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HIGHER RANDOMNESS AND GENERICITY

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Abstract

We use concepts of continuous higher randomness, developed in Bienvenu *et al.* [‘Continuous higher randomness’, *J. Math. Log.* **17**(1) (2017).], to investigate Π^1_1-randomness. We discuss lowness for Π^1_1-randomness, cupping with Π^1_1-random sequences, and an analogue of the Hirschfeldt–Miller characterization of weak 2-randomness. We also consider analogous questions for Cohen forcing, concentrating on the class of Σ^1_1-generic reals.

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1. Background

Mathematical objects often have a general definition which has no regard for any method or procedure that can describe it. For instance, a function is defined as an arbitrary correspondence between objects, but nothing in the definition requires that we are given a way to construct the correspondence. Nonetheless, when the modern definition of functions (often credited to Dirichlet) appeared, it was obvious that all the actual functions that were studied in practice were determined by simple analytic expressions, such as explicit formulae or infinite series.

In the early days of logic, some mathematicians tried to delineate the functions which could be defined by such accepted methods and they searched for their characteristic properties, presumably nice properties not shared by all functions. Baire was first to introduce in his thesis [Bai99] what we now call Baire functions, the smallest set which contains all continuous functions and is closed under the taking of (pointwise) limits. His work was then pursued by Lebesgue [Leb05], who initiated the first systematic study of definable functions. According to Moschovakis [Mos87], Lebesgue's paper truly started the subject of descriptive set theory.

At the time, the modern notions of computability and definability were yet to appear, but we can see, through the work of Borel, Baire and Lebesgue, the necessity of giving a precise meaning to the intuition we have of objects we can 'describe' or 'understand'. A couple of years later, Gödel's work around his incompleteness theorems constituted a key step leading to the understanding of what is a computable object and to the understanding of definability in general. This work was then pursued in the thirties, by Church with Lambda calculus, and by Turing with his eponymous machine. The modern notion of computable function was made clear and researchers were soon convinced of the rather philosophical following statement, known as the Turing–Church thesis: 'A function is computable (using any of the numerous possible equivalent mathematical definitions) if and only if its values can be found by some purely mechanical process'.

Let us now go back to the early days of descriptive set theory. The study of the hierarchy of functions initiated by Baire and pursued by Lebesgue naturally led to the notion of Borel sets. One goal here was again to refine the very general definition of sets (say of reals) in order to work with objects we can understand and describe. The notion of Borel sets takes care of one aspect of sets complexity, their complexity with respect to their 'shape': The sets of reals with simplest shape complexity are the open sets ($\boldsymbol{\Sigma}^0_1$ sets) and their complement, the closed sets ($\boldsymbol{\Pi}^0_1$ sets). The first ones are merely unions of interval and the second ones complements of unions of interval. We then obtain sets of higher and higher complexity by taking countable unions or countable intersections of sets of lower complexity. We obtain a hierarchy of sets, each of them having nice properties, such as for instance being measurable or having the Baire property. However, this hierarchy of complexity is still unsatisfactory, because even a set of simple shape, like an open set, can be very complex from the viewpoint of effectiveness: a set may be open, but there may be no way to describe the intervals which compose it. Kleene, a student of Church, reintroduced computability in the study of Borel sets. We now want to work only with open sets that can be described in some effective way. Then when we consider a countable intersection or a countable union, we also want to be able to describe in some effective way which sets take part in this union or intersection. This led to the very nice and beautiful theory of effectively Borel sets, and of effectively analytic and coanalytic sets, which constitute one of the main material of this paper.

Computability and definability could be used successfully in the study of sets of reals. But they were primarily designed to study sets of integers. Interestingly, the effective sets of reals proved themselves useful to conduct a study of the sets of integers which are far from being describable or understandable as single objects. This is the purpose of, for instance, algorithmic randomness. This field tries to resolve an apparent paradox that probability theory is helpless with: if one flip a fair coin twenty times in a row, a result like this 01001011011010101110 will seem rather ‘normal’, whereas a result like this one : 00000000000000000000 will appear as nonrandom and extraordinary, to the point that one would probably check if the coin is valid. However, these two outcomes have the same probability of occurrence. So why one of them seems more random than the other one? It is simply because one is hard to describe whereas the other one is simple to describe. This is an extreme case, and it is not always the case that strings which seem nonrandom (with respect to a fair-coin flipping) are simple to describe. Consider for instance a long string with twice more 0’s than 1’s, but chaotic enough with regards to any other aspect you could think of. This string is not necessarily simple to describe, but it belongs to a small set that is simple to describe : the set of strings with twice more 0’s than 1’s, which has small measure by the concentration inequalities, like the Chernoff bounds. The mathematical formalization of this idea was a long process throughout the 20th century, started by Kolmorogov and Solomonov [**Sol64**, **Kol65**]. Martin-Löf was the first, in 1966 [**ML66**], to use the above paradigm to define randomness of infinite binary sequences: such a sequence is random if it belongs to no set of measure 0, for a given class of set which should be describable in some way. Whichever notion of ‘being describable’ is used, the only requirement is that at most countably many sets are describable for this notion. This way the set of randoms still has measure one, by the countable additivity of measures.

There are other approaches to the study of sets of integers which are typical. In 1966 Cohen showed that the continuum hypothesis was independent of the standard axioms of set theory (ZFC). To do so he devised his famous forcing method, which should latter have numerous various applications in mathematical logic in general. The first example of forcing given by Cohen is forcing with the dense open sets in a countable model of ZFC. With respect to that forcing, a set of integers is called Cohen generic if it belongs to none of the meagre sets definable in the model. Just as countable additivity of measures is used to ensure that the set of random elements has measure 1, here we use the fact that in a Baire space, a countable union of meagre sets is still a meagre set. Therefore, the sets of generic elements is comeagre. The study of Cohen generics was latter pursued by several authors [**Joc80**, **Kur82**, **Kur83**], by lowering the effective complexity of meagre sets which are used: we do not consider all the meagre sets in a countable model of ZFC, but only some of them. We can for instance keep only the closed sets of empty interior whose complement can be enumerated by a Turing machines.

There are a lot of similarities between Cohen generics and random sequences. This is because Cohen generics are for category theory what randoms are for measure theory: in both case we have a notion of ‘small set’, for randomness a set is small if it has measure 0 and for categoricity a set is small if it is meagre. Also in both case we declare an element ‘typical’ if it belongs to no small set among a countable selection of them.

This paper deals with both randomness and genericity at certain various levels of effectiveness or describability. We mainly deal with what is called Π^1_1-randomness and Σ^1_1-genericity. The notion of Π^1_1-randomness goes back to Sacks [**Sac90**] and Kechris [**Kec75**], and it started to be studied formally by Hjorth and Nies [**HN07**]. It is a notion of interest because of some remarkable properties shared with no other randomness notion. For instance there is a largest Π^1_1 set of measure 0. This notion was so far not very well understood, and we unveil in this paper most of its mysteries. Our work provides insight about its inner mechanisms: Π^1_1-randomness becomes with this paper a well understood notion.

As for Σ^1_1-genericity, the notion was at first built by the authors to mimic on the categorical side the phenomena that occur on the measure theoretical side with Π^1_1-randomness. We conduct a study of various genericity notions lying next to Σ^1_1-genericity, and we show that it has a lot of similarities with Π^1_1-randomness.

2. Introduction

Interest in Π^1_1-randomness comes from both above and below. From ‘above’, effective descriptive set theory attempts to understand the computable content of basic facts about definable sets of real numbers. Lightface investigations shed new light on classical results; for an example we can take Spector’s proof of the measurability of Π^1_1 sets, originally established by Lusin. The ordinal analysis of Π^1_1 sets allows us to consider them as being in some sense enumerable. For sets of natural numbers, this is made precise by using admissible computability over $L_{\omega_1^{\mathrm{ck}}}$. Of course measure plays a central role in descriptive set theory, and so null Π^1_1 sets are a natural object to study.

From ‘below’, investigation of higher notions of algorithmic randomness were started by Martin-Löf [**ML66**], who considered Δ^1_1-randomness, mostly because it satisfies better closure properties than the computably enumerable notion. Sacks (see [**Sac90**, IV2.5]) was the first to define the notion of Π^1_1-randomness and show it is distinct from Δ^1_1-randomness. An important advance in the theory of ‘higher randomness’ was made by Hjorth and Nies in [**HN07**]. They used the analogy between computably enumerable and Π^1_1 sets of numbers to define higher analogues of notions of algorithmic randomness, the most central being Π^1_1-ML-randomness. The theory was then further developed by Chong *et al.* [CNY08], by Chong and Yu [CY] and by Bienvenu *et al.* [BGM].

These contributions enriched various aspects of the theory, but very little was discovered about the key notion of Π^1_1-randomness. This concept is very natural. It is simply defined (avoiding all null Π^1_1 sets), and has a universal test (a greatest null Π^1_1 set); and unlike ML-randomness, the universal test occurs without having to encumber the definition with extra conditions (the speed of convergence of the measure to 0). On the other hand it is a singularity among higher randomness notions, in that it is not the higher analogue of any ‘lower’ notion of randomness: Δ^1_1-randomness is higher Schnorr randomness, and other notions are direct analogues: the main one is Π^1_1-ML-randomness, but also higher weak 2-randomness (introduced by Nies [Nie09, 9.2.17], studied by Chong and Yu [CY] and later in [BGM]), and higher Kurtz randomness. It was not clear how to use computability-theoretic tools to tackle Π^1_1-randomness.

A breakthrough was made by the second author in [Mon14], who showed that the set of Π^1_1-randoms is Π^0_3, a Borel rank much lower than expected earlier. In this paper we use his work to continue the effective study of Π^1_1-randomness, and in particular answer some questions that have been left open for more than a decade. For example, we show that lowness for Π^1_1-randomness coincides with being hyperarithmetic, and prove a similar result about cupping with Π^1_1-random sequences. We also identify and investigate the category analogue of Π^1_1-randomness, which is Σ^1_1-genericity.

2.1. Π^1_1-randomness, lowness and cupping. As mentioned above, there is a greatest null Π^1_1 set (Stern and independently Kechris [Ste75, Kec75], and later rediscovered in [HN07]). In fact, this greatest set can be described succinctly. Recall that a sequence is Δ^1_1-random if it avoids all null Δ^1_1 (hyperarithmetic) sets. We say that a real X *collapses* ω_1^{ck} if $\omega_1^X > \omega_1^{\mathrm{ck}}$; otherwise it *preserves* ω_1^{ck}. The following characterization was first proved by Stern [Ste73, Ste75] and rediscovered later by Chong *et al.* [CNY08]:

THEOREM 2.1. *A sequence is Π^1_1-random if and only if it is Δ^1_1-random and preserves ω_1^{ck}.*

In this paper we answer the question of lowness for Π^1_1-randomness, first stated in [HN07]. The idea of lowness has been extensively studied in algorithmic randomness: for a given randomness notion Γ, we say that a set X is *low for Γ* if X cannot derandomize any Γ-random: every Γ-random is also $\Gamma(X)$-random. In particular the class of reals low for ML-randomness has been central in algorithmic randomness, with many equivalent characterizations. The higher analogue of this class was studied in [HN07, BGM].

Any Δ^1_1 set is low for Π^1_1-randomness. In this paper (Theorem 4.1) we prove that these are the only ones.

We also consider the question of *cupping* with Π^1_1-random sequences. A fundamental result in the study of both the local and global Turing degrees is the Posner–Robinson theorem, showing that any noncomputable real can be joined above $\varnothing'$ with a 1-generic sequence. The cupping question for incomplete randoms was settled by Day and Miller [DM14] using tools of effective analysis. Their solution gives yet another characterization of lowness for ML-randomness. Limits on cupping with random sequences were established by Day and Dzhafarov [DD13].

In the higher setting, Kleene's O, the complete Π^1_1 set of numbers, often plays the role of $\varnothing'$. Here the problem of cupping can be rephrased, since a real X is hyperarithmetically above O if and only if it collapses ω_1^{ck}. Hence for cupping partner for a real A we are searching for a real X which preserves ω_1^{ck} but such that $A \oplus X$ collapses ω_1^{ck}. Kumabe–Slaman forcing can be used to show that any nonhyperarithmetic real can be nontrivially cupped (for Kumabe–Slaman forcing see [SS99]). Theorem 2.1 shows that for random sequences, the random cupping partners desired are precisely the Π^1_1-random sequences. We show that any nonhyperarithmetic real can in fact be cupped by a Π^1_1-random sequence (Theorem 4.3).

2.2. Continuous higher randomness, and an analogue of Hirschfeldt–Miller.

We use concepts, terminology and notation from [BGM]. The main theme of the paper is the centrality of continuous reductions in algorithmic randomness. Hyperarithmetic reducibility is too coarse for many arguments to go through. A central concept introduced in [BGM] is a higher analogue of Turing reducibility that allows us to lift many arguments to the higher setting. The idea is to take the definition of Turing reducibility in terms of functionals and allow the functionals to be Π^1_1 rather than c.e. We give the details in Section 3 below. Higher Turing reducibility requires any output to be determined by only finitely many bits of the oracle. If an oracle Y collapses ω_1^{ck}, then hyperarithmetic reducibility gives Y extra computational power simply because enumerations processes with oracle Y are carried out over more than ω_1^{ck} many steps; higher Turing reducibility does not allow that.

Hirschfeldt and Miller gave the following characterization of weak 2-randomness (see for example [Nie09, Theorem 5.3.15]).

THEOREM 2.2. *Let X be* ML*-random. The following are equivalent:*

(1) *X forms a minimal pair with $\varnothing'$.*

(2) *X does not compute any noncomputable c.e. set.*

(3) *X is weakly 2-random.*

In the higher setting, a modified version (involving enumerating Δ_2 sets) was shown to characterize the class MLR$[O]$, which is strictly smaller than the Π^1_1-randoms. For higher weak 2-randomness, or even Π^1_1-randomness, (1) of the theorem fails, since there is a Π^1_1-random which is computable from O. However, we show here that using the continuous notion of higher computability, (2) characterizes Π^1_1-randomness (Theorem 4.6) and not higher weak 2-randomness. The direction (3) → (2) does not work in the higher setting as it uses what we call a 'time trick': the number of stages of computation is the same as the length of the oracle. The fact that in the higher setting, (2) characterizes Π^1_1-randomness instead shows that reliance on this trick is fundamental.

2.3. A higher arithmetical hierarchy. Yu showed [Nie14] that the set of Π^1_1-randoms is not $\boldsymbol{\Sigma}^0_3$. As mentioned above, the second author showed later [Mon14] that the set of Π^1_1-randoms is $\boldsymbol{\Pi}^0_3$, which is optimal by Yu's result. One can ask how effective this is. The strong analogy between c.e. and Π^1_1 allows us to define a new hierarchy which is the higher analogue of the arithmetical hierarchy (for sets of reals). Namely a subset of Cantor space is *higher effectively open* (higher Σ^0_1) if it is Π^1_1 open, and higher effectively closed (higher Π^0_1) if it is Σ^1_1 closed. To continue we take effective ω-unions. So for example, a set is higher Π^0_2 if it is of the form $\bigcap \mathcal{U}_n$, where each $\mathcal{U}_n$ is Π^1_1 open, uniformly in n; higher Σ^0_3 if it is of the form $\bigcup \mathcal{Q}_n$, where each $\mathcal{Q}_n$ is higher Π^0_2 (uniformly in n); higher Σ^0_2 if it is of the form $\bigcup \mathcal{P}_n$, where each $\mathcal{P}_n$ is Σ^1_1 closed (uniformly); and so on. This definition is motivated by Nies's higher analogue of weak 2-randomness, defined as avoiding all null higher Π^0_2 sets. For brevity, we use the notation Π^{ck}_n and Σ^{ck}_n to denote the levels in this hierarchy.

