

Contents

Acknowledgements

page xiii

0	Introduction: glimpses of the theory beneath Monstrous Moonshine	1
0.1	Modular functions	1
0.2	The McKay equations	3
0.3	Twisted #0: the Thompson trick	4
0.4	Monstrous Moonshine	5
0.5	The Moonshine of E_8 and the Leech	6
0.6	Moonshine beyond the Monster	8
0.7	Physics and Moonshine	9
0.8	Braided #0: the meaning of Moonshine	11
0.9	The book	11
1	Classical algebra	14
1.1	Discrete groups and their representations	14
1.1.1	Basic definitions	15
1.1.2	Finite simple groups	17
1.1.3	Representations	20
1.1.4	Braided #1: the braid groups	26
1.2	Elementary geometry	29
1.2.1	Lattices	29
1.2.2	Manifolds	32
1.2.3	Loops	40
1.3	Elementary functional analysis	44
1.3.1	Hilbert spaces	45
1.3.2	Factors	49
1.4	Lie groups and Lie algebras	52
1.4.1	Definition and examples of Lie algebras	53
1.4.2	Their motivation: Lie groups	55
1.4.3	Simple Lie algebras	59
1.5	Representations of simple Lie algebras	65
1.5.1	Definitions and examples	65
1.5.2	The structure of simple Lie algebras	68
1.5.3	Weyl characters	73

1.5.4	Twisted #1: automorphisms and characters	78
1.5.5	Representations of Lie groups	82
1.6	Category theory	87
1.6.1	General philosophy	87
1.6.2	Braided monoidal categories	88
1.7	Elementary algebraic number theory	95
1.7.1	Algebraic numbers	95
1.7.2	Galois	98
1.7.3	Cyclotomic fields	101
2	Modular stuff	104
2.1	The underlying geometry	104
2.1.1	The hyperbolic plane	104
2.1.2	Riemann surfaces	110
2.1.3	Functions and differential forms	116
2.1.4	Moduli	119
2.2	Modular forms and functions	126
2.2.1	Definition and motivation	126
2.2.2	Theta and eta	131
2.2.3	Poisson summation	135
2.2.4	Hauptmoduls	138
2.3	Further developments	140
2.3.1	Dirichlet series	140
2.3.2	Jacobi forms	142
2.3.3	Twisted #2: shifts and twists	144
2.3.4	The remarkable heat kernel	147
2.3.5	Siegel forms	150
2.4	Representations and modular forms	154
2.4.1	Automorphic forms	154
2.4.2	Theta functions as matrix entries	159
2.4.3	Braided #2: from the trefoil to Dedekind	164
2.5	Meta-patterns in mathematics	168
2.5.1	Twenty-four	168
2.5.2	$A-D-E$	169
3	Gold and brass: affine algebras and generalisations	176
3.1	Modularity from the circle	176
3.1.1	Central extensions	176
3.1.2	The Virasoro algebra	180
3.2	Affine algebras and their representations	187
3.2.1	Motivation	187
3.2.2	Construction and structure	189
3.2.3	Representations	192

3.2.4	Braided #3: braids and affine algebras	200
3.2.5	Singularities and Lie algebras	204
3.2.6	Loop groups	206
3.3	Generalisations of the affine algebras	208
3.3.1	Kac–Moody algebras	209
3.3.2	Borcherds’ algebras	212
3.3.3	Toroidal algebras	215
3.3.4	Lie algebras and Riemann surfaces	216
3.4	Variations on a theme of character	218
3.4.1	Twisted #3: twisted representations	218
3.4.2	Denominator identities	220
3.4.3	Automorphic products	223
4	Conformal field theory: the physics of Moonshine	226
4.1	Classical physics	227
4.1.1	Nonrelativistic classical mechanics	227
4.1.2	Special relativity	233
4.1.3	Classical field theory	237
4.2	Quantum physics	240
4.2.1	Nonrelativistic quantum mechanics	241
4.2.2	Informal quantum field theory	252
4.2.3	The meaning of regularisation	270
4.2.4	Mathematical formulations of quantum field theory	271
4.3	From strings to conformal field theory	276
4.3.1	String theory	277
4.3.2	Informal conformal field theory	280
4.3.3	Monodromy in CFT	290
4.3.4	Twisted #4: the orbifold construction	292
4.3.5	Braided #4: the braid group in quantum field theory	295
4.4	Mathematical formulations of conformal field theory	298
4.4.1	Categories	298
4.4.2	Groups are decorated surfaces	303
4.4.3	Topological field theory	305
4.4.4	From amplitudes to algebra	308
5	Vertex operator algebras	311
5.1	The definition and motivation	311
5.1.1	Vertex operators	311
5.1.2	Formal power series	312
5.1.3	Axioms	317
5.2	Basic theory	323
5.2.1	Basic definitions and properties	324
5.2.2	Examples	325

5.3	Representation theory: the algebraic meaning of Moonshine	329
5.3.1	Fundamentals	330
5.3.2	Zhu's algebra	333
5.3.3	The characters of VOAs	337
5.3.4	Braided #5: the physics of modularity	339
5.3.5	The modularity of VOA characters	342
5.3.6	Twisted #5: twisted modules and orbifolds	345
5.4	Geometric incarnations	348
5.4.1	Vertex operator algebras and Riemann surfaces	348
5.4.2	Vertex operator superalgebras and manifolds	351
6	Modular group representations throughout the realm	354
6.1	Combinatorial rational conformal field theory	354
6.1.1	Fusion rings	354
6.1.2	Modular data	359
6.1.3	Modular invariants	361
6.1.4	The generators and relations of RCFT	362
6.2	Examples	368
6.2.1	Affine algebras	368
6.2.2	Vertex operator algebras	375
6.2.3	Quantum groups	378
6.2.4	Twisted #6: finite group modular data	381
6.2.5	Knots	383
6.2.6	Subfactors	386
6.3	Hints of things to come	392
6.3.1	Higher-genus considerations	392
6.3.2	Complex multiplication and Fermat	392
6.3.3	Braided #6: the absolute Galois group	395
7	Monstrous Moonshine	402
7.1	The Monstrous Moonshine Conjectures	402
7.1.1	The Monster revisited	403
7.1.2	Conway and Norton's fundamental conjecture	407
7.1.3	E_8 and the Leech	408
7.1.4	Replicable functions	409
7.2	Proof of the Monstrous Moonshine conjectures	412
7.2.1	The Moonshine module V^\natural	413
7.2.2	The Monster Lie algebra \mathfrak{m}	415
7.2.3	The algebraic meaning of genus 0	416
7.2.4	Braided #7: speculations on a second proof	419
7.3	More Monstrous Moonshine	422
7.3.1	Mini-Moonshine	422
7.3.2	Twisted #7: Maxi-Moonshine	424

7.3.3	Why the Monster?	426
7.3.4	Genus 0 revisited	428
7.3.5	Modular Moonshine	428
7.3.6	McKay on Dynkin diagrams	430
7.3.7	Hirzebruch's prize question	431
7.3.8	Mirror Moonshine	432
7.3.9	Physics and Moonshine	433
	<i>Epilogue, or the squirrel who got away?</i>	435
	<i>Notation</i>	436
	<i>References</i>	445
	<i>Index</i>	464

