cos\theta_c)$. Since the amplitudes of these sub-processes are being added, these two contributions would exactly cancel if the u- and c-quark masses were equal.

That is, the application of the 4-momentum conservation in all vertices straightforwardly implies that one of the (internal) 4-momenta remains undetermined, and its integration remains unrestricted. We may always choose this to be the 4-momentum shown as circulating in the central loop/box and which was denoted "q." The $\int d^4$q-integral is dominated by contributions that stem from the $|\mathrm{q}| \gtrsim (m_W c) = 80.403\,\mathrm{GeV}/c$ regime, which is far in excess of m_u, m_c. The u- and c-quark mass dependence of the amplitudes must therefore be fairly soft, causes a very small ultimate difference between the two amplitudes, and guarantees their approximate cancellation. One expects the amplitude $\mathfrak{M}$ to be a function of $m_c - m_u$, and $\mathfrak{M} \propto (m_W)^{-2}$, owing to the two W-propagators. Thus, this estimate $|\mathfrak{M}|^2 \propto |\frac{(m_c-m_u)^2}{m_w^2}|^2 \sim 10^{-8}$ is already amazingly close to the experimental result (7.58) [293].

It may further be shown that the GIM mechanism actually guarantees the approximate cancellation of all possible contributions to the Z^0-mediated weak processes where $\triangle S = \pm 1$, and so guarantees good agreement between the Cabibbo–GIM model with four quarks and the experimental data. Nevertheless, the postulation of a new particle so as to preserve the logical consistency of the model was still regarded an extravagant "solution" of a problem of the otherwise (in the early 1970s) experimentally insufficiently justified quark model [243].

7.2.3 $U(1)_A$ anomaly

The existence of the fourth, c-quark was experimentally confirmed in 1974, but even before that, an extraordinarily strong but "purely theoretical" argument for its existence was known – separate from the GIM mechanism, but just as often ignored as "idle theory."

In the classical (non-quantum) version of the quark model, the functions used to represent the various particles satisfy their equations of motion:

$$\begin{aligned} i\hbar\partial_\mu[\overline{\Psi}_1\boldsymbol{\gamma}^\mu\Psi_2] &= (i\hbar\partial_\mu\overline{\Psi}_1\boldsymbol{\gamma}^\mu)\Psi_2 + \overline{\Psi}_1\boldsymbol{\gamma}^\mu(i\hbar\partial_\mu\Psi_2) = -\overline{(i\hbar\not{\partial}\Psi_1)}\Psi_2 + \overline{\Psi}_1(i\hbar\not{\partial}\Psi_2) \\ &= -\overline{(m_1 c\Psi_1)}\Psi_2 + \overline{\Psi}_1(m_2 c\Psi_2) = (m_2 - m_1)c\,\overline{\Psi}_1\Psi_2, \end{aligned} \tag{7.61}$$

since the quark functions Ψ satisfy the Dirac equation (5.34). Analogously,

$$\begin{aligned} i\hbar\partial_\mu[\overline{\Psi}_1\widehat{\boldsymbol{\gamma}}\boldsymbol{\gamma}^\mu\Psi_2] &= \big(i\hbar\partial_\mu\overline{\Psi}_1(-\boldsymbol{\gamma}^\mu\widehat{\boldsymbol{\gamma}})\big)\Psi_2 + \overline{\Psi}_1\widehat{\boldsymbol{\gamma}}\boldsymbol{\gamma}^\mu(i\hbar\partial_\mu\Psi_2) = \overline{(i\hbar\not{\partial}\Psi_1)}\widehat{\boldsymbol{\gamma}}\Psi_2 + \overline{\Psi}_1\widehat{\boldsymbol{\gamma}}(i\hbar\not{\partial}\Psi_2) \\ &= \overline{(m_1 c\Psi_1)}\widehat{\boldsymbol{\gamma}}\Psi_2 + \overline{\Psi}_1\widehat{\boldsymbol{\gamma}}(m_2 c\Psi_2) = (m_1 + m_2)c\,\overline{\Psi}_1\widehat{\boldsymbol{\gamma}}\Psi_2. \end{aligned} \tag{7.62}$$

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We then have:

Theorem 7.2 *For spinors Ψ_i that satisfy the Dirac equation $[i\hbar\not{\partial} - m_i c]\Psi_i = 0$:*

1. *the 4-vector current $\mathfrak{J}^{\mu}_{ij} := [\overline{\Psi}_i \boldsymbol{\gamma}^{\mu} \Psi_j]$ satisfies the continuity equation*
$$\partial_{\mu}\,\mathfrak{J}^{\mu}_{ij} = 0 \qquad \textit{precisely when} \quad m_i = m_j. \tag{7.63}$$
2. *The pseudo (axial) 4-vector current $\widehat{\mathfrak{J}}^{\mu}_{ij} := [\overline{\Psi}_i \widehat{\boldsymbol{\gamma}} \boldsymbol{\gamma}^{\mu} \Psi_j]$ satisfies the continuity equation*
$$\partial_{\mu}\,\widehat{\mathfrak{J}}^{\mu}_{ij} = 0 \qquad \textit{precisely when} \quad m_i = m_j = 0. \tag{7.64}$$

The continuity equations (7.63) and (7.64) guarantee that the "charges"

$$Q_{ij} := \int \mathrm{d}^3\vec{r}\; \mathfrak{J}^{0}_{ij} \qquad \text{and} \qquad \widehat{Q}_{ij} := \int \mathrm{d}^3\vec{r}\; \widehat{\mathfrak{J}}^{0}_{ij} \tag{7.65}$$

are conserved in all classical processes. For example, if we select i, j to count all quarks, then let $j = i$ and sum, $\sum_i Q_{ii}$ represents the quark number, and the expression $(3\sum_i Q_{ii})$ equals the baryon number [☞ Section 2.4.2, especially p. 76]. *Conversely* to Noether's theorem A.1 on p. 461, each current density that satisfies the equation of continuity defines a symmetry, and the "charges" Q_{ij} and $\widehat{Q}_{ij}$ are the formal generators of these corresponding symmetries. These are the *classical* symmetries of the system.

However, quantum effects in principle need not preserve classical symmetries, which then causes the appearance of quantum contributions that "spoil" the continuity equations

$$\partial_{\mu}\,\mathfrak{J}^{\mu}_{ij} = \mathfrak{A}_{ij} \qquad \text{and} \qquad \partial_{\mu}\,\widehat{\mathfrak{J}}^{\mu}_{ij} = \widehat{\mathfrak{A}}_{ij}, \tag{7.66}$$

where $\mathfrak{A}_{ij}$ and $\widehat{\mathfrak{A}}_{ij}$ are (quantum) *anomalies* of the current 4-vector densities $\mathfrak{J}^{\mu}_{ij}$ and $\widehat{\mathfrak{J}}^{\mu}_{ij}$, respectively, i.e., of the symmetries corresponding to these currents, whereby the *anomalies* $\mathfrak{A}_{ij}$ and $\widehat{\mathfrak{A}}_{ij}$ measure the quantum breaking of these symmetries.

It is paramount to realize the general nature of this phenomenon! We distinguish the following cases:

Approximate symmetries, as is the case with the "axial" currents (7.64), which are approximately conserved only in the specific regime of energies, 3-momenta and precision where we may neglect the differences between the masses of the particles amongst which the considered approximate symmetries operate. Even classically, such a current satisfies the continuity equation only approximately; its breaking produces a so-called pseudo-Goldstone mode, the mass of which is of the order of the resolution of the assumed approximation.

Global symmetries, such as the baryon number, for which the formal charge (7.65) is given by $(3\sum_i Q_{ii})$ and where the sum extends over all quark flavors. For that case, quantum chromodynamics yields $\mathfrak{A} \propto \vartheta\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}_{\rho\sigma}]$, where ϑ is a free parameter for which experiments indicate $\vartheta < 3\times10^{-10}$, the tininess of which has no complete theoretical explanation [☞ Section 6.3.1]✌.

Gauge symmetries, for which the appearance of anomaly indicates an essential contradiction. That is, models with anomalous gauge symmetry simply make no sense – unless they can be extended so as to cancel all gauge anomalies.

The analysis of precisely this last type of anomaly (S. Adler, and independently J. S. Bell and R. Jackiw) in 1969 pointed to the appearance of an anomalous quantum contribution in the continuity equation to the familiar electromagnetic current, owing to the coupling with the axial current [425, 586]. All amplitude contributions for any concrete process that leads to the appearance of an anomaly are products of a single, characteristic and incurably divergent type of integral and an indicative numerical factor. The algebraic sum of these contributions, the amplitude is thus a

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product of this characteristic and divergent integral and the sum of these indicative numeric factors. Such a result makes sense only if the sum of the indicative numeric factors identically cancels, as is the case, e.g., with the sum of electric charges within the family $\{u, d; \nu_e, e^-\}$ of fermions[10]

$$\sum_i Q_i = 3\big[\big(+\tfrac{2}{3}\big) + \big(-\tfrac{1}{3}\big)\big] + (0) + (-1) = 0, \tag{7.67}$$

where the explicit pre-factor "3" stems from summing over the three colors of the u- and d-quarks. The identical cancellation of this sum – and the corresponding absence of the quantum anomaly in electric charge conservation – has the following implications:

1. Every lepton pair $\{\nu_\ell, \ell^-\}$ requires a corresponding quark pair with (color-averaged) charges $+\frac{2}{3}$ and $-\frac{1}{3}$.
2. Quarks with electric charges $+\frac{2}{3}$ and $-\frac{1}{3}$ *must* occur in triples. Alternatively, the integrally charged quarks of the Han–Nambu model (5.212a) also occur in triples.

The latter of these two consequences confirms the necessity of the existence of quark colors.

However, more importantly, the first of these two consequences implies that the existence of the muon necessarily predicts the existence not only of the s-quark (with $-\frac{1}{3}$ charge) but also of the c-quark (with $+\frac{2}{3}$ charge). Since the neutrino has no electric charge, the unavoidable need for a consistent and complete cancellation of the quantum anomaly of the electric current had by 1969 predicted the existence of the fourth quark. However, it was not clear at the time that this conclusion was absolutely inevitable, and even the theoretical motivations for predicting the fourth quark, such as the GIM mechanism, originally did not include the anomaly analysis.

Digression 7.2 The lesson from Pauli's prediction of the neutrino [☞ Section 2.3.9] so as to save the 4-momentum conservation law seems not to have been learned well enough. Between 1969 and 1974, several separate theoretical considerations indicated that inconsistency and contradiction within the theoretical models of particle physics could only be avoided by introducing a new particle, the c-quark. Nevertheless, few particle physicists took these arguments seriously, since the discovery of the J/ψ particle, the lowest-energy $c\bar{c}$-bound state, came as a surprise to most.

It behooves us to finally learn that logical consistency and absence of self-contradiction is a terrific tool of theoretical physics.

The benefit of hindsight today of course permits complete certainty in limiting to quark models that include only complete quark–lepton fours (so-called "families"):

$$\underbrace{\underbrace{\underbrace{\begin{bmatrix}u\\d\end{bmatrix},\begin{bmatrix}\nu_e\\e^-\end{bmatrix}}_{};\quad \begin{bmatrix}c\\s\end{bmatrix},\begin{bmatrix}\nu_\mu\\\mu^-\end{bmatrix}}_{};\quad \begin{bmatrix}t\\b\end{bmatrix},\begin{bmatrix}\nu_\tau\\\tau^-\end{bmatrix}}_{}. \tag{7.68}$$

Including the s-quark without the c-quark or the b-quark without the t-quark is simply inconsistent, as it causes the quantum effects to ruin the $U(1)$ symmetry of quantum electrodynamics and the corresponding electric charge conservation – contradicting experiments, as well as contradicting the gauge symmetry of electromagnetism and the corresponding interactions with gauge bosons.

[10] The fundamental Standard Model fermions are typically divided into three copies of the first four: $\{u, d; \nu_e, e^-\}$, $\{c, s; \nu_\mu, \mu^-\}$ and $\{t, b; \nu_\tau, \tau^-\}$. These copies are called – figuratively – either *generations* or *families*. Without any implication or judgement about the former of these – or indeed any filial/paternal, sororal or fraternal relations, I will herein use the latter name.