An unusual feature of this hierarchy is that some higher Σ^0_1 sets are not Σ^0_2. Indeed, the sets in the classes Σ^{ck}_1, Π^{ck}_2, Σ^{ck}_3, Π^{ck}_4, ... are all Π^1_1, and some are not Σ^1_1; considering complements, sets in the classes Π^{ck}_1, Σ^{ck}_2, Π^{ck}_3, ... are all Σ^1_1, but some are not Π^1_1. See Figure 1. (The same phenomenon happens classically if one considers the Borel sets defined on some non-Polish topological space. For example consider the Gandy-Harrington topology; or, let $\mathbb{T}(2^\omega)$ be the set of open sets of 2^ω and consider the topology on $\mathbb{T}(2^\omega)$ generated by the subbasis $[\![\sigma]\!] = \{\mathcal{U} \in \mathbb{T}(2^\omega) : [\sigma] \subseteq \mathcal{U}\}$ for any string σ. Consider the closed set $\mathcal{F} = \{\mathcal{U} \in \mathbb{T}(2^\omega) : [\sigma] - \mathcal{U} \neq \varnothing\}$ for a given string σ. As any open set in this topology contains the element $[\varepsilon] = 2^\omega$, also any intersection of open set contains $[\varepsilon]$, which is not an element of $\mathcal{F}$.)

$$\begin{tikzcd}
\Sigma^{\mathrm{ck}}_1 \arrow[dr] & \Sigma^{\mathrm{ck}}_2 \arrow[dr] & \Sigma^{\mathrm{ck}}_3 \arrow[dr] & \Sigma^{\mathrm{ck}}_4 \arrow[dr] & \Sigma^{\mathrm{ck}}_5 \arrow[dr] & \cdots \\
\Pi^{\mathrm{ck}}_1 \arrow[ur] & \Pi^{\mathrm{ck}}_2 \arrow[ur] & \Pi^{\mathrm{ck}}_3 \arrow[ur] & \Pi^{\mathrm{ck}}_4 \arrow[ur] & \Pi^{\mathrm{ck}}_5 \arrow[ur] & \cdots
\end{tikzcd}$$

Figure 1. The higher hierarchy of complexity of sets. The blue complexities correspond to Π^1_1 sets. The green complexities correspond to Σ^1_1 sets.

We can ask two questions:

(1) Which null sets in this hierarchy suffice to capture Π^1_1-randomness?

(2) Does the set of Π^1_1-randoms lie in this hierarchy?

For example, we could hope that Monin's Borel rank result is completely effective, meaning that the set of Π^1_1-randoms is higher Π^0_3. This is not so, by a result in [**BGM**] (any conull Π^{ck}_3 set in fact contains a sequence which is not higher weak 2-random). For question (1), in [**BGM**] it was shown that Π^1_1-randomness is distinct from higher weak 2-randomness, showing that the level Π^{ck}_2 is insufficient. In Section 5 we establish fairly low bounds for both questions : Π^{ck}_4 as the answer to question (1) and Π^{ck}_5 as the answer for question (2).

2.4. Σ^1_1-genericity.

What about category? Stern [**Ste75**] considered category as well as measure, showing that the largest meagre Π^1_1 set is the set of Δ^1_1-generic sequences which preserve ω^{ck}_1. This uses Feferman's result [**Fef64**] that comeagrely many reals preserve ω^{ck}_1.

Recall that for any lightface pointclass Γ, we say that a sequence $G \in 2^\omega$ is:

- *Weakly Γ-generic* if it meets all dense open sets with codes in Γ (by code we mean code for sets of strings generating the open set);
- *Γ-generic* if it either meets or avoid all open sets with codes in Γ (does not lie on the boundary of any such open set).

For example, Jockusch's familiar notions of n-genericity are Σ^0_n-genericity. The closure properties of the hyperarithmetic sets show that Δ^1_1-genericity and weak Δ^1_1-genericity coincide.

Our first result here is to capture the precise level of genericity that suffices to preserve ω^{ck}_1; this is the category analogue of Monin's result on Π^1_1-randomness.

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We show that the level is precisely Σ^1_1-genericity. This notion can be considered as a higher analogue of Π^0_1-genericity, a notion which Jockusch noticed is equivalent to 2-genericity (see [Kur82] and [Kur83]). We also investigate the intermediate notion of Π^1_1-genericity (the higher analogue of 1-genericity), and consider lowness and cupping questions. We also find a partial analogue of the equivalence of Π^0_1-genericity and 2-genericity (which is the same as 1-genericity relative to $\varnothing'$) by considering a subclass of the $\Pi^1_1(O)$ dense open sets, the *finite-change* dense open sets (see Definition 6.8). Along the way we also give a direct proof of the equivalence of lowness for tests and lowness for weak genericity, which applies to the lower setting as well.

3. Preliminaries

3.1. Higher prefix-free sets of strings, and a result of Kučera's. In 'lower' randomness, many arguments use c.e. (or even computable) prefix-free sets of strings when working with effectively open sets. However, there are higher effectively open sets which are not generated by Π^1_1 prefix-free sets of strings (this is implicit in [HN07] and formally shown in [BGHM]). In the higher setting we focus on the *weight* of a set of strings (and see that in several ways it is the more fundamental concept). Recall that for a set of strings W, the weight $\mathrm{wt}(W)$ of W is $\sum_{\sigma\in W} 2^{-|\sigma|}$. Instead of prefix-free generating sets we obtain sets of weight as close as we like to the measure of the set in question. The technique used in the proof of the following lemma was already used in [BGM, Lemmas 3.1 and 3.3]. It relies on the existence of a 'projectum function': a ω_1^{ck}-computable ($\Delta_1(L_{\omega_1^{\mathrm{ck}}})$-definable) injective function $p\colon \omega_1^{\mathrm{ck}} \to \omega$. Recall that a set of strings W *generates* (or describes, or codes, or defines) the open set

$$\mathcal{W} = [W]^{\prec} = \bigcup_{\sigma\in W} [\sigma] = \{X \in 2^{\omega} : \exists \sigma \prec X \ (\sigma \in W)\}.$$

LEMMA 3.1. *For any higher effectively open set $\mathcal{U}$ and $\varepsilon > 0$ there is a Π^1_1 set of strings W generating $\mathcal{U}$ such that* $\mathrm{wt}(W) \leqslant \lambda(\mathcal{U}) + \varepsilon$.

Though we will not use it, we note that an index for W can be obtained uniformly from ε and an index for $\mathcal{U}$.

Proof. Let U be a Π^1_1 set of strings generating $\mathcal{U}$; let $\langle U_s\rangle_{s<\omega_1^{\mathrm{ck}}}$ be a higher enumeration of U. We can assume that at most one string enters U at each stage: this means that for all $s < \omega_1^{\mathrm{ck}}$, $U_{s+1} - U_s$ contains at most one element, and for all limit $s < \omega_1^{\mathrm{ck}}$, $U_s = U_{<s} = \bigcup_{t<s} U_t$.

At a stage $s < \omega_1^{\mathrm{ck}}$, if σ enters U_{s+1}, we find a clopen set $\mathcal{C}_s \subseteq [\sigma]$ such that:

- $[\sigma] \subseteq \mathcal{W}_s \cup \mathcal{C}_s$; and
- $\lambda(\mathcal{W}_s \cap \mathcal{C}_s) \leqslant \varepsilon \cdot 2^{-p(s)}$.

We then add a (finite) set of strings generating $\mathcal{C}_s$ (whose total weight will be $\lambda(\mathcal{C}_s)$) to W_{s+1}. At limit stages s we let $W_s = \bigcup_{t<s} W_t$.

By construction, $\mathcal{U} = [W]^{\prec}$. To bound the weight of W, we observe that if $s < t < \omega_1^{\mathrm{ck}}$ then the sets $\mathcal{C}_t - \mathcal{W}_t$ and $\mathcal{C}_s - \mathcal{W}_s$ are disjoint (as $\mathcal{C}_s \subseteq \mathcal{W}_t$); these sets are subsets of $\mathcal{U}$, and so

$$\sum_{s<\omega_1^{\mathrm{ck}}} \lambda(\mathcal{C}_s - \mathcal{W}_s) \leqslant \lambda(\mathcal{U}).$$

Also,

$$\mathrm{wt}(\mathcal{C}_s) = \lambda(\mathcal{C}_s) = \lambda(\mathcal{C}_s - \mathcal{W}_s) \cup \lambda(\mathcal{C}_s \cap \mathcal{W}_s),$$

and so

$$\begin{aligned} \mathrm{wt}(W) = \textstyle\sum_{s<\omega_1^{\mathrm{ck}}} \mathrm{wt}(\mathcal{C}_s) &\leqslant \lambda(\mathcal{U}) + \textstyle\sum_{s<\omega_1^{\mathrm{ck}}} \lambda(\mathcal{W}_s \cap \mathcal{C}_s) \\ &\leqslant \lambda(\mathcal{U}) + \varepsilon \textstyle\sum_{s<\omega_1^{\mathrm{ck}}} 2^{-p(s)} \leqslant \lambda(\mathcal{U}) + \varepsilon. \end{aligned}$$ □

As a result, we get a characterization of higher ML-randomness, an analogue of a result of Kučera's [**Kuč85**].

PROPOSITION 3.2. *A sequence Z is Π_1^1-ML-random if and only if Z has a tail in every non-null Π_1^{ck} set.*

Proof. Suppose that Z is not Π_1^1-ML-random. Then every tail of Z is not Π_1^1-ML-random, so Z and all of its tails miss every Π_1^{ck} set consisting only of Π_1^1-ML-random sequences (for example, complements of components of the universal Π_1^1-ML-test).

Suppose that Z is Π_1^1-ML-random. Let $\mathcal{P}$ be Π_1^{ck} and non-null, and let $\mathcal{V}$ be the complement of $\mathcal{P}$. By Lemma 3.1, let V be a Π_1^1 set of strings which generates $\mathcal{V}$ and has weight smaller than 1. We let $\mathcal{V}^m = [V^m]^{\prec}$, where V^m is the set of concatenations of m strings, all from V. The weight of V^m is bounded by $(\mathrm{wt}(V))^m$, and the measure of $\mathcal{V}^m$ is bounded by the weight of V^m. The important point is that $\lambda(\mathcal{V}^m)$ goes to 0 computably, so $\langle \mathcal{V}^m \rangle$ is a Π_1^1-ML-test. Let m be least such that $X \notin \mathcal{V}^m$; as $\mathcal{V}^0 = 2^{\omega}$, $m > 0$. Let $\sigma \in V^{m-1}$ which is a prefix of X; let $Y = X - \sigma$ (so $X = \sigma\hat{}Y$). Then $Y \in \mathcal{P}$. □

3.2. Consistency in higher functionals. Let us define higher Turing reducibility. Below we use it to compute not only elements of 2^ω (or ω^ω) but also of $(\omega_1^{\mathrm{ck}})^\omega$, so we give a general definition. A *higher Turing functional* is a ω_1^{ck}-c.e. set of triples $(\sigma, n, \alpha) \in 2^{<\omega} \times \omega \times \omega_1^{\mathrm{ck}}$. Recall that ω_1^{ck}-c.e. means $\Sigma_1(L_{\omega_1^{\mathrm{ck}}})$-definable; if the functional is a subset of $2^{<\omega} \times \omega \times \omega$ (or $2^{<\omega} \times \omega \times 2$) then this is the same as being Π_1^1. The 'axiom' (σ, n, α) indicates that with an oracle $Y \in 2^\omega$ extending σ, on input n, we output α. For a higher functional Φ and an oracle $Y \in 2^\omega$ we let $\Phi(Y)$ be the function that Φ computes with oracle Y; formally, identifying a function as a set of pairs,

$$\Phi(Y) = \{(n, \alpha) : \exists \sigma \prec Y\ ((\sigma, n, \alpha) \in \Phi)\}.$$

Here we must note something important. Unlike the usual definitions of 'lower' functionals, we do not require that a higher Turing functional is consistent. That is, we do not require that if (σ_0, n, α_0) and (σ_1, n, α_1) are both in Φ, and σ_0 and σ_1 are compatible, then $\alpha_0 = \alpha_1$. We thus have to regard $\Phi(Y)$ as a multi-valued function. For $f \in (\omega_1^{\mathrm{ck}})^\omega$ and $Y \in 2^\omega$, we write $f \leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} Y$ if $f = \Phi(Y)$ for some higher functional Φ (and say that Y *higher computes* f). That is, on the oracle Y we require that Φ gives only consistent answers (and is total), but we do not require that $\Phi(Z)$ be consistent on other oracles Z. Indeed, in [BGHM] we show that there is a higher ML-random sequence (a Π_1^1-ML-random) which higher Turing computes O but does not compute it via a functional consistent on all oracles. So the inconsistency cannot be completely removed. However, it can be 'reduced' by as much as we want, in a measure theoretic way; and this will be useful for some results of this paper.

Let us fix some notation. For a functional Φ and an oracle Y we write $\Phi(Y, n)\downarrow$ if $n \in \operatorname{dom} \Phi(Y)$: that is, at least one value is given. If more than one value is given then we anyway write $\Phi(Y, n) = \alpha_0$ and $\Phi(Y, n) = \alpha_1$. We say that $\Phi(Y)$ is *total* if $\operatorname{dom} \Phi(Y) = \omega$, that is, if $\Phi(Y, n)\downarrow$ for all n. The totality set of Φ is Π_2^{ck}. The inconsistency set of Φ (the set of Y for which for some n, $\Phi(Y, n)$ obtains more than one value) is Σ_1^{ck} (higher effectively open).

The proof of the next lemma again uses the projectum function $p\colon \omega_1^{\mathrm{ck}} \to \omega$.

LEMMA 3.3. *For any higher Turing functional Φ and $\varepsilon > 0$ there is a higher functional Ψ so that:*

(1) *Every Ψ-computation arises from a Φ-computation: for all n, α and Y, if $\Psi(Y, n) = \alpha$ then $\Phi(Y, n) = \alpha$.*

(2) *For all Y, if $\Psi(Y)$ is consistent then* $\operatorname{dom} \Psi(Y) = \operatorname{dom} \Phi(Y)$.

(3) *The measure of the inconsistency set of Ψ is smaller than ε.*

Further, an index for Ψ can be obtained uniformly from an index for Φ and from ε.

Note that (1) and (2) imply that the correct Φ-computations are unchanged in Ψ: for all $Y \in 2^\omega$, if $\Phi(Y)$ is total and consistent then so is $\Psi(Y)$, and $\Psi(Y) = \Phi(Y)$.

Proof. Given Φ and ε we enumerate Ψ. We ensure that for all s, every Ψ_s-computation arises from a Φ_s-computation. We can assume that at most one 'axiom' enters Φ at each stage. At stage $s < \omega_1^{\mathrm{ck}}$ suppose that an axiom (σ, n, α) enters Φ_{s+1}. Let $\mathcal{E}_s$ be the inconsistency set of the functional $\Psi_s \cup \{(\sigma, n, \alpha)\}$. This set is Δ_1^1 open (uniformly in s). We find a clopen set $\mathcal{C}_s \subseteq [\sigma]$ such that $[\sigma] \subseteq \mathcal{C}_s \cup \mathcal{E}_s$ and such that $\lambda(\mathcal{C}_s \cap \mathcal{E}_s) \leqslant 2^{-p(s)}\varepsilon$. We then enumerate into Ψ_{s+1} axioms which ensure that $\Psi_{s+1}(Y, n) = \alpha$ for all $Y \in \mathcal{C}_s$. Since $\mathcal{C}_s \subseteq [\sigma]$, every Ψ_{s+1}-computation arises from a Φ_{s+1}-computation; this establishes (1).

Let us see that (2) and (3) are satisfied. Suppose that $\Psi(Y)$ is consistent, and that $n \in \operatorname{dom} \Phi(Y)$; say an axiom (σ, n, α) enters Φ_{s+1}, where $\sigma \prec Y$. If $Y \in \mathcal{C}_s$ then $n \in \operatorname{dom} \Psi(Y)$. Otherwise, the functional $\Psi_s \cup \{(\sigma, n, \alpha)\}$ is inconsistent on Y. Since $\Psi(Y)$ is consistent, this means that $\Psi_s(Y, n)\downarrow$ (to some value other than α). But this again implies that $n \in \operatorname{dom} \Psi(Y)$.

