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A NOTE ON AN ASYMPTOTIC VERSION OF A PROBLEM OF MAHLER

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Abstract

We prove that for any infinite sets of nonnegative integers $\mathcal{A}$ and $\mathcal{B}$, there exist transcendental analytic functions $f \in \mathbb{Z}\{z\}$ whose coefficients vanish for any indexes $n \notin \mathcal{A} + \mathcal{B}$ and for which $f(z)$ is algebraic whenever z is algebraic and $|z| < 1$. As a consequence, we provide an affirmative answer for an asymptotic version of Mahler's problem A.

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1. Introduction

The most interesting classes of numbers for which transcendence has been proved are given by values of suitable analytic transcendental functions. Weierstrass initiated the investigation of the set of algebraic numbers for which a given transcendental function f takes algebraic values. Since that time, many mathematicians have studied such questions.

In one of his books, Mahler [2, Ch. 3] posed three problems on the arithmetic behaviour of transcendental functions at algebraic points, calling them Problems A, B and C. Problems B and C were solved completely by Marques and Moreira [3, 4], but Problem A remains open. As usual, $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers and $\mathbb{Z}\{z\}$ denotes the set of the power series analytic in the unit ball $B(0, 1)$ and with integer coefficients. Let us state this remaining unsolved problem.

MAHLER'S PROBLEM A. *Does there exist a transcendental function $f \in \mathbb{Z}\{z\}$ with bounded coefficients and such that $f(z)$ is algebraic whenever z is algebraic and $|z| < 1$?*

We remark that Mahler himself showed the existence of a function $f \in \mathbb{Z}\{z\}$ such that $f(\overline{\mathbb{Q}} \cap B(0, 1)) \subseteq \overline{\mathbb{Q}}$. Problem A was the only one for which Mahler made some prediction: 'I conjecture that this problem has a negative answer'. Recently, Marques

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and Moreira [5] proved the existence of a transcendental function $f \in \mathbb{Z}\{z\}$ whose coefficients have only 2 and 3 as prime factors and such that $f(\overline{\mathbb{Q}} \cap B(0,1)) \subseteq \overline{\mathbb{Q}}$. We refer the reader to [2, 7] (and references therein) for more results related to the arithmetic behaviour of transcendental analytic functions.

For a given power series $f(z) = \sum_{n\geq 0} a_n z^n$ and $M \geq 0$, we denote by $L(f, M)$ the set of indexes $n \geq 0$ such that $|a_n| \leq M$, that is,

$$L(f, M) = \{n \in \mathbb{N} : |a_n| \leq M\}.$$

We can rewrite Problem A as follows.

Problem A*. *Does there exist a transcendental function $f \in \mathbb{Z}\{z\}$ and an integer M such that $L(f, M) = \mathbb{Z}_{\geq 0}$ and $f(\overline{\mathbb{Q}} \cap B(0,1)) \subseteq \overline{\mathbb{Q}}$?*

This suggests a less demanding problem where we ask for $f \in \mathbb{Z}\{z\}$ having almost all its coefficients bounded. As usual, we write $\mathcal{A}(x) := \mathcal{A} \cap [0, x]$ for $x > 0$ and $\delta(\mathcal{A})$ denotes the *natural density* of a set $\mathcal{A} \subseteq \mathbb{Z}_{\geq 0}$, that is, the limit (if it exists)

$$\delta(\mathcal{A}) := \lim_{x\to\infty} \frac{\#\mathcal{A}(x)}{x}.$$

Asymptotic Problem A. *Does there exist a transcendental function $f \in \mathbb{Z}\{z\}$ and an integer M such that $\delta(L(f, M)) = 1$ and $f(\overline{\mathbb{Q}} \cap B(0,1)) \subseteq \overline{\mathbb{Q}}$?*

We give an affirmative answer for this questions and prove the following more general result.

