2

Defining Equivariant Cohomology

We will introduce our definition of equivariant cohomology using finite-dimensional algebraic varieties, constructing a contravariant functor from spaces with G-action to rings, and compute several examples of Λ_G from this definition. First we need some basic facts about principal bundles, which predate equivariant cohomology and to some extent motivate its original construction.

2.1 Principal bundles

Before discussing the general setup, here is a special case which may be familiar. Suppose E is a complex vector bundle of rank n on a space Y, so it is trivialized by some open cover U_α. The transition functions (from $U_\alpha \cap U_\beta$ to GL_n) can be used to construct a principal GL_n-bundle. Explicitly, let

$$p\colon \mathrm{Fr}(E) \to Y$$

be the *frame bundle* of E, whose fiber over $y \in Y$ is the set of all ordered bases $(v_1, \ldots, v_n)$ of E_y. (This is also known as the *Stiefel variety* of E.) There is a natural right action of GL_n on $\mathrm{Fr}(E)$, given by

$$(v_1, \ldots, v_n) \cdot g = (w_1, \ldots, w_n), \qquad \text{where } w_j = \sum_{i=1}^{n} g_{ij}\, v_i,$$

and over an open set $U \subseteq Y$ where E is trivial, the isomorphism

$$\begin{tikzcd} E|_U \arrow[rr, "\sim"] \arrow[dr] & & U \times \mathbb{C}^n \arrow[dl] \\ & U & \end{tikzcd}$$

gives rise to

$$\begin{tikzcd} \mathrm{Fr}(E|_U) = p^{-1}(U) \arrow[rr, "\sim"] \arrow[dr] & & U \times GL_n, \arrow[dl] \\ & U & \end{tikzcd}$$

so $\mathrm{Fr}(E)/GL_n = Y$. This bundle $p\colon \mathrm{Fr}(E) \to Y$, together with its GL_n action, is called the *associated principal bundle* to the vector bundle E. One can recover E from its associated principal bundle via isomorphisms

$$\begin{tikzcd} \mathrm{Fr}(E) \times^{GL_n} \mathbb{C}^n \arrow[r, "\sim"] \arrow[d] & E \arrow[d] \\ \mathrm{Fr}(E) \times^{GL_n} \mathrm{pt} \arrow[r, "\sim"] & Y, \end{tikzcd} \qquad (v_1, \ldots, v_n) \times (z_1, \ldots, z_n) \mapsto \sum_{i=1}^{n} z_i v_i.$$

Here we are using the *balanced product* notation introduced in Chapter 1 when describing the Borel construction: in general, if G acts on the right on a space X, and on the left on a space Y, then

$$X \times^G Y$$

is the quotient of $X \times Y$ by the relation $(x \cdot g, y) \sim (x, g \cdot y)$.

The associated bundle can be used to construct other bundles. Multilinear constructions on the standard GL_n-representation $\mathbb{C}^n$ lead to analogous ones on E. For instance, one has

$$\mathrm{Fr}(E) \times^{GL_n} (\mathbb{C}^n)^\vee \cong E^\vee,$$
$$\mathrm{Fr}(E) \times^{GL_n} \textstyle\bigwedge^d \mathbb{C}^n \cong \textstyle\bigwedge^d E,$$
$$\mathrm{Fr}(E) \times^{GL_n} \mathrm{Sym}^d\, \mathbb{C}^n \cong \mathrm{Sym}^d\, E.$$

Using the (left) action of GL_n on projective space $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$, the Grassmannian $Gr(d, \mathbb{C}^n)$, or flag variety $Fl(\mathbb{C}^n)$, one obtains projective bundles, Grassmann bundles, and flag bundles:

$$\mathrm{Fr}(E) \times^{GL_n} \mathbb{P}(\mathbb{C}^n) \cong \mathbb{P}(E),$$
$$\mathrm{Fr}(E) \times^{GL_n} Gr(d, \mathbb{C}^n) \cong \mathbf{Gr}(d, E),$$
$$\mathrm{Fr}(E) \times^{GL_n} Fl(\mathbb{C}^n) \cong \mathbf{Fl}(E).$$

In fact, any space X with a left GL_n-action produces a bundle

$$\mathrm{Fr}(E) \times^{GL_n} X \to Y$$

which is locally trivial with fiber X. Special cases of this construction will give us $H^k_{GL_n}(X)$, at least if $\widetilde{H}^i(\mathrm{Fr}(E)) = 0$ for $i \leq k$. Often it is simpler and more natural to study bundles in general, keeping in mind that this special case recovers equivariant cohomology.