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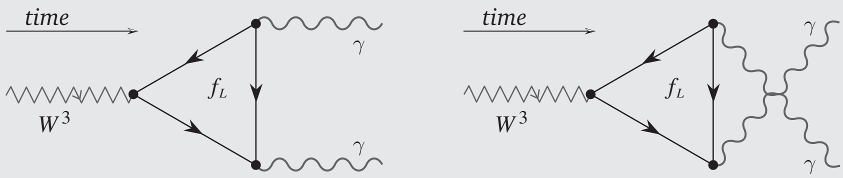
Example 7.1 The anomaly analysis from Section 7.2.3 may be applied to the pair of Feynman diagrams where $f_L \in \{u_L, d_L; \nu_{eL}, e^-_L; \overline{u}_L = \overline{u_R}, \overline{d}_L{=}\overline{d_R}; e^+_L = \overline{e^-_R}; \cdots\}$:



(7.69)

the amplitudes of which contain terms proportional to the sum $\sum_{f_L} I_w(f_L)\,\big(Q(f_L)\big)^2$. Summing over the fermions of only the first family [☞ Table 7.1 on p. 275, as well as Refs. [425, 586, Chapter 19] for details],

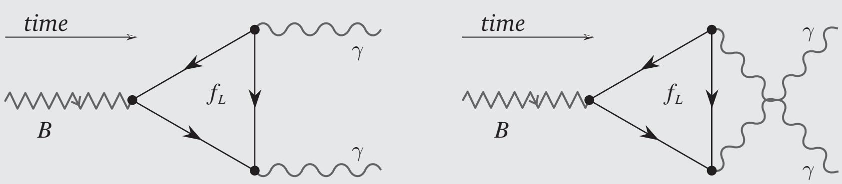
$$\begin{aligned}\sum_{f_L} I_w(f_L)\big(Q(f_L)\big)^2 &= 3\Big[\big(+\tfrac{1}{2}\big)\big(+\tfrac{2}{3}\big)^2 + \big(-\tfrac{1}{2}\big)\big(-\tfrac{1}{3}\big)^2\Big] + \big(+\tfrac{1}{2}\big)(0)^2 + \big(-\tfrac{1}{2}\big)(-1)^2 \\ &= 3\Big[+\tfrac{2}{9} - \tfrac{1}{18}\Big] - \tfrac{1}{2} = 3\Big(+\tfrac{3}{18}\Big) - \tfrac{1}{2} = 0. \end{aligned} \tag{7.70}$$

The complete computation shows that the contributions of the Feynman diagrams (7.69) in fact diverge. Thus, the contributions of the Feynman diagrams (7.69) to the amplitudes that contain the $W^3 \to 2\gamma$ factor are finite (and in fact vanish) if and only if the virtual fermions forming the triangle loops include complete families $\{u, d; \nu_e, e^-\}$, $\{c, s; \nu_\mu, \mu^-\}$, etc. Without the cancellation (7.70), models that include these Feynman diagrams simply make no sense. Notice that the same computation for the Han–Nambu model (5.212a) of integrally charged quarks,

$$\begin{aligned}&\sum_{f_L} I_w(f_L)\big(Q(f_L)\big)^2 \\ &= \Big[\big(+\tfrac{1}{2}\big)\big((+1)^2{+}(+1)^2{+}(0)^2\big) + \big(-\tfrac{1}{2}\big)\big((-1)^2{+}(0)^2{+}(0)^2\big)\Big] \\ &\qquad + \big(+\tfrac{1}{2}\big)(0)^2 + \big(-\tfrac{1}{2}\big)(-1)^2 \\ &= \Big[\big(+\tfrac{1}{2}\big)2 + \big(-\tfrac{1}{2}\big)1\Big] + \big(-\tfrac{1}{2}\big)(-1)^2 = \big[+1 - \tfrac{1}{2}\big] - \tfrac{1}{2} = 0, \end{aligned} \tag{7.71}$$

implies that it too is free of this gauge anomaly.

Example 7.2 Akin to Example 7.1, we may analyze the pair of Feynman diagrams where the unobserved fermion in the loop is again $f_L \in \{u_L, d_L; \nu_{eL}, e^-_L; \overline{u_R}{=}\overline{u}_L, \overline{d}_L; e^+_L; \ldots\}$:



(7.72)

the amplitudes of which contain terms proportional to the sum $\sum_{f_L} Y_w(f_L)\,\big(Q(f_L)\big)^2$. Summing over the fermions of only the first family [☞ Table 7.1 on p. 275, as well as Refs. [425, 586, Chapter 19] for details],

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$$\sum_{f_L} Y_w(f_L)\big(Q(f_L)\big)^2 = 3\Big[(+\tfrac{1}{3})\Big((+\tfrac{2}{3})^2+(-\tfrac{1}{3})^2\Big)\Big] + (-1)(0)^2 + (-1)(-1)^2$$
$$+3\Big[(-\tfrac{4}{3})(-\tfrac{2}{3})^2 + (+\tfrac{2}{3})(+\tfrac{1}{3})^2\Big] + (+2)(+1)^2 + (0)(0)^2$$
$$= 3\Big(\tfrac{1}{3}\cdot\tfrac{4+1}{9} - \tfrac{4}{3}\cdot\tfrac{4}{9} + \tfrac{2}{3}\cdot\tfrac{1}{9}\Big) - 1 + 2 = \tfrac{5-16+2}{9} + 1 = 0. \tag{7.73}$$

As in the previous example, the complete computation shows that the contributions of the Feynman diagrams (7.72) to the amplitude of the $B \to 2\gamma$ process in fact diverge. Again, this result makes sense only if the virtual fermions depicted by the triangular loops include complete families $\{u, d; \nu_e, e^-\}$, $\{c, s; \nu_\mu, \mu^-\}$, etc. Without a cancellation such as in (7.73), models that include these Feynman diagrams simply make no sense. Notice that the same computation for the Han–Nambu model (5.212a) of integrally charged quarks,

$$\sum_{f_L} Y_w(f_L)\big(Q(f_L)\big)^2$$
$$= \Big[(+\tfrac{1}{3})\big((+1)^2+(+1)^2+(0)^2 + (-1)^2+(0)^2+(0)^2\big)\Big] + (+1)(0)^2 + (-1)(-1)^2$$
$$+ \Big[(-\tfrac{4}{3})\big((+1)^2+(+1)^2+(0)^2\big) + (+\tfrac{2}{3})\big((-1)^2+(0)^2+(0)^2\big)\Big]$$
$$+ (+2)(+1)^2 + (0)(0)^2$$
$$= \big[\tfrac{1}{3}\cdot 3 - \tfrac{4}{3}\cdot 2 + \tfrac{2}{3}\cdot 1\big] - 1 + 2 = \tfrac{3-8+2}{3} + 1 = 0, \tag{7.74}$$

implies that it is also free of this gauge anomaly.

Conclusion 7.6 *Since the joint contributions of the Feynman diagram pairs (7.69) vanish, as they also do for the diagram pair (7.72), the joint contributions then also vanish for the linear combination $Z^0 = \cos(\theta_w)W^3 - \sin(\theta_w)B$.[11] The same holds if in these diagrams the W^3- and B-particle, respectively (which are the normal modes in the $SU(2)_w \times U(1)_y$ symmetric phase) are replaced with the Z^0-particle, one of the two normal modes after the $SU(2)_w \times U(1)_y \to U(1)_Q$ symmetry breaking.*

In the general case, the anomaly of any symmetry must remain conserved through any phase transition, and so also through the $SU(2)_w \times U(1)_y \to U(1)_Q$ electroweak symmetry breaking. Anomalies of gauge symmetries of course must vanish (cancel), but the conservation of anomalies of other (including approximate, and exact but global) symmetries is a useful "sum rule" in the study of all phase transitions.

Further details on this technique, both conceptual and practical and technical, may be found in standard field theory textbooks, and the interested Reader is directed to Refs. [12, 224, 75, 261, 425, 554, 555, 206, 484, 496, 589, 586, 590].

7.2.4 The weak (Weinberg) angle

Although both the $W^\pm$- and Z^0-particles are gauge bosons of weak interactions, their masses are not equal [☞ Table C.2 on p. 526]. This is a consequence of the fact that the Z^0-boson

[11] The angle θ_w is usually called "weak" or the Weinberg angle (although it was Glashow who introduced it); experimentally, $\theta_w \approx 28.75°$.

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and the photon are linear combinations of the $SU(2)_w$-partner of the $W^{\pm}$-boson and the $U(1)_y$-gauge boson. This effect is well described in the Glashow–Weinberg–Salam model of electroweak interactions.

The conclusions of Sections 7.2.1–7.2.3 indicate a finer structure among the particles in Table 2.3 on p. 67, of which all matter consists. That is, weak interactions may be described by a non-abelian (non-commutative) gauge model in which, owing to the relation (7.49), the left- and the right-handed fermions are treated differently. Akin to the GNN formula (2.44b), the weak isospin I_w and the weak hypercharge Y_w are defined so as to satisfy the relation

$$Q = I_w + \tfrac{1}{2} Y_w. \tag{7.75}$$

Table 7.1 The weak isospin, the weak hypercharge and the electric charge of the elementary fermions are related by equation (7.75). The values are, however, different for fermions of left-handed and right-handed chirality.

	Fermion family			**Charges**		
	1	2	3	Q	I_w	Y_w
$\underbrace{\Psi_- = \boldsymbol{\gamma}_- \Psi}_{\text{left-handed}}$	$\begin{bmatrix} u \\ d \end{bmatrix}_L$	$\begin{bmatrix} c \\ s \end{bmatrix}_L$	$\begin{bmatrix} t \\ b \end{bmatrix}_L$	$\begin{matrix} +\frac{2}{3} \\ -\frac{1}{3} \end{matrix}$	$\begin{matrix} +\frac{1}{2} \\ -\frac{1}{2} \end{matrix}$	$\begin{matrix} +\frac{1}{3} \\ +\frac{1}{3} \end{matrix}$
	$\begin{bmatrix} \nu_e \\ e^- \end{bmatrix}_L$	$\begin{bmatrix} \nu_\mu \\ \mu^- \end{bmatrix}_L$	$\begin{bmatrix} \nu_\tau \\ \tau^- \end{bmatrix}_L$	$\begin{matrix} 0 \\ -1 \end{matrix}$	$\begin{matrix} +\frac{1}{2} \\ -\frac{1}{2} \end{matrix}$	$\begin{matrix} -1 \\ -1 \end{matrix}$
$\underbrace{\Psi_+ = \boldsymbol{\gamma}_+ \Psi}_{\text{right-handed}}$	u_R	c_R	t_R	$+\frac{2}{3}$	0	$+\frac{4}{3}$
	d_R	s_R	b_R	$-\frac{1}{3}$	0	$-\frac{2}{3}$
	e_R^-	μ_R^-	τ_R^-	-1	0	-2
	ν_{eR}	$\nu_{\mu R}$	$\nu_{\tau R}$	0	0	0

It must be emphasized that the weak isospin and the weak hypercharge are defined akin to the previously defined and similarly named quantities, and so that they satisfy the familiar formula (2.44b). However, Table 7.1 shows that these quantities coincide with the "old" values (2.44a) only for the left-handed eigenfunctions of chirality and not for the right-handed ones – which have no weak isospin and so are invariant with respect to $SU(2)_w$. In this way, the weak isospin and $SU(2)_w$ play the role, respectively, of the charge and the symmetry for the gauge model of weak interactions.