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers and $\mathcal{S} := \mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A},\ b \in \mathcal{B}\}$. Then there exist uncountably many transcendental functions $f(z) = \sum_{n\in\mathcal{S}} a_n z^n \in \mathbb{Z}\{z\}$ such that $f(\overline{\mathbb{Q}} \cap B(0,1)) \subseteq \overline{\mathbb{Q}}$.*

As an immediate consequence, this gives a positive answer to Asymptotic Problem A (for $M = 0$). For example, we can choose $\mathcal{A} = \mathcal{B} = \mathbb{Z}^2_{\geq 0}$, since $S = \mathcal{A} + \mathcal{B} = \{m^2 + n^2 : m, n \in \mathbb{Z}_{\geq 0}\}$ has asymptotic density zero (see, for example, [1, page 24]).

Remark 1.2. We point out that there exist sets A which do not contain any sum set of two infinite sets (for example, take $A = \{2^k : k \geq 0\}$). However, if $\delta(A) > 0$, then $B + C \subseteq A$ for some infinite sets B and C. This was proved by Moreira *et al.* [6] settling a conjecture of Erdős.

2. The proof of Theorem 1.1

2.1. A key lemma. In this section, we shall prove a result which will be an essential ingredient in the proof of Theorem 1.1.

Lemma 2.1. *Let $P(z) \in \mathbb{Z}[z]$ be a polynomial of degree $d \geq 1$ and $\mathcal{S}$ an infinite set of positive integers. Then there exists a nonzero polynomial $Q(z) \in \mathbb{Z}[z]$ with degree m,*

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say, such that the product $PQ \in \mathbb{Z}[z]$ *is a polynomial of the form*

$$\sum_{n \in \mathcal{S}(d+m)} a_n z^n.$$

PROOF. Define

$$Q(z) := \sum_{i=0}^{L} q_i z^i,$$

where L and the q_i values will be chosen later. The polynomial PQ has degree at most $L + d$ and its coefficients are linear forms in $q_0, \ldots, q_L$. So, it suffices to prove that it is possible to choose the coefficients of $Q(z)$ to eliminate the terms z^n in $(PQ)(z)$, for which $n \notin \mathcal{S}(L + d)$ (observe that there are $L + d + 1 - \#\mathcal{S}(L + d)$ of these terms). By equating the corresponding coefficients of PQ to zero, we obtain a homogeneous linear system with $L + d + 1 - \#\mathcal{S}(L + d)$ equations in the $L + 1$ variables q_i, $i \in [0, L]$. This system has a nontrivial integer solution provided that

$$L + 1 > L + d + 1 - \#\mathcal{S}(L + d),$$

that is, $\#\mathcal{S}(L + d) > d$. Since $\mathcal{S}$ is an infinite set, this inequality holds for all sufficiently large integers L. If we set $m := \max\{i \leq L : q_i \neq 0\}$, then

$$Q(z) = \sum_{i=0}^{m} q_i z^i \in \mathbb{Z}[z]$$

is the desired polynomial. This finishes the proof. □

2.2. Proof of Theorem 1.1. Let $\{\alpha_1, \alpha_2, \ldots\}$ be an enumeration of $\overline{\mathbb{Q}} \cap B(0, 1)$ and let $P_i(z)$ of degree d_i be the minimal polynomial (over $\mathbb{Z}$) of the algebraic number α_i. We apply Lemma 2.1 to the polynomials

$$P_1(z),\ P_1(z)P_2(z),\ P_1(z)P_2(z)P_3(z),\ \ldots.$$

Since $\mathcal{B}$ is an infinite set, Lemma 2.1 ensures the existence (for any $k \geq 1$) of a polynomial $Q_k(z) \in \mathbb{Z}[z]$ of degree m_k such that

$$Q_k(z)P_1(z) \ldots P_k(z) = \sum_{n \in \mathcal{B}(m_k + D_k)} a_{k,n} z^n,$$

where $D_k := \sum_{i=1}^{k} d_i$. Now, we define recursively the sequence $(t_k)_{k \geq 1}$ by choosing $t_1 = \min \mathcal{A}$ and $t_{k+1} \in \mathcal{A}$ satisfying

$$t_{k+1} \geq \max\{k(t_k + D_k + m_k) + 1, \ell(Q_{k+1}P_1 \ldots P_{k+1}) + (k + 1)\}.$$