Exercise 2.1.1. For $d \leq n$, let $\mathrm{Fr}(d, E) \to Y$ be the bundle whose fiber over y is

$$\Big\{(v_1, \ldots, v_d) \,\Big|\, v_1, \ldots, v_d \text{ are linearly independent in the fiber } E_y\Big\}.$$

There is a right GL_d-action, as before. Show that $\mathrm{Fr}(d, E) \times^{GL_d} \mathbb{C}^d$ is naturally identified with the tautological rank d subbundle $\mathbb{S} \subseteq E_{\mathbf{Gr}}$ on the Grassmann bundle $\pi\colon \mathbf{Gr}(d, E) \to Y$, where $E_{\mathbf{Gr}} = \pi^* E$ is the pullback vector bundle.

Exercise 2.1.2. Note that $\mathrm{Fr}(d, E)$ is an open subspace of the Hom bundle $\mathrm{Hom}(\mathbb{C}^d_Y, E)$, where $\mathbb{C}^d_Y = Y \times \mathbb{C}^d$ is the trivial bundle. Use a similar open subset of $\mathrm{Hom}(E, \mathbb{C}^{n-d}_Y)$ to construct the tautological rank $n-d$ quotient bundle $E_{\mathbf{Gr}} \twoheadrightarrow \mathbb{Q} = E_{\mathbf{Gr}}/\mathbb{S}$ on $\mathbf{Gr}(d, E)$.

Generally, for a Lie group G, a *(right) principal G-bundle* is

$$p\colon \mathbb{E} \to \mathbb{B},$$

where G acts freely on $\mathbb{E}$ (on the right) and the map p is isomorphic to the quotient map $\mathbb{E} \to \mathbb{E}/G$. We will always assume such bundles are *locally trivial*, so that $\mathbb{B}$ is covered by open sets U, with G-equivariant isomorphisms $p^{-1}U \cong U \times G$, where G acts on $U \times G$ by right multiplication on itself.

Exercise 2.1.3. With G acting by right multiplication on itself, trivially on a space $\mathbb{B}$, and on the left on a space X, show that there is a canonical isomorphism

$$(\mathbb{B} \times G) \times^G X \cong \mathbb{B} \times X.$$

Exercise 2.1.4. Suppose G acts on the right on $\mathbb{E}$ and on the left on X, and H acts on the right on X and on the left on Y, compatibly so that

$$g \cdot (x \cdot h) = (g \cdot x) \cdot h$$

for all $g \in G$, $x \in X$, and $h \in H$. Show that there is a canonical isomorphism

$$(\mathbb{E} \times^G X) \times^H Y \cong \mathbb{E} \times^G (X \times^H Y).$$

Remark. If one restricts to a category of paracompact and Hausdorff spaces, there is a *universal principal G-bundle* $\mathbb{E}G \to \mathbb{B}G$, with the property that any principal bundle $\mathbb{E} \to \mathbb{B}$ comes from the universal one by a pullback

$$\begin{tikzcd} \mathbb{E} \arrow[r] \arrow[d] & \mathbb{E}G \arrow[d] \\ \mathbb{B} \arrow[r] & \mathbb{B}G \end{tikzcd}$$

for some map $\mathbb{B} \to \mathbb{B}G$, uniquely defined up to homotopy. The base $\mathbb{B}G$ of such a bundle is called a *classifying space* for G. In fact, a principal bundle $\mathbb{E} \to \mathbb{B}$ is universal if and only if $\mathbb{E}$ is contractible.

The conditions paracompact and Hausdorff guarantee that partitions of unity exist, which is what is needed to construct the classifying map $\mathbb{B} \to \mathbb{B}G$. Any complex algebraic variety has these properties. On the other hand, for most groups G, there is no (finite-dimensional) algebraic variety $\mathbb{E}$ which is contractible and admits a free G-action, so the classifying space $\mathbb{B}G$ cannot be represented by any algebraic variety. See Appendix E for an algebraic approach to this universal property.

We will not need the universal construction in our approach to equivariant cohomology. Instead, we construct finite-dimensional algebraic varieties which "approximate" $\mathbb{E}G \to \mathbb{B}G$, and suffice to compute cohomology in any finite degree.