In the gauge $SU(2)_w \times U(1)_y$ model (Glashow, Weinberg and Salam) one introduces the gauge bosons $W_\mu^{\pm}$ and W_μ^3 for the $SU(2)_w$ factor, and B_μ for the $U(1)_y$ factor. The weak isospin and the weak hypercharge [☞ Table 7.1] determine the interaction intensity between these gauge bosons and the fermions $\{u, d; \nu_e, e^-; c, s; \nu_\mu, \mu^-; \dots\}$, so we know that the interaction terms in the Lagrangian density are, in order:

$$\mathscr{L}_{\text{GWS}} \ni g_w \big(W_\mu^+ J_+^\mu + W_\mu^- J_-^\mu + W_\mu^3 J_3^\mu \big) + g_y B_\mu J_y^\mu, \tag{7.76a}$$

$$J_+^\mu := \big\{ [\overline{u_L}\, \boldsymbol{\gamma}^\mu\, d_{wL}] + [\overline{c_L}\, \boldsymbol{\gamma}^\mu\, s_{wL}] + [\overline{t_L}\, \boldsymbol{\gamma}^\mu\, b_{wL}] \big\}, \tag{7.76b}$$

$$J_-^\mu := \big\{ [\overline{d_{wL}}\, \boldsymbol{\gamma}^\mu\, u_L] + [\overline{s_{wL}}\, \boldsymbol{\gamma}^\mu\, c_L] + [\overline{b_{wL}}\, \boldsymbol{\gamma}^\mu\, t_L] \big\}, \tag{7.76c}$$

$$\begin{aligned} J_3^\mu := \Big\{ &\tfrac{1}{2} \Big([\overline{u_L}\, \boldsymbol{\gamma}^\mu\, u_L] + [\overline{c_L}\, \boldsymbol{\gamma}^\mu\, c_L] + [\overline{t_L}\, \boldsymbol{\gamma}^\mu\, t_L] + [\overline{\nu_{eL}}\, \boldsymbol{\gamma}^\mu\, \nu_{eL}] + [\overline{\nu_{\mu L}}\, \boldsymbol{\gamma}^\mu\, \nu_{\mu L}] + [\overline{\nu_{\tau L}}\, \boldsymbol{\gamma}^\mu\, \nu_{\tau L}] \Big) \\ &- \tfrac{1}{2} \Big([\overline{d_L}\, \boldsymbol{\gamma}^\mu\, d_L] + [\overline{s_L}\, \boldsymbol{\gamma}^\mu\, s_L] + [\overline{b_L}\, \boldsymbol{\gamma}^\mu\, b_L] + [\overline{e_L^-}\, \boldsymbol{\gamma}^\mu\, e_L^-] + [\overline{\mu_L^-}\, \boldsymbol{\gamma}^\mu\, \mu_L^-] + [\overline{\tau_L^-}\, \boldsymbol{\gamma}^\mu\, \tau_L^-] \Big) \Big\}, \end{aligned} \tag{7.76d}$$

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$$
\begin{aligned}
J^\mu_y := \Big\{ &\tfrac{1}{6}\Big([\overline{u_L}\,\boldsymbol{\gamma}^\mu\, u_L] + [\overline{c_L}\,\boldsymbol{\gamma}^\mu\, c_L] + [\overline{t_L}\,\boldsymbol{\gamma}^\mu\, t_L] + [\overline{d_L}\,\boldsymbol{\gamma}^\mu\, d_L] + [\overline{s_L}\,\boldsymbol{\gamma}^\mu\, s_L] + [\overline{b_L}\,\boldsymbol{\gamma}^\mu\, b_L] \Big) \\
&- \tfrac{1}{2}\Big([\overline{\nu_{eL}}\,\boldsymbol{\gamma}^\mu\, \nu_{eL}] + [\overline{\nu_{\mu L}}\,\boldsymbol{\gamma}^\mu\, \nu_{\mu L}] + [\overline{\nu_{\tau L}}\,\boldsymbol{\gamma}^\mu\, \nu_{\tau L}] + [\overline{e^-_L}\,\boldsymbol{\gamma}^\mu\, e^-_L] + [\overline{\mu^-_L}\,\boldsymbol{\gamma}^\mu\, \mu^-_L] + [\overline{\tau^-_L}\,\boldsymbol{\gamma}^\mu\, \tau^-_L] \Big) \\
&+ \tfrac{2}{3}\Big([\overline{u_R}\,\boldsymbol{\gamma}^\mu\, u_R] + [\overline{c_R}\,\boldsymbol{\gamma}^\mu\, c_R] + [\overline{t_R}\,\boldsymbol{\gamma}^\mu\, t_R] \Big) - \tfrac{1}{3}\Big([\overline{d_R}\,\boldsymbol{\gamma}^\mu\, d_R] + [\overline{s_R}\,\boldsymbol{\gamma}^\mu\, s_R] + [\overline{b_R}\,\boldsymbol{\gamma}^\mu\, b_R] \Big) \\
&- \Big([\overline{e^-_R}\,\boldsymbol{\gamma}^\mu\, e^-_R] + [\overline{\mu^-_R}\,\boldsymbol{\gamma}^\mu\, \mu^-_R] + [\overline{\tau^-_R}\,\boldsymbol{\gamma}^\mu\, \tau^-_R] \Big) \Big\},
\end{aligned}
\tag{7.76e}
$$

where d_w, s_w and b_w are the quark states defined by the Cabibbo–Kobayashi–Maskawa (CKM) mixing (2.53)–(2.55), the subscript "L" denotes the projection to the left-handed chirality, and where the expression for J^μ_y includes the factor $\frac{1}{2}$ from the formula $Q = I_w + \frac{1}{2}Y_w$, modeled on the original GNN formula (2.44b).

For the purposes of $SU(2)_w \times U(1)_y \to U(1)_Q$ symmetry breaking, Weinberg and Salam[12] introduced a doublet of complex Higgs fields:

$$
\mathbb{H} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad \text{with} \quad \begin{cases} I_w(H_1) = +\frac{1}{2} & Y_w(H_1) = +1 \quad Q(H_1) = +1, \\ I_w(H_2) = -\frac{1}{2} & Y_w(H_2) = +1 \quad Q(H_2) = 0. \end{cases}
\tag{7.77}
$$

We thus identify $H_1 = H^+$, $(H_1)^\dagger = H^-$, $H_2 = H^0$ and $(H_2)^\dagger = \overline{H}{}^0$.

Besides, $W^\pm_\mu, W^3_\mu$ and B_μ also interact with the complex Higgs field doublet, $\mathbb{H}$,

$$
\mathscr{L}_{\mathbb{H}} = \big\|(\partial_\mu - i g_w W^\alpha_\mu\, \tfrac{1}{2}\boldsymbol{\sigma}_\alpha - i g_y B_\mu\, \tfrac{1}{2}\mathbb{1})\mathbb{H}\big\|^2_\eta + \tfrac{1}{2}\big(\tfrac{\mu c}{\hbar}\big)^2 (\mathbb{H}^\dagger\mathbb{H}) - \tfrac{1}{4}\lambda\,(\mathbb{H}^\dagger\mathbb{H})^2,
\tag{7.78}
$$

where the index α is summed over the values $1, 2, 3$, and where

$$
\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\tag{7.79}
$$

With the sign of the quadratic term as in equation (7.78), the minimum of the potential lies in the values of the field $\mathbb{H}$ that satisfy

$$
|H_1|^2 + |H_2|^2 = H^2_{1r} + H^2_{1i} + H^2_{2r} + H^2_{2i} = \big(\tfrac{\mu c}{\lambda\hbar}\big)^2
\tag{7.80}
$$

and which form a 3-sphere $S^3 \subset \mathbb{R}^4 \approx \mathbb{C}^2$. One such value is $\mathbb{H} = \big(\frac{\mu c}{\lambda\hbar}\big)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Digression 7.3 That is, with the standard choice of the Higgs field (7.77), $\langle H_1 \rangle = \langle H^+ \rangle \neq 0$ would imply that the vacuum has the electric charge $+1$ and that the $U(1)$ gauge symmetry of the electromagnetic interaction is broken – which is not the case! Of

[12] S. L. Glashow had already in 1958, in his PhD dissertation mentored by J. Schwinger, proposed an electro-weak unification based only on the $SU(2)_w$ group, where the photon corresponds to the diagonal generator J_3, and the $W^\pm$-bosons correspond to the generators $J_\pm$ [☞ relations (A.38)]. The model was worked out in collaboration with H. Georgi and it turned out that this cannot be made to agree with experiments [209]. It became clear in the early 1960s that the gauge group $SU(2)_w \times U(1)_y$ is a better choice, so that the photon (7.85) would interact with fermions with an intensity equal to the electric charge obtained from the GNN formula (7.75). The mass of the $W^\pm$- and the Z^0-bosons had, however, remained a mystery: Simply added "by hand" (as Glashow advocated), the mass of the gauge bosons explicitly breaks the gauge invariance but also the renormalizability (and then also the self-consistency) of the model. In 1967–8, Weinberg and, independently, Salam showed that the Higgs mechanism may be applied and produces the desired mass. G. 't Hooft (1971), B. W. Lee and J. Zinn–Justin, and finally G. 't Hooft and M. Veltman (1972) proved the renormalizability of the Glashow–Weinberg–Salam model of electroweak interactions, and D. J. Gross and R. Jackiw, and then C. Bouchiat, J. Iliopoulos and P. Meyer showed the same year (1972) that all anomalies cancel in this model [209, 552, 473].

course, the choice $\langle H_1 \rangle \neq 0$ and $\langle H_2 \rangle = 0$ would only imply that the remaining massless field is not A_μ but Z_μ amongst the linear combinations (7.85)–(7.86), and that the corresponding $U(1) \subset SU(2)_w \times U(1)_y$ remains the exact gauge symmetry. This group $U(1)$ and this field would then have to be identified, respectively, with the gauge symmetry of electromagnetism and the photon.

After redefining the Higgs field,

$$\widetilde{\mathbb{H}} := \mathbb{H} - \langle \mathbb{H} \rangle, \qquad \langle \mathbb{H} \rangle = \left(\tfrac{\mu c}{\lambda \hbar}\right) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \tag{7.81}$$

it follows that $\widetilde{H}_{1r}, \widetilde{H}_{1i}, \widetilde{H}_{2r}, \widetilde{H}_{2i}, W_\mu^1, W_\mu^2, W^3$ and B_μ are not the normal modes – just as in the Lagrangian density (7.36)–(7.39) – and one must again diagonalize the fields. The identification of normal modes is fairly simple. From equation (7.78), we have

$$\begin{aligned}
&\big[(\partial_\mu - i g_w W_\mu^\alpha \, \tfrac{1}{2}\boldsymbol{\sigma}_\alpha - i g_y B_\mu \, \tfrac{1}{2}\mathbb{1})\mathbb{H}\big]^\dagger \eta^{\mu\nu} \big[(\partial_\nu - i g_w W_\nu^\beta \, \tfrac{1}{2}\boldsymbol{\sigma}_\beta - i g_y B_\nu \, \tfrac{1}{2}\mathbb{1})\mathbb{H}\big] \\
&\quad = \cdots + \left(\tfrac{\mu c}{\lambda \hbar}\right)^2 \Big[(-i g_w W_\mu^\alpha \, \tfrac{1}{2}\boldsymbol{\sigma}_\alpha - i g_y B_\mu \, \tfrac{1}{2}\mathbb{1}) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\Big]^\dagger \eta^{\mu\nu} \Big[(-i g_w W_\nu^\beta \, \tfrac{1}{2}\boldsymbol{\sigma}_\beta - i g_y B_\nu \, \tfrac{1}{2}\mathbb{1}) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\Big] + \cdots \\
&\quad = \cdots + \tfrac{1}{4} \left(\tfrac{\mu c}{\lambda \hbar}\right)^2 (g_w W_\mu^3 - g_y B_\mu)^\dagger \eta^{\mu\nu} (g_w W_\nu^3 - g_y B_\nu) + \cdots .
\end{aligned} \tag{7.82}$$

Using the "weak angle"

$$\theta_w = \arccos\left(\tfrac{g_w}{\sqrt{g_w^2 + g_y^2}}\right), \qquad \text{so} \quad \cos\theta_w = \tfrac{g_w}{\sqrt{g_w^2 + g_y^2}} \quad \text{and} \quad \sin\theta_w = \tfrac{g_y}{\sqrt{g_w^2 + g_y^2}}, \tag{7.83}$$

the expression (7.82) becomes

$$\cdots + \tfrac{1}{2} \left(\tfrac{\mu c}{\sqrt{2}\lambda \hbar}\right)^2 (g_w^2 + g_y^2) \Big\| \big(\cos(\theta_w) W_\mu^3 - \sin(\theta_w) B_\mu\big) \Big\|_\eta^2 + \cdots . \tag{7.84}$$