For (3), suppose that $\Psi(Y)$ is inconsistent. Let s be the stage at which Y enters the inconsistency set of Ψ: $\Psi_s(Y)$ is consistent but $\Psi_{s+1}(Y)$ is not. [There is such a stage; if s is a limit stage and $\Psi_t(Y)$ is consistent for all $t < s$, then $\Psi_s(Y)$ is consistent.] A new axiom applying to Y is enumerated into Ψ_{s+1}, so $Y \in \mathcal{C}_s$. The fact that this new axiom makes $\Psi_{s+1}(Y)$ inconsistent also implies that $Y \in \mathcal{E}_s$. So the inconsistency set of Ψ is a subset of $\bigcup_{s<\omega_1^{\mathrm{ck}}}(\mathcal{C}_s \cap \mathcal{E}_s)$; (2) follows as in the previous proof, since $\lambda(\bigcup_s(\mathcal{C}_s \cap \mathcal{E}_s)) \leqslant \sum_{s<\omega_1^{\mathrm{ck}}} 2^{-p(s)}\varepsilon \leqslant \varepsilon$. ☐

3.3. Π_1^1-randomness and forcing.

The heart of Monin's proof that the Π_1^1-randoms form a $\boldsymbol{\Pi}_3^0$ set goes through an analysis of forcing with Π_1^{ck} sets of positive measure. This is in analogy to forcing with Π_1^0 closed sets of positive measure, which Monin shows yields computably dominated weakly 2-random sequences. The precise level resembles genericity.

THEOREM 3.4 [Mon14]. *Let X be Δ_1^1-random. The following are equivalent:*

(1) *X is Π_1^1-random.*

(2) *For any Σ_2^{ck} set $\mathcal{H}$, either $X \in \mathcal{H}$, or X is an element of some Π_1^{ck} set (necessarily of positive measure) which is disjoint from $\mathcal{H}$.*

We present a proof of Monin's theorem in a language and notation which is aligned with the rest of this paper.

Proof. For a Π^{ck}_2 set $\mathcal{G}$, we let $\mathcal{G}^*$ denote the union of all Π^{ck}_1 subsets of $\mathcal{G}$. So we need to show that if X is Δ^1_1-random, then X collapses ω^{ck}_1 (fails to be Π^1_1-random) if and only if $X \in \mathcal{G} - \mathcal{G}^*$ for some Π^{ck}_2 set $\mathcal{G}$.

Recall that a Π^1_1 set $\mathcal{A}$ is the union $\bigcup_{s<\omega_1} \mathcal{A}_s$, where each A_s is Δ^1_1 in any code for s; in particular, for $s < \omega^{\mathrm{ck}}_1$, $\mathcal{A}_s$ is Δ^1_1, uniformly in s. We let $\mathcal{A}_{<\omega^{\mathrm{ck}}_1} = \bigcup_{s<\omega^{\mathrm{ck}}_1} \mathcal{A}_s$. If $\mathcal{U}$ is Π^1_1 open, then $\mathcal{U} = \mathcal{U}_{<\omega^{\mathrm{ck}}_1}$; but in general, $\lambda(\mathcal{A}) = \lambda(\mathcal{A}_{<\omega^{\mathrm{ck}}_1})$ for any Π^1_1 set. If $\mathcal{G} = \bigcap_n \mathcal{U}_n$ is Π^{ck}_2 then $\mathcal{G} = \mathcal{G}_{\omega^{\mathrm{ck}}_1}$ but may not equal $\mathcal{G}_{<\omega^{\mathrm{ck}}_1}$; the elements of $\mathcal{G}_{\omega^{\mathrm{ck}}_1} - \mathcal{G}_{<\omega^{\mathrm{ck}}_1}$ are those which are enumerated into each $\mathcal{U}_n$ at stages $s_n < \omega^{\mathrm{ck}}_1$ such that the sequence $\langle s_n \rangle$ is unbounded in ω^{ck}_1. Note that the sequence $\langle s_n \rangle$ is $\Delta_1(L_{\omega^{\mathrm{ck}}_1}(X))$-definable, so such X collapses ω^{ck}_1.

We show:

(a) For any $\mathcal{G} \in \Pi^{\mathrm{ck}}_2$, $\mathcal{G}^* \subseteq \mathcal{G}_{<\omega^{\mathrm{ck}}_1}$.

(b) For any $\mathcal{G} \in \Pi^{\mathrm{ck}}_2$, $\mathcal{G}_{<\omega^{\mathrm{ck}}_1} - \mathcal{G}^*$ is null, indeed does not contain Δ^1_1-random sequences.

(c) If X is Δ^1_1-random and collapses ω^{ck}_1, then $X \in \mathcal{G} - \mathcal{G}_{<\omega^{\mathrm{ck}}_1}$ for some Π^{ck}_2 set $\mathcal{G}$.

Then (a) + (c) establish the direction (2) $\Longrightarrow$ (1) of the theorem; and (a) + (b) establish (1) $\Longrightarrow$ (2), as we already observed that any $X \in \mathcal{G} - \mathcal{G}_{<\omega^{\mathrm{ck}}_1}$ collapses ω^{ck}_1.

For (a), let $\mathcal{F} \subseteq \mathcal{G}$ be higher effectively closed. Say $\mathcal{G} = \bigcap_n \mathcal{U}_n$. Just like in the lower setting, by compactness, for each n, there is some $s < \omega^{\mathrm{ck}}_1$ such that $\mathcal{F}_s \subseteq \mathcal{U}_{n,s}$. Observing this fact is $\Delta_1(L_{\omega^{\mathrm{ck}}_1})$, so by admissibility, there is some s such that for all n, $\mathcal{F}_s \subseteq \mathcal{U}_{n,s}$, that is, $\mathcal{F}_s \subseteq \mathcal{G}_s$, yielding $\mathcal{F} \subseteq \mathcal{F}_s \subseteq \mathcal{G}_s \subseteq \mathcal{G}_{<\omega^{\mathrm{ck}}_1}$.

Both (b) and (c) rely on effective regularity of Lebesgue measure. Recall that for any Δ^1_1 set $\mathcal{C}$ we can find a Δ^1_1, G_δ set $\mathcal{G}$ such that $\mathcal{C} \subseteq \mathcal{G}$ and $\mathcal{C} =^* \mathcal{G}$, that is, $\lambda(\mathcal{G} - \mathcal{C}) = 0$. (In fact this can be done within the same level of the hyperarithmetic hierarchy, yielding the equivalence of α-randomness with ML-randomness relative to $\varnothing^{(\alpha)}$ (As usual replace α by $\alpha - 1$ for $\alpha < \omega$.).) Of course, taking complements, we can find a Δ^1_1, F_σ set $\mathcal{F} \subseteq \mathcal{C}$ such that $\mathcal{F} =^* \mathcal{C}$.

For (b), let $X \in \mathcal{G}_{<\omega^{\mathrm{ck}}_1} - \mathcal{G}^*$. Let $s < \omega^{\mathrm{ck}}_1$ such that $X \in \mathcal{G}_s$. Since $\mathcal{G}_s$ is Δ^1_1, find a Δ^1_1, F_σ set $\mathcal{Q} \subseteq \mathcal{G}_s$ such that $\mathcal{Q} =^* \mathcal{G}_s$. Since $\mathcal{Q}$ is a union of Δ^1_1 closed sets and $X \notin \mathcal{G}^*$, $X \notin \mathcal{Q}$. So X is an element of the Δ^1_1 null set $\mathcal{G}_s - \mathcal{Q}$, and so is not Δ^1_1-random.

Finally we prove (c). Let X be a Δ^1_1-random which collapses ω^{ck}_1. Let Ψ be a computable operator taking reals to linear orderings such that $\Psi^X \cong \omega^{\mathrm{ck}}_1$. For any Y and $n < \omega$ let $\Psi^Y(\leqslant n)$ denote the restriction of the ordering Ψ^Y to the numbers $m <_{\Psi^Y} n$. For $n < \omega$ let $\mathcal{A}_n$ consist of the reals Y such that $\Psi^Y(\leqslant n)$ is isomorphic to some computable ordinal. As expected we let $\mathcal{A}_{n,s} = \left\{Y : \Psi^Y \cong t \text{ for some } t < s\right\}$. Then $\mathcal{A}_n$ is Π^1_1 and $\mathcal{A}_n = \mathcal{A}_{n,<\omega^{\mathrm{ck}}_1}$, but of course is not necessarily open. Let $\mathcal{B} = \bigcap_n \mathcal{A}_n$, and for $s < \omega_1^{\mathrm{ck}}$, let $\mathcal{B}_s = \bigcap_n \mathcal{A}_{n,s}$; let $\mathcal{B}_{<\omega_1^{\mathrm{ck}}} = \bigcup_{s<\omega_1^{\mathrm{ck}}} \mathcal{B}_s$. So $X \in \mathcal{B} = \mathcal{B}_{\omega_1^{\mathrm{ck}}} - \mathcal{B}_{<\omega_1^{\mathrm{ck}}}$. We want the same thing except to replace $\mathcal{A}_n$ by open sets. We do this by approximating.

For each n and s find $\mathcal{P}_{n,s} \supseteq \mathcal{A}_{n,s}$, Δ_1^1 and G_δ, such that $\mathcal{A}_{n,s} =^* \mathcal{P}_{n,s}$. Further write $\mathcal{P}_{n,s} = \bigcap_k \mathcal{U}_{n,k,s}$, with each $\mathcal{U}_{n,k,s}$ being a Δ_1^1 open set. These can be chosen so that $\mathcal{U}_{n,k,s} \subseteq \mathcal{U}_{n,k,t}$ if $s < t$. Let $\mathcal{G} = \bigcap_{n,k} \mathcal{U}_{n,k}$; this set is Π_2^{ck} and $\mathcal{G}_{<\omega_1^{\mathrm{ck}}} = \bigcup_s \mathcal{G}_s$ where $\mathcal{G}_s = \bigcap_n \mathcal{P}_{n,s} = \bigcap_{n,k} \mathcal{U}_{n,k,s}$. Since $\mathcal{B} \subseteq \mathcal{G}$, $X \in \mathcal{G}$. For each $s < \omega_1^{\mathrm{ck}}$, $X \notin \mathcal{B}_s$ implies $X \notin \mathcal{G}_s$: otherwise for some $n < \omega$, X is an element of the Δ_1^1 null set $\mathcal{P}_{n,s} - \mathcal{A}_{n,s}$. Hence $X \in \mathcal{G} - \mathcal{G}_{<\omega_1^{\mathrm{ck}}}$, as required. □

Theorem 3.4 can be restated in the language of forcing. Let $\mathbb{P}$ be the partial order consisting of the Π_1^{ck} sets of positive measure, ordered by inclusion. Theorem 3.4 implies the following proposition. Recall that for $\mathcal{K} \subseteq 2^\omega$ we say that a sufficiently $\mathbb{P}$-generic real is in $\mathcal{K}$ if there is a countable collection of dense subsets of $\mathbb{P}$ such that for any filter $G \subseteq \mathbb{P}$ meeting these dense sets, Z_G (defined by $\bigcap G = \{Z_G\}$) is in $\mathcal{K}$. That is, if $\bigcap_n \bigcup \mathbb{D}_n \subseteq \mathcal{K}$, where each $\mathbb{D}_n$ is a dense subset of $\mathbb{P}$.

PROPOSITION 3.5 [**Mon14**]. *A sufficiently $\mathbb{P}$-generic real is Π_1^1-random.*

To prove Proposition 3.5 we observe the following (which we use later as well):

LEMMA 3.6. *Let $\mathcal{K}$ be a countable union of elements of $\mathbb{P}$, and suppose that every element of $\mathbb{P}$ intersects $\mathcal{K}$ positively (the intersection has positive measure). Then every sufficiently $\mathbb{P}$-generic real is in $\mathcal{K}$.*

Note that the union is not required to be uniform.

Proof. If $\mathcal{K} = \bigcup_n \mathcal{F}_n$, with $\mathcal{F}_n \in \mathbb{P}$, the dense set is the set of $\mathcal{F} \in \mathbb{P}$ such that $\mathcal{F} \subseteq \mathcal{F}_n$ for some n. □

In particular, Lemma 3.6 applies to all open sets (as all nonempty clopen sets are elements of $\mathbb{P}$). And Proposition 3.5 follows from Theorem 3.4, as the complement of $\mathcal{G} - \mathcal{G}^*$ (where $\mathcal{G}$ is Π_2^{ck}) is a union (nonuniform) of elements of $\mathbb{P}$, and it is conull.

4. Lowness, cupping, and computing c.e. sets

4.1. Lowness for Π_1^1-randomness. Theorem 3.4 helps us here to solve the question of lowness for Π_1^1-randomness [**Nie09**, question 9.4.11]: Is there some

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sequence A which is not Δ^1_1 and such that the largest $\Pi^1_1(A)$ set equals the largest Π^1_1 set? We answer the question by the negative, in a strong sense.

THEOREM 4.1. *If A is not hyperarithmetic, then some Π^1_1-random is not $\Pi^1_1(A)$-*ML*-random.*

We then improve this result in Theorem 4.3 by solving the cupping question for Π^1_1-randomness, showing that a nonhyperarithmetic A can be cupped above O by a Π^1_1-random sequence. However, the direct proof of Theorem 4.1 is simpler and we believe is interesting in its own right. Indeed the second proof elaborates on the simpler one. Our proof can be transfered in a straightforward way to the lower setting, simplifying the proof that a non-K-trivial is not low for weak 2-randomness [**DNWY06**].

The proof is based on a result of Hjorth and Nies: only the Δ^1_1 sets are low for higher ML-randomness. Here they use full relativization. That is, they show that if A is not hyperarithmetic then $\Pi^1_1(A)$-ML-randomness is strictly stronger than Π^1_1-ML-randomness. This does not use the continuous relativization introduced in [**BGM**] (for which the higher K-trivials are indeed low for randomness). Their argument is a dichotomy: either A is not higher K-trivial, in which case the usual arguments show that it is not low for higher ML-randomness; or it is, but in that case it collapses ω_1^{ck}, which gives it sufficient power to derandomize some Π^1_1-ML-random reals. One of the effects of the continuous relativization is to prevent K-trivials from using this extra power. In this section we only use full relativization.

Our first step is a higher version of Kjos-Hanssen's characterization of lowness for Martin-Löf randomness [**KH07**]. Given Proposition 3.2, the argument is identical; we give a proof for completeness.

LEMMA 4.2. *Suppose that A is not hyperarithmetic. Let $\mathcal{U}$ be a $\Pi^1_1(A)$ open set which contains all reals which are not $\Pi^1_1(A)$-ML-random. Then $\mathcal{U}$ positively intersects every higher effectively closed set of positive measure.*

Proof. As mentioned, we use the fact that A is not low for Π^1_1-ML-randomness. Let X be Π^1_1-ML-random which is not $\Pi^1_1(A)$-ML-random. Let $\mathcal{P}$ be a non-null Π_1^{ck} set. By Kučera's Proposition 3.2, there is a tail Y of X in $\mathcal{P}$. Since Y is not $\Pi^1_1(A)$-ML-random, $Y \in \mathcal{U}$, so $\mathcal{U} \cap \mathcal{P} \neq \varnothing$. Indeed this intersection must have positive measure; say $\sigma \prec Y$ and $[\sigma] \subseteq \mathcal{U}$; then $[\sigma] \cap \mathcal{P}$ is non-null, as it contains Y. □

Proof of Theorem 4.1. Let $A \notin \Delta^1_1$; let $\langle \mathcal{U}_n \rangle$ be the universal $\Pi^1_1(A)$-ML-test. By Lemmas 4.2, 3.6 and Proposition 3.5, a sufficiently $\mathbb{P}$-generic real is both Π^1_1-random and an element of $\bigcap_n \mathcal{U}_n$, that is, not $\Pi^1_1(A)$-ML-random. ☐

4.2. Cupping with a Π^1_1-random. Recall that a real X is *higher random-cuppable* (or Π^1_1-random-cuppable) if there is a Π^1_1-random sequence Z such that $X \oplus Z \geqslant_h O$, equivalently $X \oplus Z$ collapses ω_1^{ck}. No Δ^1_1 real is higher random-cuppable. We show here that every other real is higher random-cuppable. Note that if $A \not\geqslant_h O$, then a Π^1_1-random cupping partner of A cannot be $\Pi^1_1(A)$-random; so this result implies that only hyperarithmetics are low for Π^1_1-randomness (Theorem 4.1 gives a slightly stronger form of that). In particular the following is an improvement of the lowness result:

THEOREM 4.3. *If A is not hyperarithmetic then for all $Y \in 2^\omega$ there is some Π^1_1-random Z such that $Y \leqslant_h A \oplus Z$.*

Chong *et al.* (Together with Slaman and Harrington) **[CNY08]** proved the following relation between cuppability and lowness: A real is low for Π^1_1-randomness if and only if it is low for Δ^1_1-randomness and is not higher random-cuppable. Unfortunately, the equivalence of lowness for Π^1_1-randomness, and of Π^1_1-random noncuppability, with being hyperarithmetic, make this result less interesting. We, however, have some hope that an analogous characterization (with possibly a similar proof) will find its use with Σ^1_1-genericity; see Proposition 7.9 below.