2.2 Definitions

The equivariant cohomology groups H^i_G will be contravariant functors for G-equivariant maps $f\colon X \to Y$, and $H^*_G X = \bigoplus_{i\geq 0} H^i_G X$ will be a ring. To define $H^i_G X$ in any range $i < N$ (with N a positive integer or infinity), it suffices to find a principal G-bundle $\mathbb{E} \to \mathbb{B}$ with $\widetilde{H}^i\mathbb{E} = 0$ for $i < N$. (That is, $\mathbb{E}$ is path-connected and $H^i\mathbb{E} = 0$ for $0 < i < N$.) Then we set

$$H^i_G X := H^i(\mathbb{E} \times^G X) \qquad \text{for} \quad i < N.$$

To use this definition, we must show it is independent of choices, and we must also find spaces $\mathbb{E}$ with N arbitrarily large.

For any G which embeds as a closed subgroup of GL_n, we have an answer to the second point.

Lemma 2.2.1. *Let G be any complex linear algebraic group, and $N > 0$ an integer. There are nonsingular finite-dimensional algebraic varieties $\mathbb{E}$ and $\mathbb{B}$, with $\widetilde{H}^i\mathbb{E} = 0$ for $i < N$, and G acting freely on $\mathbb{E}$ so that $\mathbb{E} \to \mathbb{B} = \mathbb{E}/G$ is a principal G-bundle which is locally trivial in the complex topology.*

In §2.4 we will give an explicit construction of $\mathbb{E}$ making the proof of the lemma clear. Let us grant this for now, and check that the definition does not depend on the choice of $\mathbb{E}$.

Proposition 2.2.2. *If $\mathbb{E} \to \mathbb{B}$ and $\mathbb{E}' \to \mathbb{B}'$ are principal G-bundles with $\widetilde{H}^i\mathbb{E} = \widetilde{H}^i\mathbb{E}' = 0$ for $i < N$, then there are canonical isomorphisms of cohomology groups*

$$H^i(\mathbb{E} \times^G X) \cong H^i(\mathbb{E}' \times^G X)$$

for all $i < N$, and these are compatible with cup products in this range.

Proof Consider the product space $\mathbb{E} \times \mathbb{E}'$, with the diagonal action of G, so $(e, e') \cdot g = (e \cdot g, e' \cdot g)$. The projections are equivariant and give a commuting diagram

$$\begin{tikzcd}
\mathbb{E} \times X \arrow[d] & \mathbb{E} \times \mathbb{E}' \times X \arrow[l] \arrow[r] \arrow[d] & \mathbb{E}' \times X \arrow[d] \\
\mathbb{E} \times^G X & (\mathbb{E} \times \mathbb{E}') \times^G X \arrow[l] \arrow[r] & \mathbb{E}' \times^G X.
\end{tikzcd}$$

The horizontal maps to the left are locally trivial bundles with fiber $\mathbb{E}'$, and those to the right are locally trivial with fiber $\mathbb{E}$. A special case of the Leray–Hirsch theorem says that such bundle maps determine group isomorphisms

$$H^i(\mathbb{E} \times^G X) \xrightarrow{\sim} H^i((\mathbb{E} \times \mathbb{E}') \times^G X) \xleftarrow{\sim} H^i(\mathbb{E}' \times^G X)$$

for $i < N$ (see Appendix A, §A.4). Since these come from ring homomorphisms

$$H^*(\mathbb{E} \times^G X) \to H^*((\mathbb{E} \times \mathbb{E}') \times^G X) \leftarrow H^*(\mathbb{E}' \times^G X),$$

they respect cup products. □

Exercise 2.2.3. Verify that for a third principal bundle $\mathbb{E}'' \to \mathbb{B}''$ such that $\widetilde{H}^i\mathbb{E}'' = 0$ for $i < N$, the canonical isomorphisms are compatible: there is a commuting triangle

$$\begin{tikzcd} H^i(\mathbb{E} \times^G X) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & H^i(\mathbb{E}' \times^G X) \arrow[dl, "\sim"] \\ & H^i(\mathbb{E}'' \times^G X) & \end{tikzcd}$$

for $i < N$.

Exercise 2.2.4. With $\mathbb{E}$ and $\mathbb{E}'$ as above, suppose there is a G-equivariant continuous map $\varphi\colon \mathbb{E}' \to \mathbb{E}$, so $\varphi(e' \cdot g) = \varphi(e') \cdot g$ for all $e' \in \mathbb{E}'$, $g \in G$. This defines a continuous map $\mathbb{E}' \times^G X \to \mathbb{E} \times^G X$, and a pullback homomorphism $H^i(\mathbb{E} \times^G X) \to H^i(\mathbb{E}' \times^G X)$. Show that this is the same as the canonical isomorphism given above when $i < N$.