The normal modes then are the linear combinations

$$A_\mu := \quad \cos(\theta_w) B_\mu + \sin(\theta_w) W_\mu^3, \qquad \text{with the mass} \; = 0, \tag{7.85}$$

$$Z_\mu := -\sin(\theta_w) B_\mu + \cos(\theta_w) W_\mu^3, \qquad \text{with the mass} \; = \tfrac{\mu c}{\sqrt{2}\lambda \hbar} \sqrt{g_w^2 + g_y^2}. \tag{7.86}$$

The gauge boson represented by the 4-vector A_μ is identified as the photon, and the gauge boson represented by the 4-vector Z_μ^0 acquired a mass and is identified with the massive Z^0-particle. Similarly,

$$\begin{aligned}
&\big[(\partial_\mu - i g_w W_\mu^\alpha \, \tfrac{1}{2}\boldsymbol{\sigma}_\alpha - i g_y B_\mu \, \tfrac{1}{2}\mathbb{1})\mathbb{H}\big]^\dagger \eta^{\mu\nu} \big[(\partial_\nu - i g_w W_\nu^\beta \, \tfrac{1}{2}\boldsymbol{\sigma}_\beta - i g_y B_\nu \, \tfrac{1}{2}\mathbb{1})\mathbb{H}\big] \\
&\quad = \cdots + \left(\tfrac{\mu c}{\lambda \hbar}\right)^2 \Big[-i g_w (W_\mu^1 \, \tfrac{1}{2}\boldsymbol{\sigma}_1 + W_\mu^2 \, \tfrac{1}{2}\boldsymbol{\sigma}_2) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\Big]^\dagger \eta^{\mu\nu} \Big[-i g_w (W_\nu^1 \, \tfrac{1}{2}\boldsymbol{\sigma}_1 + W_\nu^2 \, \tfrac{1}{2}\boldsymbol{\sigma}_2) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\Big] + \cdots \\
&\quad = \cdots + \tfrac{1}{2} g_w^2 \left(\tfrac{\mu c}{\lambda \hbar}\right)^2 \Big[(W_\mu^+ \boldsymbol{\sigma}_- + W_\mu^- \boldsymbol{\sigma}_+) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\Big]^\dagger \eta^{\mu\nu} \Big[(W_\nu^+ \boldsymbol{\sigma}_- + W_\nu^- \boldsymbol{\sigma}_+) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\Big] + \cdots \\
&\quad = \cdots + g_w^2 \left(\tfrac{\mu c}{\sqrt{2}\lambda \hbar}\right)^2 W_\mu^+ \eta^{\mu\nu} W_\nu^- + \cdots ,
\end{aligned} \tag{7.87}$$

where

$$W_\mu^\pm := \tfrac{1}{\sqrt{2}} (W_\mu^1 \pm i W_\mu^2) \qquad \text{and} \qquad \boldsymbol{\sigma}_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{7.88}$$

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This shows that the mass of the $W^{\pm}$-bosons equals $g_w\big(\frac{\mu c}{\sqrt{2}\lambda\hbar}\big)$, and using the definition (7.83) and the results (7.84) and (7.87) we have

$$M_W = \cos(\theta_w)\,M_Z, \tag{7.89a}$$

$$\text{since}\quad \Big[g_w\big(\tfrac{\mu c}{\sqrt{2}\lambda\hbar}\big)\Big] = \frac{g_w}{\sqrt{g_w^2+g_y^2}}\Big[\sqrt{g_w^2+g_y^2}\big(\tfrac{\mu c}{\sqrt{2}\lambda\hbar}\big)\Big]. \tag{7.89b}$$

Conclusion 7.7 *Note that the gauge fields B_μ and W^3_μ couple, respectively, to the corresponding "charges" Y_w and I_w, and that the gauge field A_μ – the photon – couples to the electric charge Q. The linear relation (7.85) then corresponds to the "weak" version of the GNN formula, $Q = I_w + \frac{1}{2}Y_w$, which holds for the values of these charges as they are given in Table 7.1 on p. 275.*

The fermion currents that interact with the gauge fields $W^{\pm}$ remain the same as in (7.76b)–(7.76c), and the A_μ and the Z^0_μ fields respectively interact with the fermion currents:

$$J^\mu_{\text{em}} := [J^\mu_3 + J^\mu_y] = [J^\mu_{\text{em}\,L} + J^\mu_{\text{em}\,R}], \tag{7.90}$$

$$J^\mu_Z := \frac{1}{\cos(\theta_w)}\big[J^\mu_3 - \sin^2(\theta_w)J^\mu_{\text{em}\,L}\big] = \frac{1}{\cos(\theta_w)}\big[\cos^2(\theta_w)J^\mu_3 - \sin^2(\theta_w)J^\mu_y\big], \tag{7.91}$$

where

$$\begin{aligned} J^\mu_{\text{em}\,i} := \sum_{i=L,R}\Big\{ &+\tfrac{2}{3}\Big([\overline{u_i}\,\boldsymbol{\gamma}^\mu\,u_i] + [\overline{c_i}\,\boldsymbol{\gamma}^\mu\,c_i] + [\overline{t_i}\,\boldsymbol{\gamma}^\mu\,t_i]\Big) - \tfrac{1}{3}\Big([\overline{d_i}\,\boldsymbol{\gamma}^\mu\,d_i] + [\overline{s_i}\,\boldsymbol{\gamma}^\mu\,s_i] + [\overline{b_i}\,\boldsymbol{\gamma}^\mu\,b_i]\Big) \\ &-1\Big([\overline{e_i^-}\,\boldsymbol{\gamma}^\mu\,e_i^-] + [\overline{\mu_i^-}\,\boldsymbol{\gamma}^\mu\,\mu_i^-] + [\overline{\tau_i^-}\,\boldsymbol{\gamma}^\mu\,\tau_i^-]\Big)\Big\}. \end{aligned} \tag{7.92}$$

Digression 7.4 That is, we have that

$$\begin{aligned} g_w\,W^3_\mu\,J^\mu_3 + g_y\,B_\mu\,J^\mu_y &= g_w\Big[\sin(\theta_w)A_\mu + \cos(\theta_w)Z_\mu\Big]J^\mu_3 + g_y\Big[\cos(\theta_w)A_\mu - \sin(\theta_w)Z_\mu\Big]J^\mu_y \\ &= \Big[g_w\sin(\theta_w)\,J^\mu_3 + g_y\cos(\theta_w)\,J^\mu_y\Big]A_\mu + \Big[g_w\cos(\theta_w)\,J^\mu_3 - g_y\sin(\theta_w)\,J^\mu_y\Big]Z_\mu, \end{aligned} \tag{7.93a}$$

where, of course, we know that

$$\Big[g_w\sin(\theta_w)\,J^\mu_3 + g_y\cos(\theta_w)\,J^\mu_y\Big] = \Big[\frac{g_w\,g_y}{\sqrt{g_w^2+g_y^2}}J^\mu_3 + \frac{g_y\,g_w}{\sqrt{g_w^2+g_y^2}}J^\mu_y\Big] = g_e\,J^\mu_{\text{em}}. \tag{7.93b}$$

This recovers the original GNN formula (2.30), i.e., (2.44b):

$$J^\mu_{\text{em}} = J^\mu_3 + J^\mu_y, \tag{7.93c}$$

since the $\frac{1}{2}$ factor in the GNN formula (2.30) is built into the definition of J^μ_y (7.76e). Also,

$$\frac{g_w\,g_y}{\sqrt{g_w^2+g_y^2}} \overset{(7.83)}{=} g_w\sin(\theta_w) \overset{(7.83)}{=} g_y\cos(\theta_w) \overset{(7.90)}{=} g_e. \tag{7.93d}$$

In turn,

$$g_w\cos(\theta_w)\,J^\mu_3 - g_y\sin(\theta_w)\,J^\mu_y = g_z\Big[\cos^2(\theta_w)\,J^\mu_3 - \sin^2(\theta_w)\,J^\mu_{\text{em}}\Big] \tag{7.93e}$$

recovers equation (7.91), where

$$g_z = g_w/\cos(\theta_w) = \sqrt{g_w^2+g_y^2}. \tag{7.93f}$$

Note that $g_z = g_w/\cos(\theta_w) > g_w\sin(\theta_w) = g_e$, and $\frac{g_e}{g_z} = \frac{1}{2}\sin(2\theta_w)$.

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Already from the expansions (7.76) and (7.90)–(7.91), we see that the complete Lagrangian density contains very many terms. There exist several different "economical" ways of writing that "pack" of the myriads of summands in different ways. For example, we may write

$$J^\mu_{\rm em} = \sum_n \Big\{ \tfrac{2}{3}[\overline{U}_n \boldsymbol{\gamma}^\mu U_n] - \tfrac{1}{3}[\overline{D}_n \boldsymbol{\gamma}^\mu D_n] - [\bar{\ell}_n \boldsymbol{\gamma}^\mu \ell_n] \Big\}, \qquad n = 1,2,3, \tag{7.94}$$

where $U_1 = u$, $U_2 = c$, $U_3 = t$, $D_1 = d$, $D_2 = s$, $D_3 = b$, $\ell_1 = e^-$, $\ell_2 = \mu^-$ and $\ell_3 = \tau^-$, and omitting the projections to left-handed chirality of a particle indicates the inclusion of both left- and right-handed particles in the sum.

For concrete computations, it is however more convenient to simply list the amplitude contributions of each possible vertex and line, as done in the next section.

7.2.5 Feynman's rules for weak interactions

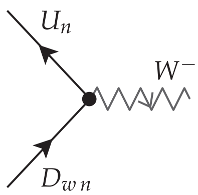
Interactions of the $W^\pm$-bosons with elementary Standard Model fermions are simple as compared to the interactions of the Z^0-boson. It is important, however, to keep in mind that the d_w-, s_w- and b_w-quark states, which interact by weak interactions, are defined as the CKM combinations (2.53)–(2.55):

$$\begin{bmatrix} |d_w\rangle \\ |s_w\rangle \\ |b_w\rangle \end{bmatrix} := \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \begin{bmatrix} |d\rangle \\ |s\rangle \\ |b\rangle \end{bmatrix}, \tag{7.95a}$$

$$= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\, e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\, e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}\, e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\, e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\, e^{i\delta_{13}} & c_{23}c_{13} \end{bmatrix} \begin{bmatrix} |d\rangle \\ |s\rangle \\ |b\rangle \end{bmatrix}, \tag{7.95b}$$

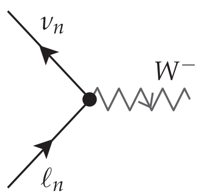
$$\text{where} \qquad c_{ij} := \cos(\theta_{ij}), \qquad s_{ij} := \sin(\theta_{ij}), \quad i,j = 1,2,3 = d,s,b,$$

and where $|d\rangle$, $|s\rangle$ and $|b\rangle$ are the eigenstates of the "free" Hamiltonian, i.e., the states with the well-defined mass.[13] This permits writing



$$\longmapsto \quad -\tfrac{ig_w}{2\sqrt{2}} \boldsymbol{\gamma}^\mu [\mathbb{1} - \widehat{\boldsymbol{\gamma}}]\, \delta^{n'}_n \qquad \begin{bmatrix} U_n \\ D_{w\,n} \end{bmatrix} = \underbrace{\begin{bmatrix} u \\ d_w \end{bmatrix}}_{n=1}, \underbrace{\begin{bmatrix} c \\ s_w \end{bmatrix}}_{n=2}, \underbrace{\begin{bmatrix} t \\ b_w \end{bmatrix}}_{n=3}, \tag{7.96}$$

and



$$\longmapsto \quad -\tfrac{ig_w}{2\sqrt{2}} \boldsymbol{\gamma}^\mu [\mathbb{1} - \widehat{\boldsymbol{\gamma}}]\, \delta^{n'}_n \qquad \begin{bmatrix} \nu_n \\ \ell_n \end{bmatrix} = \underbrace{\begin{bmatrix} \nu_e \\ e^- \end{bmatrix}}_{n=1}, \underbrace{\begin{bmatrix} \nu_\mu \\ \mu^- \end{bmatrix}}_{n=2}, \underbrace{\begin{bmatrix} \nu_\tau \\ \tau^- \end{bmatrix}}_{n=3}, \tag{7.97}$$

which of course implies all processes that may be obtained from $D_{w\,n} \to W^- + U_n$ and $\ell_n \to W^- + \nu_n$ using the crossing symmetry and the principle of detailed balance [☞ Section 2.3.8].