The cupping result is very similar to another cupping result of Greenberg *et al.* **[GMMTar]**; they show that if $A \not\leqslant_{\mathrm{LR}} B$ then A can be cupped (in the Turing degrees) with B-ML-randoms arbitrarily high.

As usual in the higher setting, we need to deal with the fact that a Π^1_1 open set does not necessarily have a Π^1_1 prefix-free representation, but we need something different from Lemma 3.1.

Let us consider the general plan. We are given A which is not hyperarithmetic and some $Y \in 2^\omega$. We construct Z as a sequence $Y(0)\sigma_0 Y(1)\sigma_1 \cdots$ with each $[\sigma_n] \subseteq \mathcal{U}$, where $\mathcal{U}$ is a $\Pi^1_1(A)$ open set of small measure (say less than 0.1) which contains all reals which are not $\Pi^1_1(A)$-ML-random, say a component of the universal $\Pi^1_1(A)$-ML-test. To make Z Π^1_1-random we use Theorem 3.4. We construct a sequence $\mathcal{P}_0 \supseteq \mathcal{P}_1 \supseteq \cdots$ of Π^{ck}_1 sets of positive measure and ensure that $Z \in \bigcap_n \mathcal{P}_n$. The sequence $\langle \mathcal{P}_n \rangle$ will generate a filter in $\mathbb{P}$ (the partial ordering of all Π^{ck}_1 sets of positive measure), sufficiently generic as to ensure that Z is Π^1_1-random.

Let $\tau_n = Y(0)\sigma_0 Y(1)\sigma_1 \cdots Y(n-1)\sigma_{n-1} Y(n)$ (so $\tau_0 = Y(0)$). The inductive hypothesis is:

$$\lambda(\mathcal{P}_n \mid \tau_n) > 0.1 \tag{$\star$}$$

(where recall that $\lambda(\mathcal{R} \mid \tau)$ is the conditional probability of $\mathcal{R}$ given τ, namely $\lambda(\mathcal{R} \cap [\tau])/2^{-|\tau|}$). We start with $\mathcal{P}_0 = 2^\omega$ so equation ($\star$) holds for $n = 0$. Given τ_n, to define σ_n we use the following claim, which is identical to one proved in [**GMMTar**]:

Claim 4.4. *Let U be a Π^1_1 set of strings generating $\mathcal{U}$. For any string τ and any Π^{ck}_1 set $\mathcal{P}$ such that $\lambda(\mathcal{P} \mid \tau) > 0.1$ there is some σ such that $\sigma \in U$ and $\lambda(\mathcal{P} \mid \tau\sigma) \geqslant 0.8$.*

Proof. First we find an extension ρ of τ such that $[\rho] \nsubseteq \tau\mathcal{U}$ (the latter is of course $\{\tau X : X \in \mathcal{U}\}$), and such that $\lambda(\mathcal{P} \mid \rho) > 0.9$. This is done with the Lebesgue density theorem. Letting $\mathcal{G} = 2^\omega - \tau\mathcal{U}$, as $\lambda(\mathcal{G} \mid \tau) > 0.9$ and $\lambda(\mathcal{P} \mid \tau) > 0.1$, we must have $\lambda(\mathcal{G} \cap \mathcal{P} \mid \tau) > 0$ and by Lebesgue density theorem there is an extension ρ of τ such that $\lambda(\mathcal{G} \cap \mathcal{P} \mid \rho) > 0.9$. In particular we must have $\lambda(\mathcal{P} \mid \rho) > 0.9$ and $\mathcal{G} \cap [\rho]$ is nonempty.

Next we find an extension ν of ρ such that $[\nu] \subseteq \tau\mathcal{U}$ and $\lambda(\mathcal{P} \mid \nu) \geqslant 0.8$ as required. We let $\mathcal{Q}$ be the Π^{ck}_1 subset obtained from $\mathcal{P} \cap [\rho]$ by removing all cylinders in which the measure of $\mathcal{P}$ drops below 0.8. Formally

$$\mathcal{Q} = \{X \in \mathcal{P} \cap [\rho] : \forall n \geqslant |\rho|\ (\lambda(\mathcal{P} \mid X{\upharpoonright}_n) \geqslant 0.8)\}.$$

By considering the antichain of minimal strings removed we see that $\lambda(\mathcal{P} - \mathcal{Q} \mid \rho) \leqslant 0.8$. Since $\lambda(\mathcal{P} \mid \rho) > 0.9$ we see that $\lambda(\mathcal{Q} \mid \rho) > 0.1$. In particular, $\mathcal{Q}$ is a positive measure Π^{ck}_1 subset of $[\tau]$, and so by the choice of $\mathcal{U}$ and Lemma 4.2, $\tau\mathcal{U}$ intersects $\mathcal{Q}$. Choose $\nu \supseteq \rho$ such that $\nu = \tau\sigma$ for some $\sigma \in U$ and such that $[\nu] \cap \mathcal{Q} \neq \varnothing$. By the definition of $\mathcal{Q}$, $\lambda(\mathcal{P} \mid \nu) \geqslant 0.8$. □

Now the idea would be to take two steps. First, given τ_n, by equation ($\star$) and Claim 4.4 we find some σ_n such that $\sigma_n \in U$ and $\lambda(\mathcal{P}_n \mid \tau_n\sigma_n) \geqslant 0.8$. This determines τ_{n+1}. Then to define $\mathcal{P}_{n+1}$ we consider the next set in a list $\mathcal{S}_1, \mathcal{S}_2, \ldots$ of $\boldsymbol{\Sigma}^0_2$ sets which are each the union of Π^{ck}_1 sets, conull, and such that $\bigcap_k \mathcal{S}_k$ contains only Π^1_1-random sequences; this is given by Theorem 3.4. We then let $\mathcal{P}_{n+1} = \mathcal{P}_n \cap \mathcal{R}$, where $\mathcal{R} \subseteq \mathcal{S}_n$ is a Π^{ck}_1 set of sufficiently large measure so that $\lambda(\mathcal{P}_{n+1} \mid \tau_n\sigma_n) \geqslant 0.7$ equation ($\star$) for $n+1$ follows.

So far the construction is the same as in [**GMMTar**] (except that instead of Σ^{ck}_2 sets we use nonuniform unions of Π^{ck}_1 sets. This improvement, and Monin's analysis of forcing with Π^0_1 sets of positive measure, shows that the cupping partner built in that argument can be made not only weakly 2-random but also of hyperimmune-free degree.) However, we also need to show that $Y \leqslant_h A \oplus Z$. In [GMMTar] this is done by using a c.e. antichain which generates $\mathcal{U}$; then at each step the string σ_n is made to be an element of that antichain, and is so determined by Z (and using A as an oracle to enumerate this antichain). Here we need a new ingredient.

LEMMA 4.5. *Let $\mathcal{U}$ be a Π^1_1 open set. Then for every $\varepsilon > 0$ there is a Π^1_1 set of strings W (and a higher effective enumeration $\langle W_s \rangle$ of W) such that:*

- *$\mathcal{W} =^* \mathcal{U}$; and*
- *For every $s < \omega_1^{\mathrm{ck}}$, if $\sigma \in W_{s+1} - W_s$ then $\lambda(\mathcal{W}_s \mid [\sigma]) < \varepsilon$.*

(As usual, W (and its enumeration) can be obtained uniformly, but we do not use this.) To complete the proof of Theorem 4.3, we relativize Lemma 4.5 to A, apply it to $\mathcal{U}$ and $\varepsilon = 0.1$, and apply Claim 4.4 to $\mathcal{W}$ instead of $\mathcal{U}$; since $\mathcal{W} =^* \mathcal{U}$ it is still the case that $\mathcal{W}$ intersects all Π^{ck}_1 sets of positive measure. We further note that applying the lemma we can take $\sigma \in W$: examining the proof of the lemma, we can take ν to be any extension of ρ such that $[\nu] \subseteq \mathcal{W}$ and $[\nu] \cap \mathcal{Q} \neq \varnothing$. The plan then would be to throw $\tau_n \mathcal{W}_{s_n}$ out of $\mathcal{P}_{n+1}$ (where $\sigma_n \in W_{s_n+1} - W_{s_n}$); this will determine σ_n given Z.

Proof of Lemma 4.5. Let U be a Π^1_1 set of strings generating $\mathcal{U}$. As above we assume that at most one string enters U at each stage. We enumerate W: say $\sigma \in U_{s+1} - U_s$. Let

$$G_s = \{\tau \succcurlyeq \sigma \ : \ \lambda(\mathcal{U}_s \mid \tau) < \varepsilon\}.$$

This is Δ^1_1. We let $W_{s+1} - W_s$ consist of a Δ^1_1 prefix-free set of strings which generates $\mathcal{G}_s$ (for example the minimal strings in G_s). Note that $\mathcal{W}_s \subseteq \mathcal{U}_s$ (and so $\mathcal{W} \subseteq \mathcal{U}$).

By induction on s we show that $\lambda(\mathcal{U}_s - \mathcal{W}_s) = 0$. It suffices to show that for $\sigma \in \mathcal{U}_{s+1} - \mathcal{U}_s$,

$$[\sigma] =^* \mathcal{G}_s \cup (\mathcal{W}_s \cap [\sigma]) .$$

Suppose not. Then by the Lebesgue density theorem there is some $\tau \succcurlyeq \sigma$ such that $\lambda(\mathcal{G}_s \cup \mathcal{W}_s \mid \tau) < \varepsilon$. Since $\mathcal{W}_s \subseteq \mathcal{U}_s$, we see that $\tau \in G_s$, which is impossible.

It remains to show that $\lambda(\mathcal{W}_s \mid \tau) < \varepsilon$ for any $\tau \in W_{s+1} - W_s$. But such τ is an element of G_s, so $\lambda(\mathcal{U}_s \mid \tau) < \varepsilon$; and $\mathcal{U}_s =^* \mathcal{W}_s$. □

Proof of Theorem 4.3. We briefly give the rest of the details. Let $\mathcal{W}$ and $\mathcal{S}_1$, $\mathcal{S}_2, \ldots$ as discussed above. We define the sequence $\sigma_0, \sigma_1, \ldots$ as above, which determines τ_n. We also let s_n be the stage s such that $\sigma_n \in W_{s+1} - W_s$. In addition to equation ($\star$) we ensure that for all n,

$$\mathcal{P}_{n+1} \cap \tau_n \mathcal{W}_{s_n} = \varnothing. \tag{$\star\star$}$$

The only modification to the construction discussed above is the definition of $\mathcal{P}_{n+1}$. Given σ_n, because $\lambda(\mathcal{W}_{s_n} \mid \sigma_n) < 0.1$, we know that $\lambda(\mathcal{P}_n - \tau_n \mathcal{W}_{s_n} \mid \tau_n \sigma_n) \geqslant 0.7$, and we let $\mathcal{P}_{n+1} = (\mathcal{P}_n - \tau_n \mathcal{W}_{s_n}) \cap \mathcal{Q}$, where $\mathcal{Q} \subseteq \mathcal{S}_n$ is sufficiently large so that $\lambda(\mathcal{P}_{n+1} \mid \tau_n \sigma_n) \geqslant 0.69$; then equation ($\star$) still holds, and equation ($\star\star$) as well.

Now to recover Y from $A \oplus Z$ in a hyperarithmetic way, we observe that no initial segment of $Z - \tau_n$ is enumerated into W prior to stage $s_n + 1$, and so σ_n is the first initial segment of $Z - \tau_n$ enumerated into W. □

4.3. Hirschfeldt–Miller for Π^1_1-randomness.

Here we prove the following analogue of the Hirschfeldt–Miller characterization of weak 2-randomness.

THEOREM 4.6. *Let Z be higher Martin-Löf random. The following are equivalent:*

(1) *Z is Π^1_1-random.*

(2) *Z does not higher Turing compute a Π^1_1 set which is not Δ^1_1.*

Proof. (1)$\Longrightarrow$(2): This is the easy direction. It is well-known (Spector; see [Sac90, II.7.1]) that if A is any Π^1_1 set which is not Δ^1_1 then A collapses ω_1^{ck}. If $Z \geqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} A$ then $Z \geqslant_h A$ and so Z too collapses ω_1^{ck}, so is not Π^1_1-random.

(2)$\Longrightarrow$(1): The idea follows the standard Hirschfeldt–Miller construction, which can be described using cost functions. Recall that construction. We are given a ML-random set Z which is captured by some weak 2-test $\langle \mathcal{U}_n \rangle$. This gives an X-computable function $t^X(n)$: the stage at which X enters $\mathcal{U}_n$. We want to enumerate a c.e. set A whose settling-time function is bounded by t^X. That is, we want $A(n) = A_{t^X(n)}(n)$. Hence, enumerating n into A_{s+1} incurs a *cost*: in this case, the measure of $\mathcal{U}_{n,s}$. Any c.e. set obeying this cost will be Z-computable. For example, we can allow the e^{th} Friedberg–Muchnik requirement to spend 2^{-e}. So the algorithm for enumerating A is: for each e, if the e^{th} requirement is not met already, and we see some $n \in W_{e,s}$ whose cost is at most 2^{-e}, then we enumerate such n into A_{s+1} (we insist that $n \geqslant 2e$ so that A is coinfinite). The collection of oracles which are wrong on some input forms a Solovay test, and so Z will correctly compute A. The fact that the measure of $\mathcal{U}_n$ approaches 0 shows that if W_e is infinite, then it will get to act, as the cost of large n is always small.

To prove our theorem, we use Theorem 3.4: there is a Π^{ck}_2 set $\mathcal{G}$ such that $Z \in \mathcal{G}$, but Z is not an element of any Π^{ck}_1 subset of $\mathcal{G}$. Say $\mathcal{G} = \bigcap_n \mathcal{U}_n$. The measure of

$\mathcal{U}_n$ may not go to 0, but we know (in the notation of the proof of Theorem 3.4) that $\mathcal{G} - \mathcal{G}^*$ is null. So we let, for $n < \omega$ and $s < \omega_1^{\mathrm{ck}}$,

$$\mathbf{c}(n, s) = \lambda(\mathcal{U}_{n,s} - \mathcal{G}_s^*)$$

where recall that $\mathcal{G}_s = \bigcap_n \mathcal{U}_{n,s}$; from the proof of Theorem 3.4, $\mathcal{G}_s^*$ is the union of all Δ_1^1 closed subsets of $\mathcal{G}_s$. The construction is the same: let $\langle W_e \rangle$ be an effective list of all Π_1^1 subsets of ω. At stage $s < \omega_1^{\mathrm{ck}}$, the e^{th} requirement is already satisfied if $A_s \cap W_{e,s} \neq \varnothing$. If it is not already satisfied and there is some $n \geqslant 2e$ such that $\mathbf{c}(n, s) \leqslant 2^{-e}$ then we enumerate such n into A_{s+1}.