Any G-equivariant continuous map $f\colon X \to Y$ determines a continuous map $\mathbb{E} \times^G X \to \mathbb{E} \times^G Y$, by $[e, x] \mapsto [e, f(x)]$, so we get homomorphisms

$$f^*\colon H^i_G Y \to H^i_G X.$$

In particular, from the projection $X \to \mathrm{pt}$, we obtain a ring homomorphism

$$\Lambda_G := H^*_G(\mathrm{pt}) \to H^*_G X,$$

making $H^*_G X$ a graded-commutative Λ_G-algebra. (If Λ_G^{odd} is nonzero, then one needs to use the convention that for $a \in \Lambda_G^p$ and $b \in H^q_G X$, one has $b \cdot a = (-1)^{pq} a \cdot b$.) Functoriality of cohomology means that the pullback f^* is a homomorphism of Λ_G-algebras. So we have constructed a contravariant functor

$$H^*_G\colon (G\text{-spaces}) \to (\Lambda_G\text{-algebras}).$$

In Chapter 3, we will construct more general pullbacks, allowing the group to vary as well.

Exercise 2.2.5. Check that the isomorphisms verifying independence of the choice of $\mathbb{E}$ are functorial: given an equivariant map $X \to Y$, and spaces $\mathbb{E}$ and $\mathbb{E}'$ with $\widetilde{H}^i\mathbb{E} = \widetilde{H}^i\mathbb{E}' = 0$ for $i < N$, show that the diagram

$$\begin{array}{ccc} H^i(\mathbb{E} \times^G Y) & \xrightarrow{\sim} & H^i(\mathbb{E}' \times^G Y) \\ \downarrow & & \downarrow \\ H^i(\mathbb{E} \times^G X) & \xrightarrow{\sim} & H^i(\mathbb{E}' \times^G X) \end{array}$$

commutes.

As a simple and fundamental example, consider $G = \mathbb{C}^*$. This acts freely on $\mathbb{E}_m = \mathbb{C}^m \smallsetminus 0$, by $(z_1, \dots, z_m) \cdot s = (z_1 s, \dots, z_m s)$. The quotient is $\mathbb{B}_m = \mathbb{P}^{m-1}$. Since $\widetilde{H}^i\mathbb{E}_m = \widetilde{H}^i S^{2m-1} = 0$ for $i < 2m-1$, any space X with a $\mathbb{C}^*$-action has

$$H^i_{\mathbb{C}^*}X = H^i((\mathbb{C}^m \smallsetminus 0) \times^{\mathbb{C}^*} X) \qquad \text{for } i < 2m-1.$$

In particular, for the range $i < 2m-1$, one has

$$H^i_{\mathbb{C}^*}(\mathrm{pt}) = H^i(\mathbb{P}^{m-1}) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Each $H^*(\mathbb{P}^{m-1})$ is a truncated polynomial ring isomorphic to $\mathbb{Z}[t]/(t^m)$, so $H^*_{\mathbb{C}^*}(\mathrm{pt})$ is a polynomial ring:

$$\Lambda_{\mathbb{C}^*} = \mathbb{Z}[t], \qquad \text{for } t \text{ a variable of degree 2.}$$

There are two possibilities for t, differing by a sign. In fact, there is a canonical choice of sign, as we will see in the next section.

For $G = (\mathbb{C}^*)^n$, one can take $\mathbb{E}_m = (\mathbb{C}^m \smallsetminus 0)^n$, so $\mathbb{B}_m = (\mathbb{P}^{m-1})^n$ and $\Lambda_G = \mathbb{Z}[t_1, \dots, t_n]$.

In these examples one already sees a key feature of our definition of equivariant cohomology: it takes place within the world of finite-dimensional varieties.

Proposition 2.2.6. *Let G be a complex linear algebraic group acting algebraically on a variety X. For any integer $N > 0$, there is a nonsingular algebraic variety $\mathbb{E}$ so that $H^i_GX = H^i(\mathbb{E} \times^G X)$ for $i < N$, where $\mathbb{E} \times^G X$ is a complex analytic space, nonsingular whenever X is.*

Proof Quite generally, suppose Z is a complex analytic space, and $Y \to Z$ is a continous map of topological spaces which is a locally trivial fiber bundle. If the fibers F are complex analytic spaces, and the transition functions are holomorphic maps $\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ for some complex subgroup $G \subseteq \mathrm{Aut}(F)$ of holomorphic automorphisms, then Y inherits a canonical complex analytic structure by glueing. If both Z and F are complex manifolds, so is Y.