[13] These are the *stationary states*, well known to the Student who successfully covered quantum mechanics, the eigenstates of the "free" Hamiltonian, i.e., the one where the mixing and interaction terms are omitted.

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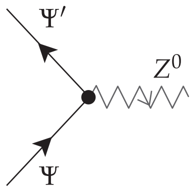
That is, using the CKM definitions (7.95), the interactions with the $W^{\pm}$-bosons do not mix the CKM-redefined "families" of quarks.

Although A_μ is a linear combination of the W^3_μ-field with the "$V-A$" type of interaction with elementary Standard Model fermions and of the B_μ-field that interacts with the fermions of both left- and right-handed chirality, the values of I_w and Y_w in Table 7.1 on p. 275 ensure that the resulting interaction with the A_μ-field is purely of the "V" type. That is, the A_μ-field interacts equally with fermions of both left- and right-handed chirality, and of course, precisely as the photon in electrodynamics [☞ Procedure 5.2 on p. 193].

The neutral Z_μ-field is the complementary linear combination of the neutral W^3_μ- and B_μ-fields, and the interactions of this Z_μ-field with the elementary Standard Model fermions are not as simple as those of the A_μ-field. Following the textbook [243], we may write



$$\longmapsto \quad -\frac{ig_w}{2\sqrt{2}}\boldsymbol{\gamma}^\mu[c_V\mathbb{1} - c_A\widehat{\boldsymbol{\gamma}}]\delta^{\Psi'}_{\Psi} \tag{7.98}$$

$\boldsymbol{\Psi}$	c_V	c_A
ν_n	$\frac{1}{2}$	$\frac{1}{2}$
ℓ_n	$-\frac{1}{2}+2\sin^2(\theta_w)$	$-\frac{1}{2}$
U_n	$\frac{1}{2}-\frac{4}{3}\sin^2(\theta_w)$	$\frac{1}{2}$
D_n	$-\frac{1}{2}+\frac{2}{3}\sin^2(\theta_w)$	$-\frac{1}{2}$

As regards the internal lines that correspond to $W^{\pm}$- and Z^0-boson exchanges, analogously to step 3 in the procedures 5.2 on p. 193, and 6.1 on p. 232, we assign

$$\longmapsto \quad -\frac{i(\eta_{\mu\nu} - q_\mu q_\nu/M^2c^2)}{\mathrm{q}^2 - M^2c^2}, \tag{7.99}$$

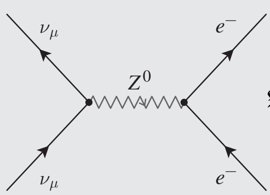
where $M = M_W$ or $M = M_Z$, depending on whether the propagator corresponds to the $W^{\pm}$- or the Z^0-boson exchange. When the exchange energies are sufficiently smaller than Mc^2, we have

$$\lim_{(|\mathrm{q}^2|/M_W^2c^2)\to 0} -\frac{i(\eta_{\mu\nu} - q_\mu q_\nu/M^2c^2)}{\mathrm{q}^2 - M^2c^2} \approx \frac{i\eta_{\mu\nu}}{M^2c^2}, \tag{7.100}$$

which is usually a good first approximation.

In addition to these definitions, the procedure for computing amplitudes of Feynman diagrams is identical to Procedures 5.2 for quantum electrodynamics on p. 193, and 6.1 for quantum chromodynamics on p. 232.

Example 7.3 The elastic collision $\nu_\mu + e^- \to \nu_\mu + e^-$ may occur, to $O(g_w^2)$ order, only mediated by a Z^0-boson exchange:



$$\mathfrak{M} = \frac{g_z{}^2}{8M_Z^2c^2}\big[\overline{\nu}_3\boldsymbol{\gamma}^\mu(\mathbb{1}-\widehat{\boldsymbol{\gamma}})\nu_1\big]\big[\overline{e}_4\boldsymbol{\gamma}^\mu(c_V\mathbb{1}-c_A\widehat{\boldsymbol{\gamma}})e_1\big], \tag{7.101}$$

where $\nu_i := \Psi_{\nu_\mu}(\mathrm{p}_i)$ and $e_i := \Psi_{e^-}(\mathrm{p}_i)$. Computing as in the case (5.131)–(5.140) we obtain

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$$\langle|\mathfrak{M}|^2\rangle = \tfrac{1}{2}\Big(\frac{g_z}{4M_Z c}\Big)^4\Big\{(c_V+c_A)^2(\mathrm{p}_1\cdot\mathrm{p}_2)(\mathrm{p}_3\cdot\mathrm{p}_4) + (c_V-c_A)^2(\mathrm{p}_1\cdot\mathrm{p}_4)(\mathrm{p}_3\cdot\mathrm{p}_2) - m_e^2c^2(c_V^2-c_A^2)(\mathrm{p}_1\cdot\mathrm{p}_3)\Big\}. \tag{7.102}$$

In the CM-system and neglecting the electron mass, $m_e \to 0$, we obtain the simpler relation

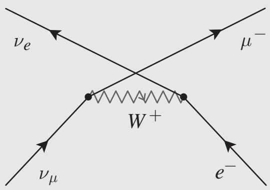
$$\langle|\mathfrak{M}|^2\rangle = \tfrac{1}{2}\Big(\frac{g_z\,E}{M_Z c^2}\Big)^4\Big\{(c_V+c_A)^2 + (c_V-c_A)^2\cos^4(\tfrac{1}{2}\theta)\Big\}, \tag{7.103}$$

where E is the energy of the electron (as well as the neutrino) in the CM-system, and θ is the electron deflection angle. Then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\Big(\frac{\hbar c}{\pi}\Big)^2\Big(\frac{g_z}{4M_Z c^2}\Big)^4 E^2\,\Big\{(c_V+c_A)^2 + (c_V-c_A)^2\cos^4(\tfrac{1}{2}\theta)\Big\}, \tag{7.104}$$

$$\begin{aligned}\sigma &= \frac{2}{3\pi}(\hbar c)^2\Big(\frac{g_z}{2M_Z c^2}\Big)^4 E^2\,(c_V^2+c_A^2+c_V\,c_A)\\ &= \frac{2}{\pi}(\hbar c)^2\Big(\frac{g_z}{2M_Z c^2}\Big)^4 E^2\,\Big(\tfrac{1}{4}-\sin^2(\theta_w)+\tfrac{4}{3}\sin^4(\theta_w)\Big).\end{aligned} \tag{7.105}$$

Comparing with the similar process $\nu_\mu + e^- \to \nu_e + \mu^-$ that involves the exchange of a W-boson:



$$\sigma = \frac{1}{8\pi}\Big[\Big(g_w M_W c^2\Big)^2\hbar c E\Big]^2\Big[1-\Big(\frac{m_\mu c^2}{2E}\Big)^2\Big]^2 \tag{7.106}$$

and at energies $E \gg m_\mu c^2$, we have (using $\theta_w = 28.75°$, from the ratio of the measured masses M_W/M_Z)

$$\frac{\sigma(\nu_\mu + e^- \to \nu_\mu + e^-)}{\sigma(\nu_\mu + e^- \to \nu_e + \mu^-)} \approx \tfrac{1}{4} - \sin^2(\theta_w) + \tfrac{4}{3}\sin^4(\theta_w) = 0.0900. \tag{7.107}$$

This agrees with the experimental value $0.11 \pm 10\%$ fairly well.

7.2.6 Exercises for Section 7.2

✎ ***7.2.1*** *Following Example 7.1, show that the sum of all amplitudes for both diagrams of the type (7.69) but with two (three) W^3-particle and one (no) photon also vanishes.*

✎ ***7.2.2*** *For the potential process $\gamma \to 2\gamma$ described by an appropriate algebraic sum of diagrams of the type (7.69) but with a photon in place of W^3, show that the symmetrization of the outgoing photons (as bosons) guarantees that the sum of the contributions of these Feynman diagrams vanishes.*

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✎ ***7.2.3*** *Following Example 7.2, show that the sum of all amplitudes for both diagrams of the type (7.72) but with n B-particles and $(3-n)$ photons also vanishes, for every $n = 0, 1, 2, 3$.*

✎ ***7.2.4*** *Complete the computation in Example 7.3.*

7.3 The Standard Model

The elaborate structure called the *Standard Model* of elementary particle physics has the following components:

1. the elementary spin-$\frac{1}{2}$ fermions in Table 7.1 on p. 275, and the data (2.44);
2. electromagnetic interactions with the $U(1)_Q$ gauge symmetry [☞ Section 5.3];
3. chromodynamic interactions with the $SU(3)_c$ gauge symmetry [☞ Section 6.1];
4. the asymmetric treatment of particles with left- and right-handed chirality [☞ discussion around the expressions (5.57)–(5.62), then Sections 7.2.1 and 7.2.4];
5. the GIM mechanism, anomaly cancellation and generalization of the GIM mechanism with the Cabibbo–Kobayashi–Maskawa quark mixing [☞ Section 7.2.2];
6. the $SU(2)_w \times U(1)_y$ gauge symmetry of the electroweak interactions, in the symmetric phase;
7. the spontaneous $SU(2)_w \times U(1)_y \rightarrow U(1)_Q$ gauge symmetry breaking of electroweak interactions in the Higgs phase [☞ Sections 7.1 and 7.2.4];
8. the very intricate and detailed structure of fermion masses [☞ Tables 4.1 on p. 152, and C.2 on p. 526].

This structure is presented in an extremely short and ultra-compact way in Table 2.3 on p. 67. However, the incremental development of the material presented in sections from Chapter 2 up to now clearly indicates that this short compactness is merely a convenient business-card to an otherwise technically very demanding and intricate Standard Model. This demanding nature should not be surprising, since this model successfully describes practically all known phenomena not only at the fundamental level of quarks and leptons, but also at the level of hadronic bound states [☞ Section 2.4.1 and Conclusion 2.4 on p. 71].

Undoubtedly, the most complex parts of the Standard Model pertain to the aspects of weak interactions, which are roughly presented in the foregoing part of this chapter. It remains to discuss (**1**) the general mechanism in the Standard Model by which fermions in Table 7.1 on p. 275 acquire a mass, and (**2**) neutrino mixing.

7.3.1 Fermion masses

The argument at the very beginning of Chapter 7 shows that the gauge bosons are massless by construction – except, as we have seen here, those corresponding to symmetries spontaneously broken via the Higgs mechanism [☞ Section 7.1.3]. The mass of these gauge bosons stems from the interaction with the Higgs field [☞ expressions (7.84) and (7.87)] and owing to the shift $\mathbb{H} \rightarrow \widetilde{\mathbb{H}} + \langle\mathbb{H}\rangle$, which is dictated by the fact that the "flipped" sign of the quadratic term in the Lagrangian density (7.78) puts the minimum of the potential energy at one of the points with $\mathbb{H}^\dagger\mathbb{H} = \left(\frac{\mu c}{\lambda\hbar}\right)^2 > 0$, so that the vacuum expectation value of the two-component Higgs field is not zero, $\langle\mathbb{H}\rangle \neq 0$.