This choice is possible, because $\mathcal{A}$ is an infinite set of nonnegative integers. As usual, $\ell(P)$ denotes the *length* of a polynomial P (that is, the sum of the absolute values of its coefficients).

We claim that the function

$$f(z) := \sum_{k\geq 1} z^{t_k} Q_k(z) P_1(z) \ldots P_k(z)$$

satisfies the conditions of the theorem. Indeed, first note that by construction, $f(z)$ can be written as $\sum_{n\in\mathcal{S}} a_n z^n$ (using the fact that $t_{k+1} > t_k + D_k + m_k$). We also have $f(\overline{\mathbb{Q}} \cap B(0,1)) \subseteq \overline{\mathbb{Q}}$, since for any $i \geq 1$,

$$f(\alpha_i) = \sum_{k=1}^{i-1} \alpha_i^{t_k} Q_k(\alpha_i) P_1(\alpha_i) \ldots P_k(\alpha_i)$$

is a finite sum of algebraic numbers. Moreover, $t_{k+1}/(t_k + D_k + m_k)$ tends to infinity as $k \to \infty$ (indeed, $t_{k+1}/(t_k + D_k + m_k) > k$), so $f(z)$ is a strongly lacunary series and hence a transcendental function (by the transcendence criterion from [2, page 40]).

To finish, it remains to show that f is an analytic function in the unit ball. For that, take $R \in (0,1)$ and $z \in \overline{B}(0,R)$. Since $|P(z)| \leq \ell(P)$ when $|z| \leq 1$, one infers that

$$|z^{t_k} Q_k(z) P_1(z) \ldots P_k(z)| \leq R^{t_k} \ell(Q_k P_1 \ldots P_k) < R^{\ell(Q_k P_1 \ldots P_k)+k} \ell(Q_k P_1 \ldots P_k),$$

since $t_k > \ell(Q_k P_1 \ldots P_k) + k$ and $R < 1$. Note now that the maximum value of the function $x \mapsto xR^x$, for real positive values of x, is attained at $x = 1/|\log R|$ and is equal to $e^{-1}/|\log R|$. This implies that

$$R^{\ell(Q_k P_1 \ldots P_k)+k} \ell(Q_k P_1 \ldots P_k) < \frac{e^{-1}}{|\log R|} R^k.$$

Summarising, we see that

$$|z^{t_k} Q_k(z) P_1(z) \ldots P_k(z)| \leq \frac{e^{-1}}{|\log R|} R^k =: M_k,$$

for all $z \in \overline{B}(0,R)$. Since $\sum_{k\geq 1} M_k$ converges, then by the Weierstrass M-test, the series $\sum_{k\geq 1} z^{t_k} Q_k(z) P_1(z) \ldots P_k(z)$ converges absolutely and uniformly on $\overline{B}(0,R)$ for any $R \in (0,1)$. Consequently, this series defines an analytic function (namely, $f(z)$) in the unit circle $B(0,1)$.

The function $f(z) = \sum_{n\in\mathcal{S}} a_n z^n$ is analytic in the unit ball and $f(\alpha) \in \overline{\mathbb{Q}}$ for all $\alpha \in \overline{\mathbb{Q}} \cap B(0,1)$. There is an ∞-ary tree of different possibilities for f. In fact, for any $k \geq 1$, t_k can be chosen in infinitely many different ways and each choice gives a different function f. Thus, we have constructed uncountably many of these functions. This completes the proof. □

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