The proposition is the special case where $Y = \mathbb{E} \times^G X$ and $Z = \mathbb{B}$, where $\mathbb{E} \to \mathbb{B}$ is chosen as in Lemma 2.2.1. □

2.3 Chern classes and fundamental classes

A *G-equivariant vector bundle* on X is a vector bundle $E \to X$ with G acting linearly on fibers, so that the projection is equivariant. (That is, G acts on E, and for all $g \in G$ and $x \in X$, and $e \in E_x$, the map $e \mapsto g \cdot e$ is a linear map of vector spaces $E_x \to E_{g \cdot x}$.) An equivariant vector bundle produces an ordinary vector bundle $\mathbb{E} \times^G E \to \mathbb{E} \times^G X$. Choosing $\mathbb{E}$ so that $\widetilde{H}^i\mathbb{E} = 0$ for $i \leq 2k$, we take the Chern classes of this bundle on $\mathbb{E} \times^G X$ to define the *equivariant Chern classes* of E:

$$c_k^G(E) := c_k(\mathbb{E} \times^G E) \qquad \text{in} \qquad H_G^{2k}X = H^{2k}(\mathbb{E} \times^G X).$$

Similarly, a G-invariant subvariety V of codimension d in a nonsingular variety X determines a subvariety $\mathbb{E} \times^G V \subseteq \mathbb{E} \times^G X$ of codimension d, and therefore an *equivariant fundamental class*

$$[V]^G = [\mathbb{E} \times^G V] \qquad \text{in} \qquad H_G^{2d}X = H^{2d}(\mathbb{E} \times^G X).$$

(Here we assume G is a complex linear algebraic group and $\mathbb{E}$ is a nonsingular algebraic variety. Then Proposition 2.2.6 says that $\mathbb{E} \times^G V$ is a complex analytic subvariety of the complex manifold $\mathbb{E} \times^G X$.)

Exercise 2.3.1. Using arguments from before, show that these definitions are independent of choices. More precisely,

$$c_k(\mathbb{E} \times^G E) \mapsto c_k(\mathbb{E}' \times^G E) \text{ under } H^{2k}(\mathbb{E} \times^G X) \xrightarrow{\sim} H^{2k}(\mathbb{E}' \times^G X),$$

when $\widetilde{H}^i\mathbb{E} = \widetilde{H}^i\mathbb{E}' = 0$ for $i \leq 2k$; and

$$[\mathbb{E} \times^G V] \mapsto [\mathbb{E}' \times^G V] \text{ under } H^{2d}(\mathbb{E} \times^G X) \xrightarrow{\sim} H^{2d}(\mathbb{E}' \times^G X),$$

when $\widetilde{H}^i\mathbb{E} = \widetilde{H}^i\mathbb{E}' = 0$ for $i \leq 2d$.

Exercise 2.3.2. Show that multilinear constructions on vector bundles are preserved by the Borel construction. For instance, if E and F are G-equivariant vector bundles on X, verify that

$$\mathbb{E} \times^G (E \oplus F) \cong (\mathbb{E} \times^G E) \oplus (\mathbb{E} \times^G F)$$

as vector bundles on $\mathbb{E} \times^G X$, where $\mathbb{E} \to \mathbb{B}$ is a principal G-bundle. Do the same for tensor products $E \otimes F$, $\bigwedge^k E$, and $\operatorname{Sym}^k E$.

The basic properties of equivariant Chern classes and fundamental classes follow directly from the corresponding properties of ordinary classes on approximation spaces; details and references can be found in Appendix A, §A.3 and §A.5. For instance, one has the following:

- For equivariant line bundles L and M, equivariant Chern classes are additive: $c_1^G(L \otimes M) = c_1^G(L) + c_1^G(M)$.
- When $0 \to E' \to E \to E'' \to 0$ is an equivariant short exact sequence, there is a Whitney formula $c^G(E) = c^G(E') \cdot c^G(E'')$.
- If E has rank e on a nonsingular variety X, and s is an equivariant section, then $Z(s) \subseteq X$ is an invariant subvariety of codimension at most e. If $\operatorname{codim}(Z(s)) = e$, then $[Z(s)]^G = c_e^G(E)$ in $H_G^{2e}X$.
- If G is connected, and two invariant subvarieties V and W of a nonsingular variety X intersect properly, with $V \cdot W = \sum m_i Z_i$ as cycles, then $[V]^G \cdot [W]^G = \sum m_i [Z_i]^G$ in $H_G^* X$. In particular, if $V \cap W = \emptyset$, then $[V]^G \cdot [W]^G = 0$.