Define $\Phi(\sigma, n) = A_s(n)$ if $[\sigma] \subseteq \mathcal{U}_{n,s} - \mathcal{U}_{n,<s}$. This defines a higher Turing reduction. Certainly $\Phi(Z, n)\downarrow$ for all n. To show that it is wrong only finitely often we enumerate a higher Solovay test $\langle \mathcal{V}_n \rangle$: if n enters A_{s+1} then we let $\mathcal{V}_n = \mathcal{U}_{n,s} - \mathcal{F}$ where $\mathcal{F} \subseteq \mathcal{G}_s^*$ is a Δ_1^1 closed set, chosen so that $\lambda(\mathcal{V}_n) \leqslant \mathbf{c}(n, s) + 2^{-n}$ (that is, we choose $\mathcal{F}$ such that $\lambda(\mathcal{G}_s^* - \mathcal{F}) \leqslant 2^{-n}$). Note that we cannot take $\mathcal{V}_n = \mathcal{U}_{n,s} - \mathcal{G}_s^*$, as this may not be open. The total weight of the test $\langle \mathcal{V}_n \rangle$ is bounded by the sum of $\sum_e 2^{-e}$ (the total costs paid by the requirements enumerating A) and $\sum_n 2^{-n}$ (the excess to the cost that we added to make $\mathcal{V}_n$ open). If $Z \notin \mathcal{V}_n$ then as $Z \notin \mathcal{G}^*$, it must be that $\Phi(Z, n) = A(n)$.

It only remains to show that each requirement is met. Again this is a measure calculation: since $\lambda(\mathcal{G} - \mathcal{G}^*) = 0$, for sufficiently large n, $\lambda(\mathcal{U}_n - \mathcal{G}^*)$ is small, and for sufficiently large s, $\lambda(\mathcal{G}^* - \mathcal{G}_s^*)$ is small as well. □

As mentioned above, in [**BGM**] it is shown that Π_1^1-randomness differs from higher weak 2-randomness. It follows that there is a higher weakly 2-random sequence which higher Turing computes a Π_1^1 set which is not Δ_1^1.

5. Randomness and the higher arithmetic hierarchy

In this section we investigate randomness notions arising from the higher arithmetical hierarchy. For a lightface pointclass Γ, say that a real is *Γ-random* if it avoids all null sets in Γ. We consider the notions of Π_n^{ck}- and Σ_n^{ck}-randomness. We see that we get exactly four randomness notions, linearly ordered by strength:

(1) Higher Kurtz randomness;

(2) Δ_1^1-randomness;

(3) higher weak 2-randomness;

(4) Π_1^1-randomness.

First, observe that we can dispense with Σ^{ck}_n-randomness. For $n = 1$, the notion is trivial, as no nonempty open sets are null. Otherwise, we easily see that $\Sigma^{\mathrm{ck}}_{n+1}$-randomness is Π^{ck}_n-randomness.

Next, recall that the higher arithmetic hierarchy is separated into two strands: the classes Π^{ck}_1, Π^{ck}_3, Π^{ck}_5, ... consisting of Σ^1_1 sets, and the classes Π^{ck}_2, Π^{ck}_4, ... consisting of Π^1_1 sets (see Figure 1).

Sacks noted that Σ^1_1-randomness is the same as Δ^1_1-randomness. Chong, Nies and Yu [**CNY08**] showed:

- Π^{ck}_1-randomness (higher Kurtz randomness) is strictly weaker than Δ^1_1-randomness; and
- Π^{ck}_3-randomness is Δ^1_1-randomness.

It follows that Π^{ck}_3, Π^{ck}_5, ...-randomness are all the same, namely Δ^1_1-randomness.

On the Π^1_1 side, higher weak 2-randomness is defined as Π^{ck}_2-randomness; as mentioned above, this is distinct from Π^1_1-randomness. The classification of Π^{ck}_n-randomness is completed by the following theorem:

THEOREM 5.1. *Π^{ck}_4-randomness is Π^1_1-randomness.*

Again, it follows that Π^{ck}_4, Π^{ck}_6, Π^{ck}_8, ...-randomness are all the same, namely Π^1_1-randomness.

5.1. The proof of Theorem 5.1. As discussed in Section 3, we use higher functionals which induce functions from 2^ω to $(\omega^{\mathrm{ck}}_1)^\omega$. We cannot guarantee that such functionals are consistent everywhere.

We need to cover the set of non-Π^1_1-random sequences by topologically simple sets, namely null Π^{ck}_4 sets. The first step is obtaining cofinal ω-sequences in ω^{ck}_1 in a continuous fashion.

LEMMA 5.2. *If Z is Π^1_1-ML-random but not Π^1_1-random then there is an ω-sequence cofinal in ω^{ck}_1 which is higher Turing reducible to Z.*

Proof. For a quick proof we use Theorem 4.6. Let A be Π^1_1 and not Δ^1_1, and let Ψ be a higher Turing functional such that $\Psi(Z) = A$. Define $\Phi(X, n) = s$ if $\Psi(X)\restriction_n\downarrow$ and s is the least such that $\Psi(X)\restriction_n = A_s\restriction_n$. Since A is not Δ^1_1, $\langle\Phi(X, n)\rangle$ is unbounded in ω^{ck}_1 (this is proved in [**BGM**]).

If we would like a more direct proof we can appeal to Theorem 3.4 (and its proof). Let $\mathcal{G} = \bigcap_n \mathcal{U}_n$ be a Π^{ck}_2 set such that $Z \in \mathcal{G} - \mathcal{G}_{<\omega^{\mathrm{ck}}_1}$. We let $\Phi(X, n) = s$ if

$X \in \mathcal{U}_{n,s} - \mathcal{U}_{n,<s}$. This may be inconsistent because we might see an initial segment of X enter U_n, and then a shorter initial segment enter U_n later. By Lemma 3.3 and its proof, uniformly in $\varepsilon > 0$ we can modify Φ to a functional Φ_ε whose inconsistency set has measure at most ε, but preserving the totality of $\Phi(Z)$. The sequence of inconsistency sets of the functionals Φ_ε forms a higher ML-test, and so $\Phi_\varepsilon(Z)$ is consistent for some ε, and since $Z \notin \mathcal{G}_{<\omega_1^{\mathrm{ck}}}$, is unbounded in ω_1^{ck}. □

For a functional Φ mapping from 2^ω to $(\omega_1^{\mathrm{ck}})^\omega$, let $\mathcal{U}(\Phi)$, the unboundedness set of Φ, be the set of X such that $\Phi(X)$ is total, consistent and unbounded in ω_1^{ck}. Note that this set is null. Also let $\mathcal{E}(\Phi)$ be the inconsistency set of Φ.

PROPOSITION 5.3. *Let Φ be a higher functional mapping from 2^ω to $(\omega_1^{\mathrm{ck}})^\omega$. Then $\mathcal{U}(\Phi) \cup \mathcal{E}(\Phi)$ is Π_4^{ck}. This is uniform in the indices.*

Proof of Theorem 5.1, given Proposition 5.3. Since every Π_4^{ck} set is Π_1^1, it suffices to show that every sequence which is not Π_1^1-random is an element of some null Π_4^{ck} set. Let $Z \in 2^\omega$ be not Π_1^1-random. If Z is not Π_1^1-ML-random then Z is contained in a null Π_2^{ck} set (determined by the universal Π_1^1-ML-test). Otherwise, by Lemma 5.2 we obtain a functional Φ such that $\Phi(Z)$ is total, consistent and cofinal in ω_1^{ck}.

For each $\varepsilon > 0$, using Lemma 3.3 we modify Φ to a functional Φ_ε preserving the total and consistent Φ-computations but restricting the inconsistency set to have measure at most ε. By Proposition 5.3,

$$\mathcal{H} = \bigcap_{\varepsilon>0} \left(\mathcal{U}(\Phi_\varepsilon) \cup \mathcal{E}(\Phi_\varepsilon)\right)$$

is Π_4^{ck}. It is null, and contains Z. □

Proof of Proposition 5.3. Suppose that $\Phi(X)$ is total, but not necessarily consistent. We let $\Phi[X]$ be the closed subset of $(\omega_1^{\mathrm{ck}})^\omega$ consisting of all possible sequences $\langle \alpha_n \rangle$ such that for each n, α_n is a possible value for $\Phi(X, n)$. We let

$$\underline{\alpha}(X) = \min \left\{ \sup_n \alpha_n : \langle \alpha_n \rangle \in \Phi[X] \right\} = \sup_n \min \{\alpha : \Phi(X, n) = \alpha\};$$

and

$$\overline{\alpha}(X) = \sup \left\{ \sup_n \alpha_n : \langle \alpha_n \rangle \in \Phi[X] \right\} = \sup \{\alpha : \exists n\, (\Phi(X, n) = \alpha)\}.$$

Of course if $\Phi(X)$ is total and consistent then $\underline{\alpha}(X) = \overline{\alpha}(X) = \sup \Phi(X)$. What we want to do is to describe the set of X such that $\underline{\alpha}(X)$ is greater than every computable ordinal. But universal quantification over computable ordinals gives a Σ^1_1, rather than Π^1_1, set. The main idea is to use overspill: allow pseudo-ordinals as well.

Namely, let R be a Harrison linear ordering, and let $\langle R_k\rangle_{k<\omega}$ be the list of all principal initial segments of R (initial segments determined by a least upper bound). The list $\langle R_k\rangle$ is a list of uniformly computable linear orderings, containing one copy of each computable ordinal, and otherwise also Harrison linear orderings (whose well-founded initial segment has order type ω_1^{ck}).

For a Harrison linear ordering R let $\operatorname{otp}(R) = \infty$ and stipulate that $\alpha < \infty$ for every ordinal α. For each k, we let

$$\mathcal{S}_k = \{X : \Phi(X) \text{ is total and } \operatorname{otp}(R_k) < \overline{\alpha}(X)\}$$

and

$$\mathcal{L}_k = \{X : \Phi(X) \text{ is total and } \operatorname{otp}(R_k) > \underline{\alpha}(X)\}.$$

The set $\mathcal{S}_k$ is Π^{ck}_2: beyond totality, to find that $X \in \mathcal{S}_k$, working in $L_{\omega_1^{\mathrm{ck}}}$, we first find an ordinal β isomorphic to R_k, and then observe that for some n, $\Phi(X, n) > \beta$ (for some possible value of $\Phi(X, n)$); so beyond totality, this is in fact a Σ^{ck}_1 condition.

The set $\mathcal{L}_k$ is Σ^{ck}_3: $X \in \mathcal{L}_k$ if and only if there is some $m < \omega$ such that for all n, for some possible value α_n of $\Phi(X, n)$, α_n is embeddable into the initial segment $R_k(\leqslant m)$ (the initial segment of R_k determined by m); note that this embedding can be found in $L_{\omega_1^{\mathrm{ck}}}$.

Hence, the set $\mathcal{S}_k \cup \mathcal{L}_k$ is Σ^{ck}_3. If R_k is a Harrison linear ordering then $\mathcal{L}_k$ is the totality set of Φ. Hence

$$\bigcap_k (\mathcal{L}_k \cup \mathcal{S}_k) = \left\{X : \Phi(X) \text{ is total, and either } \underline{\alpha}(X) < \overline{\alpha}(X) \text{ or } \underline{\alpha}(X) = \omega_1^{\mathrm{ck}}\right\}.$$

The intersection $\bigcap_k (\mathcal{L}_k \cup \mathcal{S}_k)$ is Π^{ck}_4. If $\underline{\alpha}(X) < \overline{\alpha}(X)$ then $\Phi(X)$ is inconsistent. It follows that

$$\mathcal{U}(\Phi) \cup \mathcal{E}(\Phi) = \mathcal{E}(\Phi) \cup \bigcap_k (\mathcal{L}_k \cup \mathcal{S}_k)$$

is the union of a Σ^{ck}_1 set and a Π^{ck}_4 set, and so is Π^{ck}_4. ☐

5.2. The complexity of the set of Π^1_1-randoms.

We now consider the complexity of the largest null Π^1_1 set. The following theorem says it is Σ^{ck}_5.

THEOREM 5.4. *The set of Π^1_1-randoms is Π^{ck}_5.*

Proof. The proof of Theorem 5.1 is uniform. Using the projetum function p, we can give a ω_1^{ck}-effective ω-list $\Phi_0, \Phi_1, \Phi_2, \ldots$ of all functionals mapping from 2^ω to $(\omega_1^{\mathrm{ck}})^\omega$. We then define, for each $i < \omega$ and $\varepsilon > 0$, $\Phi_{i,\varepsilon}$ as in the proof of Theorem 5.1: restricting the inconsistency set to have measure bounded by ε. We then let $\mathcal{H}_i = \bigcap_{\varepsilon>0} \left(\mathcal{U}(\Phi_{i,\varepsilon}) \cup \mathcal{E}(\Phi_{i,\varepsilon})\right)$. Also let $\mathcal{R}$ be the Π^{ck}_2 null set of non-Π^1_1-ML-randoms. Then $\mathcal{H} = \mathcal{R} \cup \bigcup_i \mathcal{H}_i$ is Σ^{ck}_5, null, and contains all non-Π^1_1-random sequences; as it is Π^1_1, it equals the set of non-Π^1_1-randoms. □

As mentioned above, the set of Π^1_1-randoms is not Π^{ck}_3 [**BGM**]: every conull Π^{ck}_3 set contains an element which collapses ω_1^{ck}. This leaves the question of whether it is Σ^{ck}_4 or not. At present we do not know how to resolve this question. It is related to whether we can improve the Π^{ck}_3 result to sets of positive measure. (In fact, every conull Π^{ck}_3 set contains a real with a finite-change approximation; such a real is not even higher weak 2-random. We do not know whether there is a Σ^{ck}_3 set of measure less than 1, containing all sequences with finite-change approximations.)

PROPOSITION 5.5. *The set of Π^1_1-randoms is Σ^{ck}_4 if and only if there is some Π^{ck}_3 set of positive measure containing only reals which preserve ω_1^{ck}.*

Proof. One direction is easy; a conull Σ^{ck}_4 set is the union of Π^{ck}_3 sets of positive measure.

Suppose that $\mathcal{H}$ is Π^{ck}_3 of positive measure, and contains only reals which preserve ω_1^{ck}. Let $\mathcal{K} = \bigcup_{\sigma \in 2^{<\omega}} \sigma\mathcal{H}$. Then $\mathcal{K}$ is Σ^{ck}_4, and by the Lebesgue density theorem has measure 1; and it contains only reals which preserve ω_1^{ck}. Intersecting with the Σ^{ck}_2 set of Π^1_1-ML-randoms, we can assume that $\mathcal{K}$ contains only Π^1_1-ML-randoms. It thus contains only Π^1_1-randoms, and is Σ^1_1. The set of Π^1_1-randoms is the smallest conull Σ^1_1 set, and so must equal $\mathcal{K}$. □

5.3. The complexity of the set of higher weak 2-randoms. What about the set of higher weakly 2-random sequences? It is not even immediately clear that this set is Σ^1_1. We know it is not Π^1_1; this follows from the fact that Σ^1_1-randomness is the same as Δ^1_1-randomness, which is strictly weaker than higher weak 2-randomness. As mentioned, every conull Π^{ck}_3 set must contain a sequence which is not higher weakly 2-random [**BGM**], so the set of higher weakly 2-randoms is not Π^{ck}_3.

THEOREM 5.6 (With Dan Turetsky). *The set of higher weakly 2-random sequences is* Π^{ck}_5.

In particular, it is indeed Σ^1_1. As with Π^1_1-randoms, we do not know if the set of higher weakly 2-random sequences is Σ^{ck}_4 or not.

Proof. We modify the proof of Theorem 5.4. We start with a modification of the direct proof of Lemma 5.2. With every Π^{ck}_2 set $\mathcal{G} = \bigcap_n \mathcal{U}_n$ we associate a higher functional Φ, defined as follows: $(\sigma, n, s) \in \Phi$ if $[\sigma] \subseteq \mathcal{U}_{n,s}$ and $\lambda(\mathcal{G}_s) = 0$. Now from an effective list $\mathcal{G}_0, \mathcal{G}_1, \ldots$ of all Π^{ck}_2 sets we obtain a list of the associated functionals $\Phi_0, \Phi_1, \ldots$. To each functional Φ_e and each $\varepsilon > 0$ we apply Lemma 3.3 to obtain a functional $\Phi_{e,\varepsilon}$. We then again let $\mathcal{H}_e = \bigcap_{\varepsilon>0} \left(\mathcal{U}(\Phi_{e,\varepsilon}) \cup \mathcal{E}(\Phi_{e,\varepsilon})\right)$ and $\mathcal{K} = \mathcal{R} \cup \bigcup_e \mathcal{H}_e$, where $\mathcal{R}$ is the set of non-Π^1_1-ML-randoms. As in the previous proof, this is Σ^{ck}_5. We want to show that $\mathcal{K}$ is the set of sequences which are not higher weakly 2-random.