Similarly, one expects that the fermion masses also stem from the Higgs field shift $\mathbb{H} \rightarrow \widetilde{\mathbb{H}} + \langle\mathbb{H}\rangle$. The expression (7.47) shows that a typical term in the Lagrangian density that provides the fermion fields with a mass must be of the form (with the customary notation $\Psi_+ = \Psi_L$ and $\Psi_- = \Psi_R$)

$$\overline{\Psi_L}\,\mathbb{H}\,\Psi_R \quad \text{and} \quad \overline{\Psi_R}\,\mathbb{H}\,\Psi_L. \tag{7.108}$$

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Such terms are possible precisely because $\mathbb{H}$ is an $SU(2)_w$-doublet, just as are the wave-functions for all fermions of left-handed chirality, whereas all right-handed fermions are invariant under $SU(2)_w$ transformations. Therefore, terms such as[14]

$$\begin{aligned} h_e\,\overline{e_R^-}\,\mathbb{H}^\dagger\left[\begin{smallmatrix}\nu_e\\ e^-\end{smallmatrix}\right]_L + h.c. &= h_e\,\overline{e_R^-}\,{}^{[H_1^*\ H_2^*]}\left[\begin{smallmatrix}\nu_e\\ e^-\end{smallmatrix}\right]_L + h.c. = h_e\,\overline{e_R^-}\big(H_1^*\nu_{eL} + H_2^* e_L^-\big) + h.c.\\ &= \Re e\big(h_e\langle H_2\rangle^*\big)\,\big(\overline{e_R^-}\,e_L^- + \overline{e_L^-}\,e_R^-\big) + \cdots \end{aligned} \tag{7.109}$$

are $SU(2)_w \times U(1)_y$-invariant and produce the electron mass, $m_e = \Re e\,\big(h_e\langle H_2\rangle\big)/c^2$. Similarly, for d-quarks one has

$$\begin{aligned} h_d\,\overline{d_R}\,\mathbb{H}^\dagger\left[\begin{smallmatrix}u\\ d\end{smallmatrix}\right]_L + h.c. &= h_d\,\overline{d_R}\,{}^{[H_1^*\ H_2^*]}\left[\begin{smallmatrix}u\\ d\end{smallmatrix}\right]_L + h.c. = h_d\,\overline{d_R}\big(H_1^* u_L + H_2^* d_L\big) + h.c.\\ &= \Re e\big(h_d\langle H_2\rangle^*\big)\,\big(\overline{d_R}\,d_L + \overline{d_L}\,d_R\big) + \cdots, \end{aligned} \tag{7.110}$$

which are also $SU(2)_w \times U(1)_y$-invariant and produce $m_d = \Re e\,\big(h_d\langle H_2\rangle\big)/c^2$, the d-quark mass. For u-quarks, an additional definition [☞ discussion of the relation (A.49)] is needed:

$$C : \mathbb{H} = \begin{bmatrix}H_1\\ H_2\end{bmatrix} \longmapsto \mathbb{H}^c := -\boldsymbol{\varepsilon}\,\mathbb{H}^* = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}H_1^*\\ H_2^*\end{bmatrix} = \begin{bmatrix}-H_2^*\\ H_1^*\end{bmatrix}, \tag{7.111}$$

which transforms, under $SU(2)_w$, the same as $\mathbb{H}$. We can therefore add to the Lagrangian density also the terms

$$\begin{aligned} -h_u\,\overline{u_R}\,(\mathbb{H}^c)^\dagger\left[\begin{smallmatrix}u\\ d\end{smallmatrix}\right]_L + h.c. &= -h_u\,\overline{u_R}\,{}^{[-H_2\ H_1]}\left[\begin{smallmatrix}u\\ d\end{smallmatrix}\right]_L + h.c. = -h_u\,\overline{u_R}\big(-H_2 u_L + H_1 d_L\big) + h.c.\\ &= \Re e\big(h_u\langle H_2\rangle\big)\,\big(\overline{u_R}\,u_L + \overline{u_L}\,u_R\big) + \cdots, \end{aligned} \tag{7.112}$$

which are also $SU(2)_w \times U(1)_y$-invariant and produce $m_u = \Re e\,\big(h_u\langle H_2\rangle\big)/c^2$, the u-quark mass.

The structure of the Standard Model neither requires nor prohibits adding the neutrino of right-handed chirality, which is noted in Table 7.1 on p. 275: ν_{iR} (with $i = e, \mu, \tau$) are included in the table but are separated from the other fermions. If one includes these right-handed neutrinos, one can include in the Lagrange density also the terms

$$\begin{aligned} -h_\nu\,\overline{\nu_{eR}}\,(\mathbb{H}^c)^\dagger\left[\begin{smallmatrix}\nu_e\\ e^-\end{smallmatrix}\right]_L + h.c. &= -h_\nu\,\overline{\nu_{eR}}\,{}^{[-H_2\ H_1]}\left[\begin{smallmatrix}\nu_e\\ e^-\end{smallmatrix}\right]_L + = -h_\nu\,\overline{\nu_{eR}}\big(-H_2\nu_{eL} + H_1 e_L^-\big) + h.c.\\ &= \Re e\big(h_\nu\langle H_2\rangle\big)\,\big(\overline{\nu_{eR}}\,\nu_{eL} + \overline{\nu_{eL}}\,\nu_{eR}\big) + \cdots, \end{aligned} \tag{7.113}$$

which are also $SU(2)_w \times U(1)_y$-invariant and produce $m_\nu = \hbar\,\Re e\,\big(h_\nu\langle H_2\rangle\big)/c$, the neutrino mass.

The quantities defined by the relations (7.109), (7.110), (7.112) and (7.113) are the so-called Dirac masses, since the variation of the Lagrangian density by fermion fields produces the Dirac equation (5.34), with the indicated masses. In addition, terms that were omitted in the expressions (7.109), (7.110), (7.112) and (7.113) are of the general form

$$h_i\,\Re e\,(H_2)\,\big(\overline{\Psi_{iR}}\,\Psi_{iL} + \overline{\Psi_{iL}}\,\Psi_{iR}\big), \tag{7.114}$$

which define interactions of the Higgs particle, $\Re e(H_2)$, with the Standard Model fermions. The remaining components of the complex Higgs doublet, $H_1 = H^+$, $H_1^* = H^-$ and $\Im m(H_2)$ have become the longitudinal components of the $W^\pm$- and the Z^0-bosons; see Section 7.1.3, Conclusion 7.5 on p. 265, and equation (7.49).

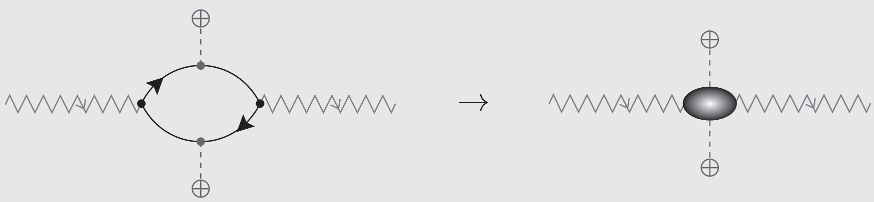
The so-obtained fermion masses (7.109), (7.110), (7.112) and (7.113) as well as the masses of the Z^0- and the $W^\pm$-bosons (7.82)–(7.87) are all proportional to the mass $\Re e\,\big(\langle H_2\rangle\big)/c^2$. The

[14] The abbreviation "$+h.c.$" is standard for adding the Hermitian conjugate terms.

Yukawa parameters h_e, h_d, h_u, h_ν (and similarly for the remaining two families) are, however, completely arbitrary parameters of the Standard Model and, besides in the fermion masses, appear only in the terms of the type (7.114) that describe the Yukawa interactions of the fermions with the Higgs particle. This then links the intensity of this interaction with the fermion masses. Of course, until the details of the interactions of the Higgs particle with the Standard Model fermions are measured sufficiently precisely, the choice of the parameters h_e, h_d, h_u, h_ν, etc., is determined only in terms of the measured fermion masses – except for the neutrinos; see the next section.

Since the Standard Model fermion masses [☞ Tables 4.1 on p. 152, and C.2 on p. 526] differ significantly from the masses of the $W^\pm$- and the Z^0-bosons, it follows that the parameters h_e, h_d, h_u, h_ν, etc., are quite far from numbers of order 1, and the structure represented by this list of parameters ought to be explained somehow. However, that is a task beyond the Standard Model☝.

Digression 7.5 Let us mention a non-standard version of the Standard Model [169], where one introduces a Higgs field that is $SU(2)_w \times U(1)_y$-invariant, but has Yukawa interactions $(\overline{\Psi}\,\widetilde{H}\,\Psi)$ with the Standard Model fermions. Shifting $\widetilde{H} \to \widetilde{H}' + \langle\widetilde{H}\rangle$, the fermions acquire a mass just as by the previously described standard method (7.114). As $SU(2)_w \times U(1)_y$ gauge bosons do not interact directly with this Higgs boson, their masses stem from perturbative corrections of the type



(7.115)

where the shaded oval in the right-hand diagram represents the resulting effective (indirect) interaction between $SU(2)_w \times U(1)_y$ gauge bosons and the Higgs field $\widetilde{H}$ that sinks into the vacuum, i.e., $\langle\widetilde{H}\rangle \neq 0$; compare with the illustration (7.43). Effectively, the so-obtained mass for the gauge bosons produces a model that differs from the Standard Model results only at energies significantly larger than $m_W, m_Z \sim 100\,\text{GeV}/c^2$. Since these masses are radiatively induced, the mass of the Higgs particle itself is expected to be larger than $100\,\text{GeV}/c^2$ – in agreement with the recent LHC results at CERN [25, 109, 293]. Only detailed measurements of the interactions of the Higgs particle with the other Standard Model particles can distinguish this possibility from the original version, or other generalizations and extensions☝.

7.3.2 Neutrino mixing

It was noted in the early 1990s that amongst the neutrinos that arrive at the Earth's surface there are fewer muon neutrinos, ν_μ, than expected. That is, neutrinos are produced in the atmosphere mainly through the decay of pions and muons:

$$\pi^+ \to \mu^+ + \nu_\mu, \quad \to (e^+ + \nu_e + \overline{\nu}_\mu) + \nu_\mu, \tag{7.116}$$

$$\pi^- \to \mu^- + \overline{\nu}_\mu, \quad \to (e^- + \overline{\nu}_e + \nu_\mu) + \overline{\nu}_\mu. \tag{7.117}$$

Evidently, one expects about twice as many muon (anti)neutrinos than electron (anti)neutrinos to reach the Earth's surface. However, experimental results of the KamiokaNDE installation showed

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that the atmospheric muon-to-electron (anti)neutrino number ratio depends on the direction of their arrival: Among the (anti)neutrinos arriving at the Earth's surface well-nigh vertically, the ratio was really close to 2:1. However, amongst the neutrinos arriving at a large angle from the vertical, this ratio is closer to 1:1. This indicates that the muon (anti)neutrinos somehow vanish whilst passing through the atmosphere, much faster than the electron (anti)neutrinos, and certainly much faster than would be expected from the known fact that the effective cross-section of the interaction between neutrinos and other matter is extremely small. These experimental results were later confirmed in the Super-KamiokaNDE installation.

In turn, the mechanisms that produce the enormous energy of a star such as our Sun had been subject to research from the beginning of the nineteenth century, when Lord Rayleigh showed that – with the *then* generally accepted assumption that the Sun's energy stems from gravitational contraction – the Sun could not be as old as the geological finds (of Earth) indicate and as needed for the process of evolution. However, Becquerel discovered radioactivity in 1896, and by about 1920 the atomic weights were measured sufficiently precisely to make it possible for Arthur Eddington to notice that four hydrogen atoms are a little heavier than the helium atom. According to Einstein's relation $E_0 = mc^2$, the difference $(4m_{\rm H} - m_{\rm He})$ indicates that fusing four hydrogen atoms into an atom of helium should release energy.