(In the last item, connectedness of G is needed to guarantee that each Z_i is also G-invariant.)

As usual, the basic case $X = \mathrm{pt}$ offers plenty to study. Here a G-equivariant vector bundle is just a representation of G, so each representation V of G has Chern classes $c_i^G(V) \in H_G^{2i}(\mathrm{pt}) = \Lambda_G^{2i}$.

Example 2.3.3. For each integer a, $\mathbb{C}^*$ has the one-dimensional representation $\mathbb{C}_a$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by $z \cdot v = z^a v$. So $\mathbb{C}_1$ is the *standard representation*. In Exercise 2.1.1, we saw that

$$\begin{array}{ccc}
(\mathbb{C}^m \smallsetminus 0) \times^{\mathbb{C}^*} \mathbb{C}_1 & \xrightarrow{\sim} & \mathcal{O}(-1) \\
\big\downarrow & & \big\downarrow \\
(\mathbb{C}^m \smallsetminus 0) \times^{\mathbb{C}^*} \mathrm{pt} & \xrightarrow{\sim} & \mathbb{P}^{m-1},
\end{array}$$

so $\mathbb{E} \times^{\mathbb{C}^*} \mathbb{C}_1$ gets identified with the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{B}$. Taking $t = c_1^{\mathbb{C}^*}(\mathbb{C}_1) = c_1(\mathcal{O}(-1))$ as a generator for $\Lambda_{\mathbb{C}^*} = \mathbb{Z}[t]$, we see

> $\Lambda_{\mathbb{C}^*}$ is a polynomial ring generated by the Chern class of the standard representation.

More generally, since $\mathbb{C}_a \otimes \mathbb{C}_b = \mathbb{C}_{a+b}$, we have $c_1^{\mathbb{C}^*}(\mathbb{C}_a) = at$. One can also see this from an identification $\mathbb{E} \times^{\mathbb{C}^*} \mathbb{C}_a \cong \mathcal{O}(-a)$.

Example 2.3.4. Consider $T = (\mathbb{C}^*)^n$ acting on $\mathbb{C}^n = V$ by the standard action scaling coordinates. For $1 \leq i \leq n$, we have one-dimensional representations $\mathbb{C}_{t_i}$, where $z \cdot v = z_i v$. Then

$$c_i^T(V) = e_i(t_1, \ldots, t_n) \quad \text{in} \quad \Lambda_T = \mathbb{Z}[t_1, \ldots, t_n],$$

where $t_i = c_1^T(\mathbb{C}_{t_i})$ and e_i is the elementary symmetric polynomial. Using $\mathbb{E} = (\mathbb{C}^m \smallsetminus 0)^n$ and $\mathbb{B}_m = (\mathbb{P}^{m-1})^n$, the class t_i is identified with the Chern class of the tautological bundle from the ith factor of $\mathbb{B}_m$.

Example 2.3.5. In the equivariant setting, it is harder to move G-invariant subvarieties so that they intersect properly. For example, consider $G = \mathbb{C}^*$ acting on $\mathbb{C}$ in the standard way. Then the only invariant subvarieties are $\{0\}$ and $\mathbb{C}$. In ordinary cohomology, one could move 0 to 1 to see $[0]^2 = [0] \cdot [1] = [\{0\} \cap \{1\}] = 0$, but this is not possible equivariantly. Indeed, $([0]^T)^2 = t^2 \neq 0$ in $H^*_{\mathbb{C}^*}(\mathbb{C})$.

2.4 The general linear group

Now we will consider $G = GL(V)$, for an n-dimensional vector space V. This has its standard representation on V itself, so there are Chern classes $c_i^G(V) \in H_G^{2i}(\mathrm{pt}) = \Lambda_G^{2i}$. Our main calculation is the following.

Proposition 2.4.1. *We have*

$$\Lambda_G = \mathbb{Z}[c_1, \ldots, c_n],$$

where $c_i = c_i^G(V)$.

In other words,

> $\Lambda_{GL(V)}$ is a polynomial ring generated by the Chern classes of the standard representation.