In one direction, suppose that X is not higher weakly 2-random. Find some e such that $\lambda(\mathcal{G}_e) = 0$ and $X \in \mathcal{G}_e$. Since $\mathcal{R} \subseteq \mathcal{K}$, to show that $X \in \mathcal{K}$ we may assume that X is Δ^1_1-random. This implies that for all $s < \omega^{\mathrm{ck}}_1$, $X \notin \mathcal{G}_{e,s}$; so $X \in \mathcal{G}_e - \mathcal{G}_{e,<\omega^{\mathrm{ck}}_1}$.

Since $X \in \mathcal{G}_e$ and $\mathcal{G}_e$ is null, $\Phi_e(X)$ is total; it will be inconsistent. Let $\varepsilon > 0$. If $\Phi_{e,\varepsilon}(X)$ is consistent, then by (2) of Lemma 3.3, $\Phi_{e,\varepsilon}(X)$ is total. Since $X \notin \mathcal{G}_{e,<\omega^{\mathrm{ck}}_1}$, $\Phi_{e,\varepsilon}(X)$ is unbounded in ω^{ck}_1. So $X \in \mathcal{U}(\Phi_{e,\varepsilon}) \cup \mathcal{E}(\Phi_{e,\varepsilon})$. It follows that $X \in \mathcal{H}_e$, so $X \in \mathcal{K}$.

In the other direction, let $X \in \mathcal{K}$; we show it is not higher weakly 2-random. If $X \in \mathcal{R}$ then we are done. Suppose that $X \in \mathcal{H}_e$ for some e. Since we are assuming that X is Π^1_1-ML-random, there is some $\varepsilon > 0$ such that $X \notin \mathcal{E}(\Phi_{e,\varepsilon})$; so $X \in \mathcal{U}(\Phi_{e,\varepsilon})$. The fact that $\Phi_{e,\varepsilon}(X)$ is unbounded in ω^{ck}_1 implies that for all $s < \omega^{\mathrm{ck}}_1$, $\lambda(\mathcal{G}_{e,s}) = 0$—so $\lambda(\mathcal{G}_e) = 0$; the fact that $\Phi_{e,\varepsilon}(X)$ is total implies that $X \in \mathcal{G}_e$. ☐

6. Higher generic sequences

In the introduction we recalled the concepts of Γ-genericity (for Cohen forcing) and weak Γ-genericity for lightface pointclasses Γ. In this section we investigate these notions for the classes $\Gamma = \Delta^1_1, \Pi^1_1, \Sigma^1_1$.

We see that we get three distinct genericity notions, linearly ordered by strength: Σ^1_1-genericity implies Π^1_1-genericity which implies Δ^1_1-genericity. We further characterize Σ^1_1-generic sequences as those which are Δ^1_1-generic and preserve ω^{ck}_1—the category analogue of Theorem 2.1. We summarize our results in Figure 2.

$$\begin{tikzcd}
\text{weakly } \Sigma^1_1\text{-generic} \arrow[dr, leftrightarrow, bend left] \arrow[dd, leftrightarrow, bend right] & & & & \text{weakly } \Pi^1_1\text{-generic} \arrow[dl, leftrightarrow, bend right] \arrow[dd, leftrightarrow, bend left] \\
& \Sigma^1_1\text{-generic} \arrow[r] \arrow[dl, leftrightarrow, bend left] & \Pi^1_1\text{-generic} \arrow[r] & \Delta^1_1\text{-generic} \arrow[dr, leftrightarrow, bend right] & \\
\Delta^1_1\text{-generic} \wedge \omega_1^X = \omega_1^{\mathrm{ck}} & & & & \text{weakly } \Delta^1_1\text{-generic}
\end{tikzcd}$$

Figure 2. Higher genericity

We start by proving implications and equivalences; then we prove the analogue of Theorem 2.1; and then separate between the three genericity notions. We end the section by giving a characterization of Σ^1_1-genericity using finite-change dense sets.

6.1. Implications.

6.1.1. Δ^1_1-genericity. The closure of the class Δ^1_1 under arithmetic operations shows the equivalence of Δ^1_1- and weak Δ^1_1-genericity; and so the implication from weak Π^1_1-genericity to Δ^1_1-genericity. The equivalence of Δ^1_1-genericity with weak Π^1_1-genericity is similar to the equivalence of Δ^1_1-randomness and Σ^1_1-randomness. Suppose that $D \subseteq 2^{<\omega}$ is Π^1_1 and dense; let $\langle D_s \rangle_{s<\omega_1^{\mathrm{ck}}}$ be a higher effective enumeration of D. For each $\sigma \in 2^{<\omega}$, the appearance of some extension of σ into D is a ω_1^{ck}-c.e. event; by admissibility of ω_1^{ck}, we see that there is some $s < \omega_1^{\mathrm{ck}}$ such that D_s is dense; of course D_s is Δ^1_1.

6.1.2. Weak Σ^1_1-genericity. First we prove:

PROPOSITION 6.1. *Weak Σ^1_1-genericity implies Π^1_1-genericity.*

Let us consider the lower analogue of Proposition 6.1, which is true: weak Π^0_1-genericity implies 1-genericity. The argument is simple: given a c.e. open set $\mathcal{U}$, we find a computable set U generating $\mathcal{U}$. Then the set of strings which are either in U or have no extension in U generates the union of $\mathcal{U}$ with the complement of its interior, and is Π^0_1. In the higher setting we need to overcome the absence of nice generating sets for Π^1_1 open sets.

Proof of Proposition 6.1. Let $\mathcal{U}$ be Π^1_1 open; let $\langle U_s\rangle_{s<\omega_1^{\mathrm{ck}}}$ be a higher effective enumeration of a Π^1_1 set of strings U generating $\mathcal{U}$. By restraining some strings from entering U, we can modify the set U and its enumeration to ensure that for all $s < \omega_1^{\mathrm{ck}}$,

$$\text{for all } \sigma \in U_s, \text{ no proper extension of } \sigma \text{ is enumerated into } U_{s+1}. \qquad (\star)$$

As usual we also assume that at most one string is enumerated at each stage. Let F be the set of strings, no extension of which is ever enumerated into U. The set F is Σ^1_1. It is dense: suppose that $\sigma \notin F$. Let s be the least stage at which some extension of σ is enumerated into U; say that extension is τ. Then no proper extension of τ is ever enumerated into U, so for example $\tau 0 \in F$. Finally, suppose that $\sigma \in F$. If some predecessor ρ of σ is in U then $[\sigma] \subset \mathcal{U}$. Otherwise, by definition of F, $[\sigma]$ is a subset of the complement of $\mathcal{U}$. Hence every sequence meeting $\mathcal{F}$, also meets or avoids $\mathcal{U}$. □

We next use Proposition 6.1 to show the following:

PROPOSITION 6.2. *Weak Σ^1_1-genericity is equivalent to Σ^1_1-genericity.*

Proof. What we really prove is that the conjunction of weak Σ^1_1-genericity and Π^1_1-genericity implies Σ^1_1-genericity, and then appeal to Proposition 6.1. Suppose that G is weakly Σ^1_1-generic; let $\mathcal{F}$ be a Σ^1_1 open set (an open set generated by a Σ^1_1 set of strings F). An admissibility argument shows that the set W of strings which have no extension in F is Π^1_1: if every extension of σ is eventually extracted from F, we see this at a computable stage. If G meets $\mathcal{W}$ then it avoids $\mathcal{F}$. Otherwise it avoids $\mathcal{W}$: there is some $\sigma \prec G$ with no extension in W; this means that $\mathcal{F}$ is dense in $[\sigma]$. Since G is weakly Σ^1_1-generic and $\sigma \prec G$, it must meet $\mathcal{F}$. □

6.2. Preserving ω_1^{ck}.

Feferman [Fef64] proved that if G is sufficiently Cohen generic, then $\omega_1^G = \omega_1^{\mathrm{ck}}$. We give here the exact genericity notion that is required for G to preserves ω_1^{ck}.

THEOREM 6.3. *A Δ^1_1-generic sequence preserves ω_1^{ck} if and only if it is Σ^1_1-generic.*

A weaker version of one direction of Theorem 6.3 was first observed by Slaman and the first author (unpublished), namely that if G is Δ^1_1-generic and preserves ω_1^{ck} then it is Π^1_1-generic. For if W is dense along G, then the fact that

$\omega_1^G = \omega_1^{\mathrm{ck}}$ implies that some W_s is dense along G. A similar argument yields Σ_1^1-genericity as well. Again let G be Δ_1^1-generic and suppose that it preserves ω_1^{ck}. Let F be a Σ_1^1 set of strings, and suppose that G does not meet F. Let $\langle F_s \rangle_{s<\omega_1^{\mathrm{ck}}}$ be a coenumeration of F. For each n we consider the stage at which $G\restriction_n$ leaves F; since G preserves ω_1^{ck}, there is some $s < \omega_1^{\mathrm{ck}}$ such that F_s contains $G\restriction_n$ for no n. Since G is Δ_1^1-generic and does not meet F_s, it avoids F_s; since $F \subseteq F_s$, G avoids F as well.

The other direction of Theorem 6.3 is an effectivization of Feferman's proof. We first give the proof in modern set-theoretic terminology.

Proof of the other direction of Theorem 6.3. We consider the standard, set-theoretic forcing language and forcing relation for Cohen forcing, as interpreted in $L_{\omega_1^{\mathrm{ck}}}$. We use the fact that Cohen forcing is a set forcing in this model (unlike for example forcing with Δ_1^1 sets of positive measure, or hyperarithmetic Sacks forcing). By induction on the complexity of formulae we see that for the classes $\Gamma = \Delta_0, \Pi_1, \Sigma_1$, for any formula $\varphi \in \Gamma$ in the forcing language, the relation $p \Vdash \varphi$ (as interpreted in $L_{\omega_1^{\mathrm{ck}}}$) is Γ-definable in $L_{\omega_1^{\mathrm{ck}}}$. Further, the proof of the forcing theorem holds for these levels; if G is Σ_1^1-generic (and so also Π_1^1-generic), any Σ_1 or Π_1 formula holds in $L_{\omega_1^{\mathrm{ck}}}[G]$ if and only if it is forced by some initial segment of G.

Let G be Σ_1^1-generic. We need to show that the structure $L_{\omega_1^{\mathrm{ck}}}[G]$ is Σ_1-admissible. It suffices to show that it is Δ_0-admissible. Let φ be a Δ_0 formula; suppose that in $L_{\omega_1^{\mathrm{ck}}}[G]$, φ defines a function from ω into ω_1^{ck}; we need to show that this function is bounded below ω_1^{ck}. For all n let F_n be the set of conditions $p \in 2^{<\omega}$ which force (in $L_{\omega_1^{\mathrm{ck}}}$) that there is no $\alpha < \omega_1^{\mathrm{ck}}$ such that $\varphi(n, \alpha)$ holds. This is Π_1-definable in $L_{\omega_1^{\mathrm{ck}}}$ (in other words, is Σ_1^1); and so $\bigcup_n F_n$ is Σ_1^1 as well. (Note that before we know that G preserves ω_1^{ck}, we cannot claim that the formula $\exists n \forall \alpha\, (\neg\varphi(n, \alpha))$ is equivalent to a Π_1 formula; this uses admissibility in $L_{\omega_1^{\mathrm{ck}}}[G]$.) By assumption, G does not meet $\bigcup_n F_n$, and so it avoids it; say $p \prec G$ has no extension in $\bigcup_n F_n$. This means that for all n, densely below p we can find conditions which force some value $\alpha < \omega_1^{\mathrm{ck}}$ such that $\varphi(n, \alpha)$ holds. By admissibility (ranging over the extensions of p and of n), there is some $\gamma < \omega_1^{\mathrm{ck}}$ such that for each n, densely below p we can find conditions which force that $\varphi(n, \alpha)$ holds for some $\alpha < \gamma$. That is, p forces that for all $n < \omega$ there is some $\alpha < \gamma$ such that $\varphi(n, \alpha)$ holds. But this is a Δ_0 statement, and so holds in $L_{\omega_1^{\mathrm{ck}}}[G]$. $\square$

For the benefit of computability-oriented readers who may be uncomfortable with forcing over models of KP, we translate the proof to the language of computability. The proof resembles the proof of Theorem 3.4.

Proof of the other direction of Theorem 6.3. Let G be Σ^1_1-generic. Let Ψ be a Turing functional (not a higher functional!), which maps oracles to linear orderings. It suffices to show that if for all n, $\Psi^G(\leqslant n)$ is isomorphic to a computable ordinal, then these ordinals are bounded below ω_1^{ck}. Here we use the notation of the proof of Theorem 3.4. As in that proof, we let $\mathcal{A}_{n,\alpha}$ be the set of oracles X such that $\Psi^X(\leqslant n)$ is isomorphic to an ordinal shorter than α; we let $\mathcal{A}_n = \mathcal{A}_{n,\omega_1^{\mathrm{ck}}} = \bigcup_{\alpha<\omega_1^{\mathrm{ck}}} \mathcal{A}_{n,\alpha}$ and $\mathcal{A} = \bigcap_n \mathcal{A}_n$. The sets $\mathcal{A}_n$ are Π^1_1, and the sets $\mathcal{A}_{n,\alpha}$ (for $\alpha < \omega_1^{\mathrm{ck}}$) are Δ^1_1, uniformly in α.

The computability-theoretic translation of the forcing theorem is an effectivization of Baire's category theorem. For any Δ^1_1 set $\mathcal{K}$ we can effectively find a Δ^1_1 open set $\mathcal{U}$ which is equivalent to $\mathcal{K}$ in category; that is, $\mathcal{K} \bigtriangleup \mathcal{U}$ is meagre. As G is Δ^1_1-generic, $G \in \mathcal{K}$ iff $G \in \mathcal{U}$. We apply this to the sets $\mathcal{A}_{n,\alpha}$ to get open sets $\mathcal{U}_{n,\alpha}$. For each n, $\mathcal{U}_n = \bigcup_{\alpha<\omega_1^{\mathrm{ck}}} \mathcal{U}_{n,\alpha}$ is Π^1_1 open. We assume that for all n, $G \in \mathcal{A}_n$, and so $G \in \mathcal{U}_n$.

Let $\mathcal{F}$ be the interior of the complement of $\bigcap_n \mathcal{U}_n$. This is a Σ^1_1 open set, and G does not meet it; so G avoids it. This means that there is some $\sigma \prec G$ such that $\bigcap_n \mathcal{U}_n$ is dense in $[\sigma]$. By admissibility of ω_1^{ck}, there is some $\alpha < \omega_1^{\mathrm{ck}}$ such that $\bigcap_n \mathcal{U}_{n,\alpha}$ is dense in $[\sigma]$; in other words each $\mathcal{U}_{n,\alpha}$ is dense in $[\sigma]$. Again as G is Δ^1_1-generic, we see that $G \in \bigcap_n \mathcal{U}_{n,\alpha}$, so $G \in \bigcap_n \mathcal{A}_{n,\alpha}$, as required. □

6.3. Separations.

We now turn to the separations between the three notions of genericity we have analysed so far. These separations in fact are not difficult.

6.3.1. Π^1_1- and weak Π^1_1-genericity.

Π^1_1-genericity behaves very much as the higher analogue of 1-genericity. In particular, a familiar proof translates perfectly to give the following. Recall the notion of higher relative computability ($\leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}}$) which was defined in Section 3.

LEMMA 6.4. *A Π^1_1-generic sequence does not higher compute any Π^1_1 set which is not Δ^1_1.* □

On the other hand, some weakly Π^1_1-generic sequences do higher compute Π^1_1 sets. A standard construction (see for example [Nie09, 1.8.49]) shows the existence of a left-Π^1_1, weakly Π^1_1-generic sequence. By Lemma 6.4, such a sequence cannot be Π^1_1-generic.

6.3.2. Σ^1_1- and Π^1_1-genericity.