In the early 1930s Chadwick discovered the neutron, Pauli postulated the existence of the neutrino and Fermi described the basic process of weak nuclear interaction, $n^0 \to p^+ + e^- + \bar{\nu}_e$. This opened the possibility for a realistic description of the nuclear processes that produce most of the radiation energy of the Sun. By 1938, Hans Bethe had worked out the details of the so-called carbon cycle, where the process of fusion is catalyzed by carbon, nitrogen and oxygen, and which is the dominant process in very large stars. In the Sun, which is a relatively smaller and lighter star, the basic mechanism is the so-called pp-process:

1. $$p^+ + p^+ \to d^+ + e^+ + \nu_e, \qquad \text{(continuous spectrum)} \tag{7.118a}$$
$$p^+ + p^+ + e^- \to d^+ + \nu_e, \qquad \text{(discrete spectrum)} \tag{7.118b}$$

2. $$d^+ + p^+ \to {}^3\text{He}^{++} + \gamma, \tag{7.118c}$$

3. $${}^3\text{He}^{++} + p^+ \to \alpha^{++} + e^+ + \nu_e, \qquad \text{(continuous spectrum)} \tag{7.118d}$$
$${}^3\text{He}^{++} + {}^3\text{He}^{++} \to \alpha^{++} + p^+ + p^+, \tag{7.118e}$$
$${}^3\text{He}^{++} + \alpha^{++} \to {}^7\text{Be}^{4+} + \gamma, \tag{7.118f}$$

4. $${}^7\text{Be}^{4+} + e^- \to {}^7\text{Li}^{3+} + \nu_e, \qquad \text{(discrete spectrum)} \tag{7.118g}$$
$${}^7\text{Li}^{3+} + p^+ \to \alpha^{++} + \alpha^{++}, \tag{7.118h}$$
$${}^7\text{Be}^{4+} + p^+ \to {}^8\text{B}^{5+} + \gamma, \tag{7.118i}$$
$${}^8\text{B}^{5+} \to ({}^8\text{Be}^{4+})^* + e^+ + \nu_e, \qquad \text{(continuous spectrum)} \tag{7.118j}$$
$${}^8\text{B}^{5+} + e^- \to ({}^8\text{Be}^{4+})^* + \nu_e, \qquad \text{(discrete spectrum)} \tag{7.118k}$$
$$({}^8\text{Be}^{4+})^* \to \alpha^{++} + \alpha^{++}. \tag{7.118l}$$

The processes (7.118a), (7.118d) and (7.118j) produce neutrinos with a continuous distribution of energies, while the neutrinos produced in the processes (7.118b), (7.118g) and (7.118k) have a fixed energy [☞ Section 3.2: when a collision or a decay produces only two particles, their energies are completely determined]. Most of the neutrinos are created in the process (7.118a) as the concentration of input "ingredients" (protons) is much larger than the concentration of input "ingredients" in the other processes. However, the energy of the so-produced neutrinos is no

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larger than about 400 keV, which makes their detection harder. In turn, neutrinos produced in the processes (7.118d) and (7.118j) have energies reaching over 1 MeV, where the detectors are far more sensitive.

John Bahcall's additional and detailed computations of the resulting distribution and total neutrino flux were finally verified in 1968 [355, 369]: Ray Davis's group monitored a giant tank (4,850 feet underground, in the *Homestake* gold mine in South Dakota) containing a dry-cleaning fluid with a large content of chlorine, seeking the results of the reaction

$$\nu_e + n^0 \to p^+ + e^-, \qquad \text{by way of} \qquad \nu_e + {}^{37}\text{Cl} \to {}^{37}\text{Ar} + e^-. \tag{7.119}$$

The detection of argon-37 indicated that only about one-third of electron neutrinos that the Sun emits actually arrive at the surface of the Earth. This discrepancy in the number of solar electron neutrinos was dubbed the "neutrino problem."

A little earlier, in 1967, Bruno Pontecorvo proposed (following up on a decade-earlier proposal) a simple solution of the neutrino problem, by postulating that the electron neutrinos produced in the Sun at least partially transform during their flight to the Earth into another type (muon and tau) of neutrinos or even antineutrinos. As the Davis experiment could detect only electron neutrinos, the transformed neutrinos would show up as "missed." This mechanism is, in general, called "neutrino oscillation," as it is based on an essentially simple quantum-mechanical effect.

To wit, with two eigenstates of the Hamiltonian

$$H|1\rangle = E_1|1\rangle \qquad \text{and} \qquad H|2\rangle = E_2|2\rangle, \tag{7.120}$$

the evolution of a linear combination of these two stationary states is described as

$$|\text{“}1{+}2\text{”}; t\rangle = C_1 e^{-iE_1 t/\hbar}|1\rangle + C_2 e^{-iE_2 t/\hbar}|2\rangle, \tag{7.121}$$

where the constants C_1, C_2 are determined from the initial condition. The probability that this linear combination is after the amount of time t in the state $\cos(\alpha)|1\rangle + \sin(\alpha)|2\rangle$ equals

$$\begin{aligned} P_\alpha &:= \Big|\big[\cos(\alpha)\langle 1| + \sin(\alpha)\langle 2|\big]|\text{“}1{+}2\text{”}; t\rangle\Big|^2 \\ &= |C_1|^2\cos^2(\alpha) + |C_2|^2\sin^2(\alpha) + \sin(2\alpha)\,\Re e\big[C_1 C_2^* e^{-i(E_1-E_2)t/\hbar}\big]. \end{aligned} \tag{7.122}$$

If the system was originally in the "opposite" linear combination, $\cos(\alpha)|2\rangle - \sin(\alpha)|1\rangle$ so $C_1 = -\sin(\alpha)$ and $C_2 = \cos(\alpha)$, we have that

$$P_{|\alpha+\frac{\pi}{2}\rangle \to |\alpha\rangle} = \sin^2(2\alpha)\sin^2(\tfrac{1}{2}\omega_{12}t), \qquad \omega_{12} := \tfrac{E_1-E_2}{\hbar}. \tag{7.123}$$

Therefore, the system oscillates:

$$\big(|\alpha{+}\tfrac{\pi}{2}\rangle = -\sin(\alpha)|1\rangle + \cos(\alpha)|2\rangle\big) \quad\longleftrightarrow\quad \big(|\alpha\rangle = \cos(\alpha)|1\rangle + \sin(\alpha)|2\rangle\big) \tag{7.124}$$

under the conditions that

1. the two stationary states are not degenerate: $E_1 \neq E_2$, so that $\omega_{12} \neq 0$, and
2. the system is initially in a nontrivial ($\alpha \neq 0$) linear combination of the two stationary states.

It is evident that the conceptually same phenomenon occurs in a system with three non-degenerate stationary states, but the oscillations are more complicated.

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For relativistic particles, we have that (using that $\vec{p}_1 = \vec{p}_2 = \vec{p}$)

$$E_1 - E_2 = \sqrt{|\vec{p}|^2 c^2 + {m_1}^2 c^4} - \sqrt{|\vec{p}|^2 c^2 + {m_2}^2 c^4} \approx |\vec{p}| c \left[\frac{1}{2} \frac{({m_1}^2 - {m_2}^2) c^2}{|\vec{p}|^2} + \cdots \right]$$
$$\approx \frac{({m_1}^2 - {m_2}^2) c^3}{2|\vec{p}|} + \cdots \approx \frac{({m_1}^2 - {m_2}^2) c^4}{2\overline{E}}, \tag{7.125}$$

where $\overline{E}$ is the average value of energies E_1 and E_2.

Just as $|d\rangle, |s\rangle$ and $|b\rangle$ – eigenstates of the free, kinetic Hamiltonian and thus characterized by their well-defined masses – are not the eigenstates of weak interactions, suppose that the electron-, muon- and tau-neutrinos (identified as the eigenstates of weak interactions) are not the eigenstates of the free Hamiltonian, $|\nu_i\rangle$. Then,

$$|\nu_e\rangle = -\sin(\theta_\nu)|\nu_1\rangle + \cos(\theta_\nu)|\nu_2\rangle, \qquad |\nu_\mu\rangle = \cos(\theta_\nu)|\nu_1\rangle + \sin(\theta_\nu)|\nu_2\rangle, \tag{7.126}$$

neglecting the third family. From this,

$$P_{\nu_e \to \nu_\mu} \approx \sin^2(2\theta_\nu) \sin^2\Big(\tfrac{({m_1}^2 - {m_2}^2) c^4}{4\overline{E}\hbar} t \Big) = \sin^2(2\theta_\nu) \sin^2\Big(\tfrac{({m_1}^2 - {m_2}^2) c^3}{4\overline{E}\hbar} z \Big), \tag{7.127}$$

where $z = ct$ is approximately equal to the distance that neutrinos traverse (the masses m_1, m_2 are very small, so the neutrinos propagate with speeds that are close to c). This shows that after a traversed distance of

$$(2n+1)\, z_*, \qquad \text{where} \quad z_* = \frac{2\pi \overline{E}\hbar}{({m_1}^2 - {m_2}^2) c^3}, \quad n = 0, 1, 2, \ldots \tag{7.128}$$

all electron neutrinos have converted into muon neutrinos, and at distances $2n\, z_*$ all electron neutrinos have turned back into their initial state. In other words, $2z_*$ is the wavelength of the simple oscillation between two types of neutrinos.

Of course, there do exist *three* types of neutrinos, and the oscillations are more complicated. Besides, traversing matter additionally changes the parameters of neutrino mixing. This was first described by Lincoln Wolfenstein, Stanislav Mikheyev and Alexei Smirnov, and this additional effect is named after then, the MSW effect. In 2001, the first results were published from Super-KamiokaNDE, which uses water in the detector, and which can detect all three types of neutrinos, albeit with different levels of efficiency. Independently, in the same year, the first results were published also from SNO (Sudbury Neutrino Observatory), which uses heavy water in the detector. Because of the presence of the neutron in the deuterium nuclei, SNO detects two additional processes with neutrinos that are not detected in Super-KamiokaNDE.

By April 2002, the combination of these experimental results unambiguously showed that the neutrino oscillations exist and solved the so-called “neutrino problem,” showing clearly that the neutrino stationary states, ν_1, ν_2, ν_3 have nonzero and different masses, and that the weak interaction eigenstates, the particles ν_e, ν_μ, ν_τ, are linear combinations of the stationary states ν_1, ν_2, ν_3. Experiments also give the *difference* of the squares of masses:

$$\triangle_{12}(m_\nu^2) \approx 8 \times 10^{-5}\, (\mathrm{eV}/c^2)^2, \qquad \triangle_{23}(m_\nu^2) \approx 3 \times 10^{-3}\, (\mathrm{eV}/c^2)^2, \tag{7.129}$$

but cannot show if the pattern of masses is two similar masses significantly smaller than the third one, or two similar masses significantly larger than the third one [☞ book [369], and [370] for a more recent and thorough review].

Finally, Section 2.3.10 discussed the research of R. Davis and D. S. Harmer, who concluded that ν_e and $\overline{\nu}_e$ are distinct elementary particles. However, a detailed analysis of the non-occurring

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process (2.24), i.e., $\overline{\nu}_e + n^0 \not\to p^+ + e^-$, shows that it may well be possible for ν_e and $\overline{\nu}_e$ to be the same particle – that the neutrino is its own antiparticle – but that this process (2.24) is forbidden by helicity/chirality: whereas $\nu_e + n^0 \to p^+ + e^-$ could happen with a *left-handed* neutrino, the absence of a left-handed *anti*neutrino would then prevent the process (2.24).