To prove this, we will use $\mathbb{E}_m = \mathrm{Emb}(V, \mathbb{C}^m)$, the space of linear embeddings $V \hookrightarrow \mathbb{C}^m$, for $m \geq n$. Choosing a basis, so $V \cong \mathbb{C}^n$, one identifies $\mathbb{E}_m$ with $M^\circ_{m,n}$, the space of full-rank $m \times n$ matrices. Let $\Omega_{n-1} = \mathrm{Hom}(V, \mathbb{C}^m) \smallsetminus \mathbb{E}_m$; choosing a basis identifies $\Omega_{n-1} \subseteq M_{m,n}$ with the locus of $m \times n$ matrices of rank at most $n-1$. A standard exercise in algebraic geometry computes its dimension.

Exercise 2.4.2. Consider the locus $\Omega_r \subseteq M_{m,n}$ of matrices of rank at most r. Show that this is irreducible of codimension $(m-r)(n-r)$.

Lemma 2.4.3. *We have $\widetilde{H}^i \mathbb{E}_m = 0$ for $i \leq 2(m-n)$.*

Proof From the long exact sequence in cohomology, we have $\widetilde{H}^i \mathbb{E}_m = H^{i+1}(\mathrm{Hom}(V, \mathbb{C}^m), \mathbb{E}_m) =: \overline{H}_{2mn-i-1}\Omega_{n-1}$. By the above exercise, Ω_{n-1} has (real) dimension $2mn - 2(m-n+1)$. When $i \leq 2(m-n)$, we have $2mn - i - 1 > 2mn - 2(m-n+1)$, so this Borel–Moore homology group vanishes. (See Appendix A, §A.3, for the relevant properties of Borel–Moore homology.) □

An alternative way of proving the lemma is given in Appendix A, §A.7.

We also need a coarse description of the cohomology of the Grassmannian, which says it is generated by Chern classes, with no relations in small degree.

Lemma 2.4.4. *We have*

$$H^* Gr(n, \mathbb{C}^m) = \mathbb{Z}[c_1(\mathbb{S}), \ldots, c_n(\mathbb{S})]/(R_{m-n+1}, \ldots, R_m),$$

where R_k is a relation of degree k.

The lemma can be found in standard algebraic topology texts, and it also follows from computations we will do later (see §4.5).

Now we can prove Proposition 2.4.1. Observe that $\mathbb{B}_m = \mathbb{E}_m/G = Gr(n, \mathbb{C}^m)$, where G acts on $\mathrm{Emb}(V, \mathbb{C}^m)$ by $(\varphi \cdot g)(v) = \varphi(g \cdot v)$. By Exercise 2.1.1, $\mathbb{E}_m \to \mathbb{B}_m$ is the frame bundle $\mathrm{Fr}(\mathbb{S}) \to Gr(n, \mathbb{C}^m)$ associated to the tautological $\mathbb{S} \subseteq \mathbb{C}^m_{Gr}$, and so the vector bundle $\mathbb{E}_m \times^G V$ identifies with the tautological bundle $\mathbb{S}$ itself. (The map is $(\varphi, v) \mapsto (\varphi, \varphi(v))$.) Thus $c_i^G(V)$ is identified with $c_i(\mathbb{S})$, and the proposition follows from Lemma 2.4.4. □

2.5 Some other groups

Any closed subgroup $G \subseteq GL(V)$ acts freely on $\mathbb{E}_m = \mathrm{Emb}(V, \mathbb{C}^m)$, so we can use these same approximation spaces for such G. (For computations, it is sometimes helpful to make other choices.) Let us see how far we can get using this explicit construction.

Exercise 2.5.1. Consider $G = SL(V) \subseteq GL(V)$ as the subgroup preserving the determinant $\bigwedge^n V \xrightarrow{\sim} \mathbb{C}$. Show that

$$\Lambda_{SL(V)} = \mathbb{Z}[c_1, \ldots, c_n]/(c_1) = \mathbb{Z}[c_2, \ldots, c_n],$$

where $c_i = c_i^G(V)$. (Note that $\bigwedge^n V$ is the trivial representation, so $c_1^G(V) = c_1^G(\bigwedge^n V) = 0$.)