To separate between Σ^1_1- and Π^1_1-genericity we use Theorem 6.3. In [BGM] a higher analogue of the class of ω-computably approximable (also known as ω-c.e.) functions is introduced. The higher version of Shoenfield's limit lemma states that a function is O-computable if and only if it is the pointwise limit of a ω_1^{ck}-computable approximation $\langle f_s\rangle_{s<\omega_1^{\mathrm{ck}}}$. Such a function is higher ω-c.a. if the number of mind changes of the approximation is finite and furthermore hyperarithmetically bounded. An important fact proved in [**BGM**] is that any higher ω-c.a. function collapses ω_1^{ck}. Thus the separation we are after follows from:

LEMMA 6.5. *There is an ω-c.a., Π_1^1-generic sequence.*

The proof again is obtained by inserting the word 'higher' in appropriate places in the standard construction of an ω-c.a. 1-generic sequence; see for example [**Nie09**, 1.8.52].

We note a difference between randomness and genericity here. Above we showed that a Δ_1^1-random sequence collapses ω_1^{ck} if and only if it higher computes a nonhyperarithmetic Π_1^1 set. Lemmas 6.4 and 6.5 show that this equivalence fails for Δ_1^1-generic sequences.

On the other hand, another characterization of the randoms collapsing ω_1^{ck} (Lemma 5.2) does hold for generics:

PROPOSITION 6.6. *A Δ_1^1-generic sequence collapses ω_1^{ck} if and only if it higher computes an ω-sequence cofinal in ω_1^{ck}.*

Proof. Using Theorem 6.3, the proof is essentially the proof of the first direction of that theorem. Let G be a Δ_1^1-generic sequence which collapses ω_1^{ck}. By Theorem 6.3, G is not Σ_1^1-generic. Let F be a Σ_1^1 set of strings such that $G \in \overline{\mathcal{F}} - \mathcal{F}$. Define a higher Turing functional: $\Psi(\sigma, n) = s$ if $|\sigma| = n$ and $\sigma \in F_s - F_{s+1}$. The functional Ψ is consistent everywhere; $\Psi(G)$ is total since $G \notin \mathcal{F}$; and as G is Δ_1^1-generic, $\langle \Psi(G, n)\rangle$ must be unbounded in ω_1^{ck}. ☐

COROLLARY 6.7. *There is a sequence which higher computes a cofinal ω-sequence in ω_1^{ck}, but does not higher compute a nonhyperarithmetic Π_1^1 set.*

6.4. Finite-change dense sets.

As discussed in the introduction, some of the analogy between higher and lower genericity breaks down when considering relativization. As in the higher setting, Π_1^0- and weak Π_1^0-genericity coincide, and are strictly stronger than 1-genericity. However, Π_1^0-genericity is also equivalent to 2-genericity, whereas $\Pi_1^1(O)$-genericity is much stronger than Σ_1^1-genericity.

We can, however, find a special subclass of the dense $\Pi_1^1(O)$ open sets which does characterize Σ_1^1-genericity. Again from [**BGM**], recall the notion of *finite-change* approximable functions. This is a class wider than the class of higher

ω-c.a. functions; we drop the requirement for a Δ^1_1 bound on the number of mind changes.

DEFINITION 6.8. An open set $\mathcal{U}$ is *dense finite-change* if it is generated by the range of a finite-change approximable function $f : 2^{<\omega} \to 2^{<\omega}$ satisfying $\sigma \preccurlyeq f(\sigma)$ for all $\sigma \in 2^{<\omega}$.

THEOREM 6.9. *A sequence is Σ^1_1-generic if and only if it is and element of every dense finite-change open set.*

Proof. In one direction, we observe that all dense Σ^1_1 open sets are dense finite-change sets. Namely, if F is a dense Σ^1_1 set of strings, let $f(\sigma)$ be the length-lexicographic least element of F extending σ. Of course for this direction we use the equivalence of weak and nonweak Σ^1_1-genericity.

In the other direction, let f be a finite-change function defining a dense finite-change open set; let $\langle f_s \rangle$ be a finite-change approximation of f. We may assume that for all σ and s, $f_s(\sigma) \succcurlyeq \sigma$. For each $s < \omega_1^{\mathrm{ck}}$ let F_s be the set of strings which extend some string in the range of f_s. So each F_s is dense and upward-closed (closed under taking extensions of strings). Let $F = \bigcap_{s<\omega_1^{\mathrm{ck}}} F_s$. Then F is Σ^1_1. We show that F is dense and that $\mathcal{F}$ is a subset of the open set determined by the range of f.

For the latter, we show that every string in F extends some string in the range of f. For let $\tau \in F$. Let s be a stage such that $f_s(\sigma) = f(\sigma)$ for all $\sigma \preccurlyeq \tau$. The fact that $\tau \in F_s$ implies that τ extends some string in the range of f.

It remains to show that F is dense. By induction on $s \leqslant \omega_1^{\mathrm{ck}}$ we show that $\bigcap_{t<s} F_t$ is dense. Let $s \leqslant \omega_1^{\mathrm{ck}}$ and suppose, by induction, that for all $r < s$, $\bigcap_{t<r} F_t$ is dense.

Let $\sigma \in 2^{<\omega}$. There is some $r < s$ such that $\tau = f_t(\sigma)$ is constant for all $t \in [r, s)$. This is immediate if s is a successor ordinal (let $r = s - 1$); if s is a limit ordinal, we use the fact that the approximation $\langle f_t \rangle$ is finite-change. This means that τ and all of its extensions are elements of $\bigcap_{t\in[r,s)} F_t$. Now by induction, $\bigcap_{t<r} F_t$ is dense; let ρ be an extension of τ in $\bigcap_{t<r} F_t$. Then $\rho \in \bigcap_{t<s} F_t$ and extends σ. ☐

Actually, the proof above directly gives the equivalence of weak Σ^1_1-genericity and genericity for dense finite-change sets. This in turn implies Proposition 6.1; it is not too difficult to see that if W is Π^1_1, then the union of $\mathcal{W}$ and the interior of its complement is dense finite-change (let $f(\sigma) = \sigma$ until we see an extension in W; so we change $f(\sigma)$ at most once). We can thus use this characterization to give an alternative proof of Proposition 6.2.

7. Lowness for higher genericity

We consider lowness and cupping for the genericity notions investigated above. The definition of lowness is the same as for randomness: an oracle A is low for Γ-genericity if every Γ-generic sequence is also $\Gamma(A)$-generic. As for randomness here we use full relativizations.

7.1. Lowness for Π^1_1-genericity. Lowness is related to cupping. The Posner–Robinson theorem [PR81] states that for any noncomputable A and any X there is a 1-generic G such that $X \leqslant_{\mathrm{T}} A \oplus G$. This implies that lowness for 1-genericity coincides with being computable (see [Yu06]). The analogy between 1-genericity and Π^1_1-genericity holds in this respect as well. The Posner–Robinson proof gives the higher analogue of their theorem:

PROPOSITION 7.1. *If A is not hyperarithmetic then for all X there is some Π^1_1-generic sequences G such that $X \leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} A \oplus G$.*

Relativizing Lemma 6.4 to an oracle shows that for any A, for any sequence G which is $\Pi^1_1(A)$-generic, $O^A \nleqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} A \oplus G$ (in fact we get this with the relativization of $\leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}}$ to A, which is weaker). Hence lowness for Π^1_1-genericity coincides with being hyperarithmetic.

7.2. Lowness for Δ^1_1-genericity. Recall that weak 1-genericity is the lower analogue of weak Π^1_1-genericity, which coincides with Δ^1_1-genericity. Lowness for weak 1-genericity was characterized by Stephan and Yu [SY06] as being computably dominated and not diagonally noncomputable.

What is the higher analogue of this characterization? Computable domination has an obvious analogue:

DEFINITION 7.2. An oracle X is Δ^1_1-*dominated* if every $\Delta^1_1(X)$ function is bounded by a Δ^1_1 function.

It is less clear what the higher analogue of DNC is. We use a different characterization. If X is not high (in particular, if it is computably dominated), then X is not DNC if and only if it is *semitraceable*: every X-computable function is infinitely often equal to some computable function (Kjos-Hanssen *et al.* [KHMS11]).

DEFINITION 7.3. An oracle X is Δ^1_1-*semitraceable* if for every $\Delta^1_1(X)$ function f there is a Δ^1_1 function g such that $f(n) = g(n)$ for infinitely many n.

Greenberg and Miller [**GM09**] showed that lowness for weak 1-genericity and lowness for Kurtz randomness coincided. In the higher setting, lowness for higher Kurtz randomness (Π_1^{ck}-randomness, also equivalent to Δ_1^1-Kurtz-randomness) has been settled by Kjos-Hanssen *et al.* [**KHNSY10**], who showed it coincided with being both Δ_1^1-dominated and Δ_1^1-semitraceable.

All of this would lead us to expect that lowness for Δ_1^1-genericity has the same characterization. This is indeed the case, as we show here. This fact was also known to Kihara (unpublished).

THEOREM 7.4. *An oracle is low for Δ_1^1-genericity if and only if it is both Δ_1^1-dominated and Δ_1^1-semitraceable.*

The characterization of lowness for various notions of randomness and genericity usually passes through an intermediate notion, that of lowness for tests, or dense open sets. For example, Stephan and Yu prove the equivalence of:

(1) X is low for dense c.e. open sets: every dense open set which is c.e. in X is a superset of a dense, c.e. open set.

(2) X is low for weak 1-genericity.

(3) X is computably dominated and semitraceable.

Their argument is (1) → (2) → (3) → (1). For (2) → (3), they use the fact that every Turing degree which is not computably dominated computes a weakly 1-generic sequence. The higher analogue of this fact fails, as was shown by Kihara [**Kih**]: he constructs a function f dominated by no Δ_1^1 function such that there is no Δ_1^1-generic $G \leqslant_h f$.

Thus we need a new argument. What we do is independently prove the equivalence of the higher analogues of (1) and (3) (Proposition 7.5) and then the equivalence of the higher analogues of (1) and (2) (Theorem 7.6). The latter is a general argument which holds in the lower setting as well, giving directly the equivalence of lowness for weak 1-genericity and lowness for dense c.e. open sets. The higher analogue of (1) is being *low for Δ_1^1 dense open sets*: every $\Delta_1^1(X)$ dense open set is a superset of a Δ_1^1 dense open set.

PROPOSITION 7.5. *An oracle is low for Δ_1^1 dense open sets if and only if it is Δ_1^1-dominated and Δ_1^1-semitraceable.*

Proof. The direction from right to left is identical to the proof of (3) → (1) in [**SY06**], so we omit it.

For the converse, suppose that X is low for Δ_1^1 dense open set. Let $f \leqslant_h X$.

We first want to show that f is dominated by a Δ^1_1 function g. For this we may assume that f is nondecreasing and that $f(n) > 0$ for all n. Let

$$W = \left\{\sigma 0^{f(|\sigma|)} : \sigma \in 2^{<\omega}\right\}.$$

Let V be a Δ^1_1 set of string such that $\mathcal{V} \subseteq \mathcal{W}$. Since $\Delta^1_1(X)$ is closed under arithmetic operations, we may assume that every string in V extends a string in W. Define $g\colon \omega \to \omega$ by letting $g(n) = |\tau|$, where τ is the shortest extension of the string 1^n in V; g is Δ^1_1. For every n, as $1^m 0 \perp 1^n$ for $m < n$ it must be the case that τ extends a string $\sigma 0^{f(|\sigma|)}$ for some $\sigma \succcurlyeq 1^n$. This shows that for every n we have $g(n) \geqslant f(n)$.

Next, we show that f is infinitely often equal to some Δ^1_1 function h. Again, for simplicity, we may assume that $f(n) \geqslant 1$ for all n.

Define a function $b\colon \omega^{<\omega} \to 2^{<\omega}$ by letting

$$b(k_0, k_1, \dots, k_{n-1}) = 0^{k_0} 1 0^{k_1} 1 \cdots 0^{k_{n-1}} 1.$$

The function b is injective. As 0^0 is the empty string, the range of b is the collection of finite binary strings ending with a 1 (together with the empty sequence). Now define

$$W = \left\{b(\sigma \hat{}\langle f(n)\rangle) : n < \omega \text{ and } \sigma \in \omega^n\right\}.$$

Let V be a Δ^1_1 set of strings such that every string in V extends a string in W. Effectively from V, given any lower bound m, we can obtain some $n > m$ and a function $\rho\colon n \to \omega$ such that $\rho(k) = f(k)$ for some $k \geqslant m$. Given this, the construction of the function h is done by recursion; if h is defined up to some m, then we find ρ with lower bound m, and extend by copying the values (beyond m) given by ρ.

Given m, we find some $\tau \in V$ which extends the string 1^m. Let $\rho = b^{-1}(\tau 1)$. The string τ extends $b((\rho\restriction_k)\hat{} f(k))$ for some k; so $\rho(k) = f(k)$. And $k \geqslant m$, as we assumed that $f \geqslant 1$, and $\rho\restriction_m = 0^m$. ☐

THEOREM 7.6. *An oracle is low for Δ^1_1-genericity if and only if it is low for Δ^1_1 dense open sets.*

As mentioned above, the proof translates easily to directly show the equivalence of weak 1-genericity and lowness for c.e. dense open sets.

Proof. Let $X \in 2^\omega$. Suppose that some dense $\Delta^1_1(X)$ open set $\mathcal{U}$ contains no Δ^1_1 open set. Our goal is to build a Δ^1_1-generic sequence which is not an element of some other dense $\Delta^1_1(X)$ open set $\mathcal{V}$, built from $\mathcal{U}$. The main step is building the

$\Delta^1_1(X)$ dense open set $\mathcal{V}$ with the property that for every $\sigma \in 2^{<\omega}$, the set $\mathcal{V} \cap [\sigma]$ contains no Δ^1_1 open set dense inside $[\sigma]$.

Let $V_0 = \{\tau\}$ for some $\tau \in U$. Let $k_0 = |\tau|$. At stage $n+1$, for every string σ of length k_n we do the following. Let $\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_n = \sigma$ be all the prefixes of σ of length k_i for $i \leqslant n$. Put in V_{n+1} a string $\tau \succcurlyeq \sigma$ such that $[\tau] \subseteq \mathcal{U} \cap \sigma_0\mathcal{U} \cap \cdots \cap \sigma_n\mathcal{U}$ (here recall that $\sigma\mathcal{U} = \{\sigma\hat{\ }Y : Y \in \mathcal{U}\}$). Finally let k_{n+1} be the longest length among the lengths of the strings in V_{n+1}. Let $V = \bigcup_n V_n$.

By construction, $\mathcal{V}$ is dense. Let us prove that for every string σ the set $\mathcal{V} \cap [\sigma]$ contains no Δ^1_1 open set dense in $[\sigma]$. Let n be the smallest such that k_n is bigger than $|\sigma|$. It is enough to prove that for any extension τ of σ of length k_n, the set $\mathcal{V} \cap [\tau]$ contains no Δ^1_1 open set dense in $[\tau]$. But by construction we have $\mathcal{V} \cap [\tau] \subseteq \tau\mathcal{U}$; if $\tau\mathcal{W} \subseteq \mathcal{V}$ then $\mathcal{W} \subseteq \mathcal{U}$, and so cannot be dense and Δ^1_1 open.

We can now use $\mathcal{V}$ to build a Δ^1_1-generic sequence not in $\mathcal{V}$. Let $\mathcal{W}_1, \mathcal{W}_2, \ldots$ be an ω-enumeration of the Δ^1_1 dense open sets. We define a sequence of strings $\sigma_0 \prec \sigma_1 \prec \sigma_2 \cdots$ and let $G = \bigcup \sigma_i$. We ensure that $[\sigma_n] \subseteq \mathcal{W}_n$; this will ensure that G is Δ^1_1-generic. We start with σ_0 being the empty sequence. Given σ_n, because $\mathcal{U}_{n+1} \cap [\sigma] \nsubseteq \mathcal{V}$, we let σ_{n+1} be an extension of σ_n such that $[\sigma_{n+1}] \subseteq \mathcal{U}_{n+1}$ but $[\sigma_{n+1}] \nsubseteq \mathcal{V}$. The fact that $[\sigma_n] \nsubseteq \mathcal{V}$ for all n implies that $G \notin \mathcal{V}$. ☐

7.3. Lowness for Σ^1_1-genericity.

We do not know what lowness for Σ^1_1-genericity is.