A more direct consequence of the logically possibility that $\nu_e = \overline{\nu}_e$ would be the neutrino-less double β-decay:

$$2d \to 2u + 2e^- + (2\overline{\nu}_e \to \overline{\nu}_e + \nu_e \to 0) \to 2u + 2e^-, \tag{7.130}$$

which has never been observed☝. Nevertheless, the logical possibility that $\nu_e = \overline{\nu}_e$ still attracts considerable interest as it is necessary for the so-called *see-saw mechanism*. This mechanism uses the fact that the (left-handed) neutrinos are the $I_w = +\frac{1}{2}$ components of the lepton doublets that interact by means of weak interactions, and so also with the doublets of Higgs fields. In turn, one may always add to the Standard Model the right-handed neutrino, which has no weak charge (isospin):

$$I_w(\nu_{e\,L}) = +\tfrac{1}{2}, \qquad I_w(H_2) = -\tfrac{1}{2}, \qquad I_w(\nu_{e\,R}) = 0. \tag{7.131}$$

The Standard Model Lagrangian density may then contain the terms[15]

$$m_\nu\left(\overline{\nu_{e\,R}}\,\nu_{e\,L} + \overline{\nu_{e\,L}}\,\nu_{e\,R}\right) + \tfrac{1}{2}M_\nu\,\overline{\nu_{e\,R}}\,\nu^c_{e\,R}, \tag{7.132a}$$

where m is the mass that stems from the (so-called Yukawa) interaction term (7.113), where $H_2 \to \widetilde{H}_2 + \langle H_2\rangle$ produces $m_\nu = h_\nu\langle H_2\rangle$. In the basis $(\nu_{e\,L}, \nu_{e\,R})$, the Lagrangian terms (7.132a) produce the mass matrix

$$\begin{bmatrix} 0 & m_\nu \\ m_\nu & M_\nu \end{bmatrix} \quad \overset{\text{diag.}}{\longmapsto} \quad m_\pm = \frac{1}{2}\left|M_\nu \pm \sqrt{4m_\nu^2 + M_\nu^2}\right| \approx \begin{cases} M_\nu, \\ m_\nu^2/M_\nu. \end{cases} \tag{7.132b}$$

One expects that $m_\nu \sim 10^2\,\text{GeV}/c^2$, while experiments indicate that the neutrino masses are $m_{\nu,\text{exp}} < 2\,\text{eV}$ [293]. Therefore, $M_\nu \sim (m_\nu^2/m_\pm) \gtrsim 10^{13}\,\text{GeV}/c^2$.

A mass parameter such as $M_\nu \gtrsim 10^{13}\,\text{GeV}/c^2$ must stem from effects that are beyond the Glashow–Weinberg–Salam theory of the electroweak interactions,[16] and also beyond the Standard Model, but are probably related to the so-called Grand Unification or some other phenomena expected to occur at such high characteristic energies.

It is worth mentioning that in 1962 Ziro Maki, Masami Nakagawa and Shoichi Sakata proposed a general neutrino mixing, akin to the CKM mixing of the "lower" quarks and extending a similar proposal by Bruno Pontecorvo [353]. The analogous general neutrino mixing matrix is thus called the PMNS-matrix [369, 370].

7.3.3 The Standard Model, summarized

We are finally ready to summarize the Lagrangian density for the Standard Model, using the list on p. 282:

[15] For any fermion, $\overline{\Psi}\Psi^c$ has twice every charge of $\overline{\Psi}$, i.e., Ψ^c, a Majorana mass term $M\,\overline{\Psi}\Psi^c$ requires a mass parameter M that has twice every charge of Ψ. All gauge symmetries corresponding to these charges must therefore be broken; either explicitly by introducing such a term by hand, or spontaneously if the mass parameter is the vacuum expectation of a scalar field. The Majorana mass term $\frac{1}{2}M_\nu\,\overline{\nu_{e\,R}}\,\nu^c_{e\,R}$ is possible exclusively because all the charges of a right-handed neutrino vanish, so that $\nu^c_{e\,R} := \mathsf{C}(\nu_{e\,R})$ transforms identically to $\nu_{e\,L}$ with respect to all *unbroken* Standard Model symmetries.

[16] Since the mass scale of the GWS-model is of the order of magnitude of $W^\pm$- and Z^0-bosons, $\sim 10^2\,\text{GeV}/c^2$, a mass parameter of the order of magnitude $\sim 10^{13}\,\text{GeV}/c^2$ would require a numerical coefficient of the order $\sim 10^{11}$, the kind of which never occurs in typical computations. That is, although the Standard Model contains dimensionless coefficients such as h_e, h_d, h_u, h_ν in the expressions (7.109), (7.110), (7.112) and (7.113), all these dimensionless coefficients are smaller than 1 and there is no systematic computation where a combination of them would emerge to be of the order $\sim 10^{11}$. This situation here is very similar to the discussion of the hydrogen atom in Sections 1.2.5 and 4.1, where *negative powers* of the fine structure constant (and so also of dimensionless coefficients larger than 1) do not occur.

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$$\mathscr{L}_{\mathrm{SM}} = \mathscr{L}_{\mathrm{F}} + \mathscr{L}_{\mathrm{G}} + \mathscr{L}_{\mathrm{H}} + \mathscr{L}_{\mathrm{Y}} + \mathscr{L}_{M_\nu}, \tag{7.133a}$$

$$\mathscr{L}_{\mathrm{F}} = i\hbar c \sum_n \Big[\overline{\Psi_{nL}} \not{D} \Psi_{nL} + \overline{\Psi_{nR}} \not{D} \Psi_{nR}\Big], \tag{7.133b}$$

$$D_\mu := \partial_\mu + \tfrac{i g_c}{\hbar c} G_\mu^a Q_{c\,a} + \tfrac{i g_w}{\hbar c} W_\mu^\alpha \mathbb{V}_w^\dagger I_{w\,\alpha} \mathbb{V}_w + \tfrac{i g_y}{\hbar c} B_\mu Y_w, \tag{7.133c}$$

$$\mathscr{L}_{\mathrm{G}} = -\tfrac{1}{4} \sum_{a=1}^{8} G_{\mu\nu}^a G^{a\,\mu\nu} - \tfrac{1}{4} \sum_{a=1}^{3} W_{\mu\nu}^a W^{a\,\mu\nu} - \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu}, \tag{7.133d}$$

$$\mathscr{L}_{\mathrm{H}} = \big\| [\partial_\mu - \tfrac{i g_w}{\hbar c} W_\mu^a \sigma_a - \tfrac{i g_y}{\hbar c} \mathbb{1}] \mathbb{H} \big\|_\eta^2 - \tfrac{\varkappa}{2} \big(\tfrac{\mu c}{\hbar}\big)^2 (\mathbb{H}^\dagger \mathbb{H}) - \tfrac{1}{4} \lambda (\mathbb{H}^\dagger \mathbb{H})^2, \tag{7.133e}$$

$$\mathscr{L}_{\mathrm{Y}} = \sum_n \big(h_n \overline{\Psi_{nR}} (\mathbb{H}^\dagger \Psi_{nL}) + h_n^* (\overline{\Psi_{nL}} \mathbb{H}) \Psi_{nR} \big), \tag{7.133f}$$

$$\mathscr{L}_{M_\nu} = \tfrac{1}{2} M_\nu c^2\, \overline{\nu_{e\,R}}\, \nu_{e\,R}^c. \tag{7.133g}$$

Here, the summands in the Lagrangian density (7.133d) were written akin to (5.118) and (6.23), but the gauge field tensors were denoted

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - \tfrac{g_c}{\hbar c} f^a{}_{bc} G_\mu^b G_\nu^c, \qquad a, b, c = 1, 2, \dots, 8, \tag{7.134}$$

for the $SU(3)_c$ gluon field,

$$W_{\mu\nu}^\alpha = \partial_\mu W_\nu^\alpha - \partial_\nu W_\mu^\alpha - \tfrac{g_w}{\hbar c} \epsilon^\alpha{}_{\beta\gamma} W_\mu^\beta W_\nu^\gamma, \qquad W^\pm = W^1 \pm i W^2, \;\; \alpha, \beta, \gamma = 1, 2, 3, \tag{7.135}$$

for the $SU(2)_w$ gauge field, and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{7.136}$$

for the $U(1)_y$ gauge field. As customary, convention-dependent coefficients such as $4\pi\epsilon_0$ for electromagnetism have been absorbed in the definition of the gauge field tensors and are not explicitly shown. In the expressions (7.133b), the derivative D_μ (7.133c) is covariant with respect to the *complete* $SU(3)_c \times SU(2)_w \times U(1)_y$ Standard Model gauge group action:

1. The operator Q_a is the ath generator of the chromodynamics $SU(3)_c$ gauge symmetry (6.6d), which annihilates $SU(3)_c$-invariant fields and wave-functions.
2. The operator $I_{w\,\alpha}$ is the αth (isospin) generator of the weak $SU(2)_w$ gauge symmetry. The fermions in Table 7.1 on p. 275 are the eigenstates of the generator $I_{w\,3}$, with the eigenvalues I_w; the operators $I_{w\,\pm}$ raise and lower the values of I_w by 1 [☞ relations (A.38) for the general $SU(2)$ algebra].
3. The operator Y_w produces the weak hypercharge of the field or wave-function on which it acts.

The $\mathbb{V}_w$ matrix encodes the CKM mixing of the lower, d-, s- and b-quarks [☞ relations (2.53)] and leaves the other fermions unchanged. The sum in the expression (7.133b) contains all the elementary fermions from Table 7.1 on p. 275.

As in Section 7.1.3, the parameter $\varkappa$ in the expression (7.133e) separates the symmetric ($\varkappa = +1$) and the "non-symmetric" ($\varkappa = -1$) phases. For $\varkappa = +1$, $\langle \mathbb{H} \rangle = 0$ and the $SU(3)_c \times SU(2)_w \times U(1)_y$ gauge symmetry is unbroken; for $\varkappa = -1$, $\langle \mathbb{H} \rangle \neq 0$ and the gauge symmetry is broken to $SU(3)_c \times U(1)_Q$, the normal modes of the gauge 4-vector potentials

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are (7.85)–(7.86) and the $W^{\pm}$- and Z^0-bosons acquired a mass [☞ expressions (7.84) and (7.87)]. In the non-symmetric phase, the linear combination of gauge bosons (7.85) is identified with the photon. On one hand, this linear combination remains massless; on the other, this linear combination interacts with the Standard Model elementary fermions proportionally to the g_e-multiple of the combination of charges $Q = I_w + \frac{1}{2}Y_w$, which is by construction equal to the electric charge.

Similarly, in the symmetric phase ($\varkappa = +1$), the terms (7.133f) describe only the interaction between elementary fermions and the Higgs doublet of scalar fields. In the non-symmetric phase, owing to the shift $\mathbb{H} \to \widetilde{\mathbb{H}} + \langle\mathbb{H}\rangle$ where $\langle\mathbb{H}\rangle \neq 0$, the terms (7.133f) also provide the Standard Model elementary fermions with mass. Finally, the last term (7.133g) is needed for the "see-saw mechanism" [☞ Section 7.3.2]. This models the left-handed neutrino masses – many orders of magnitude below other Standard Model elementary fermion masses – by means of new physics expected at energies that are many orders of magnitude above the Standard Model masses; for example, the masses of the right-handed neutrinos, which thereby remain not observable directly for now.

As has been widely reported, the search for the Higgs particle has been on for the past decade or so, with most of the meticulous analyses centering on the LEP (Large Electron–Positron collider) and more recently the LHC (Large Hadron Collider) experiments at CERN. These culminated recently with the "5-σ" (99.999,9%) confirmation by the ATLAS and CMS collaborations from the LHC at CERN of a new, $\approx 125.9\,\text{GeV}/c^2$ particle [293], consistent with the Standard Model Higgs particle [25, 109]. However, it is important to realize that the Higgs particle is hard to identify unambiguously in experiments, since its mass, decay modes and their branching ratios all strongly depend on the details of the Standard Model – and its variations. The data compiled from the pertinent experiments are found to be compatible with the Standard Model as described above, but do not exclude several generalizations. For a review of recent experimental results, including also supersymmetric variants of the Standard Model and models wherein the Higgs field is a composite bound state, see Refs. [25, 109], the references therein, and in particular also Refs. [160, 493, 494, 475] ☝.

7.3.4 Exercises for Section 7.3

✎ ***7.3.1*** *For the expressions (7.109), (7.110), (7.112) and (7.113) to be Lagrangian density terms, compute the physical unit-dimensions of the Yukawa coupling coefficients h_U, h_D, h_ν and h_ℓ in the $M^x L^y T^z$ format.*

✎ ***7.3.2*** *Confirm the result (7.122) by explicit computation, using equation (7.121).*

✎ ***7.3.3*** *Confirm the result (7.125) by explicit computation.*

✎ ***7.3.4*** *Confirm the result (7.132b) by explicit diagonalization.*

✎ ***7.3.5*** *Compute the simplified neutrino oscillation wavelength $2z_*$ (7.128), using one and then the other value in equations (7.129).*

✎ ***7.3.6*** *Compute the order of magnitude of M_ν so that equation (7.132b) would produce $m_- \sim 1\,\text{eV}/c^2$.*