QUESTION 7.7. *Is lowness for Σ^1_1-genericity different from being hyperarithmetic?*

Using the technique proving Theorem 7.6, we can prove that lowness for Σ^1_1-genericity coincides with lowness for finite-change dense open sets. Here again we take full relativizations. A function $f : \omega \to \omega$ is *X-finite-change* if there is an approximation $\langle f_s \rangle_{s<\omega_1^X}$, Δ_1-definable over $L_{\omega_1^X}[X]$, with only finitely many mind changes on each input.

THEOREM 7.8. *An oracle is low for Σ^1_1-genericity if and only if it is low for finite-change dense open sets.*

Proof. The idea is the same as in Theorem 7.6: given a X-finite-change dense open set $\mathcal{U}$ containing no finite-change dense open set, we define a X-finite-change dense open set $\mathcal{V}$ such that for every σ, the set $\mathcal{V} \cap [\sigma]$ contains no finite-change open set dense in σ. The second step is identical. All we have to do is to make sure is that the same construction works in this context. Let $\langle f_s \rangle_{s<\omega_1^X}$ be a finite-change approximation of a function $f : 2^{<\omega} \to 2^{<\omega}$ whose range generates $\mathcal{U}$. (If $X \geqslant_h O$ then certainly X is not low for Σ^1_1-genericity (there is an O-computable Σ^1_1-generic sequence), so we may assume that $\omega_1^X = \omega_1^{\mathrm{ck}}$.)

At every stage $s \leqslant \omega_1^X$ we apply the previous construction to the range of f_s. That is, we let $\mathcal{U}_s$ be the dense open set generated by the range of f_s; we build V_s as above. We can find a function g_s which generates V_s: we let $g_s(\varepsilon) = f_s(\varepsilon)$ (here ε is the empty string). Given σ of length $k_{n,s}$ and its initial segments $\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_n = \sigma$, each σ_i of length $k_{i,s}$, we define $g_s(\sigma)$ in $n+1$ many steps. Namely, for $i \leqslant n$ let $f^i = f^i_s(\sigma)$ be the function whose range generates $\sigma_i \mathcal{U}_s$: $f^i(\sigma_i \tau) = f_s(\tau)$. We let $g_s(\sigma) = f^0(f^1(\cdots f^n(\sigma)\cdots))$.

What we need to argue is that everything stabilizes with only finitely many mind changes. But this follows from $\langle f_s \rangle$ being finite-change. Suppose that on an interval I of stages, the values $k_{i,s}$ are stable for $i \leqslant n$ and the values $g_s(\sigma)$ are stable for every σ of length at most $k_{n-1,s}$. Then on this interval I, for each string σ of length $k_{n,s}$, each value $g_s(\sigma)$ can change at most finitely often (by induction, $f^n(\sigma)$ changes finitely often; then $f^{n-1}(f^n(\sigma))$ changes finitely often, and so on). Since there are only finitely many strings of length $k_{n,s}$, we see that $k_{n+1,s}$ changes only finitely often. □

As every Σ^1_1-generic sequence preserves ω_1^{ck}, we can also ask the question of cuppability, defined analogously here, as it was defined for Π^1_1-randomness in Section 4.2. We can prove an analogue of the characterization of Π^1_1-random cuppability in [**CNY08**].

PROPOSITION 7.9. *An oracle is low for Σ^1_1-genericity if and only if it is both low for Δ^1_1-genericity and is not Σ^1_1-generic-cuppable.*

Proof. Suppose that X is low for Δ^1_1-genericity and that $\omega_1^{X\oplus G} = \omega_1^{\mathrm{ck}}$ for every Σ^1_1-generic G. Let G be Σ^1_1-generic. Then G is $\Delta^1_1(X)$-generic and $\omega_1^{X\oplus G} = \omega_1^X$. Relativizing Theorem 6.3 to X, we see that G is $\Sigma^1_1(X)$-generic.

In the other direction, suppose that X is low for Σ^1_1-genericity. Then $\omega_1^X = \omega_1^{\mathrm{ck}}$, and again by Theorem 6.3, $\omega_1^{X\oplus G} = \omega_1^X$ for every $\Sigma^1_1(X)$-generic G, and so for every Σ^1_1-generic G. That is, X is not Σ^1_1-generic-cuppable.

We show that X is low for Δ^1_1-genericity. Suppose, for a contradiction, that some Δ^1_1-generic G fails to be $\Delta^1_1(X)$-generic. Let $\mathcal{F}$ be a $\Delta^1_1(X)$-meagre containing G; let $\mathcal{Q}$ be the set of Δ^1_1-generic sequences. This set is Σ^1_1. The set $\mathcal{F} \cap \mathcal{Q}$ is nonempty (it contains G) and $\Sigma^1_1(X)$. By the Gandy basis theorem (relativized to X), $\mathcal{F} \cap \mathcal{Q}$ contains an element H such that $\omega_1^H = \omega_1^X = \omega_1^{\mathrm{ck}}$. By Theorem 6.3, H is Σ^1_1-generic which fails to be even $\Delta^1_1(X)$-generic. □

As for lowness, Σ^1_1-cuppability remains unresolved:

QUESTION 7.10. *If A is not hyperarithmetic, is there a Σ^1_1-generic sequence G such that $A \oplus G \geqslant_h O$?*

By Theorem 6.3, the set of Σ^1_1-generic sequences is Σ^1_1. Question 7.10 is related to a more general question raised by Yu:

QUESTION 7.11. *Let $\mathcal{Q}$ be an uncountable Σ^1_1 set. If A is not hyperarithmetic, must there be some $Y \in \mathcal{Q}$ such that $A \oplus Y \geqslant_h O$?*

The closest result to date is by Chong and Yu [**CY**]: if $\mathcal{Q}$ and $\mathcal{P}$ are uncountable Σ^1_1 sets, then there are $X \in \mathcal{Q}$ and $Y \in \mathcal{P}$ such that $O \leqslant_h X \oplus Y$.

8. Equivalent test notions for Π^1_1-randomness

We saw how to capture Π^1_1-random sequence with Π^{ck}_4 sets of measure 0. We end this paper by giving two more test notions for Π^1_1-randomness.

8.1. Difference random style tests. Franklin and Ng [**FN11**] found a test notion which characterizes the incomplete Martin-Löf randoms. Informally they are exactly the sequences which are captured by sets which are Martin-Löf tests inside a Π^0_1 set. Following the same idea, Bienvenu, Greenberg and Monin [**BGM**] argue the following:

THEOREM 8.1. *For a Π^1_1-ML-random sequence Z, the following are equivalent:*

1. *Z is captured by a set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \mathcal{U}_n) \leqslant 2^{-n}$, where $\mathcal{F}$ is Π^{ck}_1 and each $\mathcal{U}_n$ is Σ^{ck}_1 (uniformly in n).*
2. *Z higher Turing computes Kleene's O.*

We shall see an analogous characterization for Π^1_1-randomness, in the same spirit as (1) in Theorem 8.1.

THEOREM 8.2. *For a sequence X, the following are equivalent:*

1. *There are a Π^{ck}_1 set $\mathcal{F}$ and a Π^{ck}_2 set $\mathcal{G}$ such that $X \in \mathcal{F} \cap \mathcal{G}$ and $\lambda(\mathcal{F} \cap \mathcal{G}) = 0$.*
2. *There are a Σ^1_1 set $\mathcal{F}$ and a Π^{ck}_2 set $\mathcal{G}$ such that $X \in \mathcal{F} \cap \mathcal{G}$ and $\lambda(\mathcal{F} \cap \mathcal{G}) = 0$.*
3. *X is not Π^1_1-random.*

Proof. (2) $\Longrightarrow$ (3): Suppose that X is captured by a null set $\mathcal{F} \cap \mathcal{G}$ as in (1). Then either $\omega_1^X > \omega_1^{\mathrm{ck}}$, in which case X is not Π_1^1-random, or there exists some $s < \omega_1^{\mathrm{ck}}$ such that $X \in \mathcal{G}_s$; so $X \in \mathcal{F} \cap \mathcal{G}_s$. The latter is a Σ_1^1 set of measure 0, implying that X is not Δ_1^1-random.

(3) $\Longrightarrow$ (1): This is similar to the Franklin–Ng argument. Suppose that X is not Π_1^1-random. If X is not Π_1^1-ML-random then (1) holds with $\mathcal{F} = 2^\omega$ and $\mathcal{G}$ the set of non-Π_1^1-ML-randoms. Otherwise, by Theorem 4.6, X higher Turing computes a Π_1^1 set A which is not hyperarithmetic, say via a higher functional Φ. By Lemma 3.3, uniformly in $\varepsilon > 0$ we find a higher functional Φ_ε such that $\Phi_\varepsilon(X) = A$ and the measure of the inconsistency set of Φ_ε is at most ε.

Let $\langle Y_s \rangle$ be a higher effective enumeration of A. For $\varepsilon > 0$ and $n < \omega$ we let

$$\mathcal{U}_{n,\varepsilon} = \bigcup_s \Phi_\varepsilon^{-1}(A_s\restriction_n) = \{Z \in 2^\omega : \exists s \ \ [Y_s\restriction_n \preccurlyeq \Phi_\varepsilon(Z)]\}$$

and let $\mathcal{G} = \bigcap_{n,\varepsilon} \mathcal{U}_{n,\varepsilon}$. We also let $\mathcal{F}$ be the set of oracles Z such that $\Phi(Z)$ does not lie to the left of A:

$$\mathcal{F} = \{Z \in 2^\omega : \neg\exists n\, (\Phi(Z, n) = 0 \text{ and } A(n) = 1)\}\,.$$

The set $\mathcal{F} \cap \mathcal{G}$ contains X, and is null. To see the latter, let $Z \in \mathcal{F} \cap \mathcal{G}$. Either $\Phi_\varepsilon(Z)$ is inconsistent for all ε. There are only null many such oracles. Otherwise, for some $\varepsilon > 0$, $\Phi_\varepsilon(Z) = A$. Since A is not hyperarithmetic, there are only null many oracles which higher compute A (the usual majority-vote argument holds, but we can also appeal to Sacks' theorem [**Sac90**, IV.2.4], which says that upper cones in the hyperdegrees are null).

(1) $\Longrightarrow$ (2) is immediate. ☐

8.2. Demuth style tests.

Bienvenu *et al.* give in [**BGM**] give a Demuth-style characterization of higher weak 2-randomness. Let $\langle \mathcal{U}_e \rangle_{e<\omega}$ be an effective list of all Σ_1^{ck} sets.

PROPOSITION 8.3. *The nested tests of the form* $\langle \mathcal{U}_{f(n)} \rangle$ *where* $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$ *and* f *has a finite-change approximation, precisely capture nonhigher weak 2-randoms.*

We now give a notion of test for Π_1^1-randomness, which has the same flavour as Proposition 8.3. Whereas Proposition 8.3 can be seen as a generalization that no sequence with a closed approximation is higher weak 2-random, the following characterization of Π_1^1-randomness can be seen as a generalization of the fact that no sequence with a collapsing approximation is Π_1^1-random.

THEOREM 8.4. *For a sequence X, the following is equivalent:*

(1) *X is not Π^1_1-random.*

(2) *X is captured by a set $\bigcap_n \mathcal{U}_{f(n)}$ with $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$, where f has a ω_1^{ck}-computable approximation $\langle f_s \rangle_{s<\omega_1^{\mathrm{ck}}}$ such that for every n, the sequence $\langle f_s(n) \rangle_{s<\omega_1^{\mathrm{ck}}}$ restricted to the stages s such that $X \in \mathcal{U}_{f_s(n)}$, changes finitely often.*

Proof. (2)$\Longrightarrow$(1): This is the easy direction. Let $\bigcap_n \mathcal{U}_{f(n)}$ be a test which captures some X following the hypothesis of (2). Note that we can always suppose that the approximation of f is partially continuous, that is for s limit, if the limit of $\langle f_t \rangle_{t<s}$ exists, then it is also equal to f_s. We can also always suppose that $\lambda(\mathcal{U}_{f_s(n)}) \leqslant 2^{-n}$ for any s and n, as it is harmless to trim $\mathcal{U}_{f_s(n)}$ if its measure becomes too big. Define $g\colon \omega \to \omega_1^{\mathrm{ck}}$ by $g(0) = 0$, and

$$g(n+1) = \min \left\{ s > g(n) : X \in \bigcap_{m \leqslant n} \mathcal{U}_{f_s(m),s} \right\}.$$

The function g is Δ_1-definable over $L_{\omega_1^{\mathrm{ck}}}[X]$. If $\sup_n g(n) = \omega_1^{\mathrm{ck}}$ then X collapses ω_1^{ck} and we have (1). Otherwise $s = \sup_n g(n) < \omega_1^{\mathrm{ck}}$. Also for each m, there exists some n such that $f_{g(n)}(m) = f_{g(k)}(m)$ for any $k \geqslant n$, as otherwise X would be in infinitely many versions of $\mathcal{U}_{f_s(m)}$. Therefore, $\lim_n f_{g(n)}$ exists and as the approximation is partially continuous, this limit is equal to f_s. But then $X \in \bigcap_m \mathcal{U}_{f_s(m)}$ and therefore it is not Π^1_1-ML-random.

(1)$\Longrightarrow$(2): Suppose that X is not Π^1_1-random. If X is not Π^1_1-ML-random then (2) holds easily. Otherwise we use Theorem 4.6 again. The sequence X higher Turing computes some nonhyperarithmetic, Π^1_1 set A, say via some functional Φ; we define the functionals Φ_ε as above; we assume that the measure of the inconsistency set of Φ_ε is strictly smaller than ε. Let, for $\varepsilon > 0$ and $\sigma \in 2^{<\omega}$,

$$\mathcal{W}(\varepsilon, \sigma) = \Phi_\varepsilon^{-1}[\sigma].$$

For $n < \omega$ and $s < \omega_1^{\mathrm{ck}}$ we let $m_s(n)$ be the least m such that

$$\lambda\left(\mathcal{W}(2^{-n}, A_s\restriction_m)_s\right) \leqslant 2^{-n}$$

and then let $\mathcal{U}_{f_s(n)} = \mathcal{W}(2^{-n}, A_s\restriction_{m_s(n)})$. (Since A is not hyperarithmetic, $\lim_{n\to\infty} \lambda(\Phi_\varepsilon^{-1}(A\restriction_n)) < \varepsilon$; by speeding up the enumeration of A, we may assume that such m exists for each n and s.)

The sequence $\langle f_s(n) \rangle$ stabilizes at a limit $f = f_{\omega_1^{\mathrm{ck}}}$, $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$ for all n, and $X \in \bigcap_n \mathcal{U}_{f(n)}$. It remains to show that for all n, there are only finitely many values of $f_s(n)$ such that $X \in \mathcal{U}_{f_s(n)}$.

Suppose that this is not the case. Let $s_0 < s_1 < \cdots$ be an ω-sequence of stages such that the values $f_{s_i}(n)$ are distinct and $X \in \mathcal{U}_{f_{s_i}(n)}$ for all i. Note that since $\langle f_s(n)\rangle$ reaches a limit, $s_\omega = \sup_i s_i < \omega_1^{\mathrm{ck}}$. We observe that the set $\{m_{s_i}(n) : i < \omega\}$ is unbounded in ω: for each m, the value of $A_s \restriction_m$ stabilizes below s_ω. For notational simplicity, we may assume that $A_{s_\omega} = \lim_{i \to \omega} A_{s_i}$.

Let $m < \omega$. There is some $i < \omega$ such that $A_{s_i} \restriction_m = A_{s_\omega} \restriction_m$ and $m_{s_i}(n) > m$. So $X \in \mathcal{U}_{f_{s_i}(n)}$ implies that $A_{s_\omega} \restriction_m \preccurlyeq \Phi(X)$. So $\Phi(X) = A_{s_\omega}$. But $\Phi(X) = A$ and A_{s_ω} is hyperarithmetic, a contradiction. ☐

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