For now, let us fix a basis, so $V = \mathbb{C}^n$. Our main example going forward will be $T = (\mathbb{C}^*)^n$, and we have already seen two possibilities for constructing its equivariant cohomology. Using $T = (GL_1)^n$, we get

$$\mathbb{E}_m = (\mathbb{C}^m \smallsetminus 0)^n = \big\{A \in M_{m,n} \,|\, \text{no column is zero}\big\},$$

with $\mathbb{B}_m = (\mathbb{P}^{m-1})^n$.

On the other hand, considering $T \subseteq GL_n$ as diagonal matrices, we have

$$\mathbb{E}_m = M^\circ_{m,n} = \big\{A \in M_{m,n} \,|\, \text{columns are linearly independent}\big\}.$$

Using this choice, we get

$$\mathbb{B}_m = M^\circ_{m,n}/T = \left\{ \begin{array}{c} V \subseteq \mathbb{C}^m \text{ of dimension } n, \\ \text{with a decomposition } V = L_1 \oplus \cdots \oplus L_n \end{array} \right\}$$

by sending a matrix to the tuple $(L_1, \ldots, L_n)$, with L_i the span of the ith column. Call this space the "split Grassmannian" $Gr^{\mathrm{split}}(n, \mathbb{C}^m)$; it comes with tautological line bundles $\mathbb{L}_1, \ldots, \mathbb{L}_n$, whose classes $t_i = c_1(\mathbb{L}_i)$ generate the cohomology ring.

There is a projection map $\pi\colon Gr^{\mathrm{split}}(n, \mathbb{C}^m) \to Gr(n, \mathbb{C}^m)$ sending $(L_1, \ldots, L_n)$ to $V = L_1 \oplus \cdots \oplus L_n \subseteq \mathbb{C}^m$.

Exercise 2.5.2. Taking m sufficiently large, show that the corresponding pullback map on cohomology gives

$$\Lambda_{GL_n} = \mathbb{Z}[c_1, \ldots, c_n] \to \mathbb{Z}[t_1, \ldots, t_n] = \Lambda_T,$$

defined by $c_i \mapsto e_i(t_1,\ldots,t_n)$, so Λ_{GL_n} embeds in Λ_T as the ring of symmetric polynomials.

Remark. The inclusion $\Lambda_{GL_n} \hookrightarrow \Lambda_T$ is a manifestation of the *splitting principle*: given a vector bundle E on a space X, one can find a map $f\colon X' \to X$, such that f^*E splits into a direct sum of line bundles on X', and the pullback homomorphism $f^*\colon H^*X \to H^*X'$ is injective. For any $d \leq n = \operatorname{rk} E$, there is a "split Grassmann" bundle $\mathbf{Gr}^{\text{split}}(d,E) \to X$, constructed as before by taking a quotient of the frame bundle, so

$$\operatorname{Fr}(E) \times^{GL_n} Gr^{\text{split}}(d,\mathbb{C}^n) \cong \mathbf{Gr}^{\text{split}}(d,E).$$

Taking $d = n = \operatorname{rk} E$ and $X' = Gr^{\text{split}}(n,E)$, the pullback of E from X to X' splits, and the cohomology of X embeds into that of X'.

Using functorial pullbacks in equivariant cohomology, exactly the same construction establishes the analogous equivariant splitting principle: for a G-equivariant vector bundle $E \to X$, there is an equivariant map $f\colon X' \to X$, such that f^*E splits into equivariant line bundles, and such that $f^*\colon H_G^*X \to H_G^*X'$ is injective.

In between the torus and GL_n, there is the Borel group B of upper-triangular matrices. Using $\mathbb{E}_m = M^\circ_{m,n}$ again, we have

$$\begin{aligned}\mathbb{B}_m = M^\circ_{m,n}/B &= \left\{\begin{array}{c} V \subseteq \mathbb{C}^m \text{ of dimension } n, \\ \text{with a filtration } V_1 \subset V_2 \subset \cdots \subset V_n = V \end{array}\right\} \\ &= Fl(1,2,\ldots,n;\mathbb{C}^m),\end{aligned}$$

the partial flag variety parametrizing chains $V_1 \subset \cdots \subset V_n \subseteq \mathbb{C}^m$, with $\dim V_i = i$. (The projection $\mathbb{E}_m \to \mathbb{B}_m$ sends a matrix to the flag where V_i is the span of the first i columns.) This comes with a tautological flag of bundles $\mathbb{S}_1 \subset \cdots \subset \mathbb{S}_n \subseteq \mathbb{C}^m_{Fl}$. The flag variety sits between $Gr^{\text{split}}(n,\mathbb{C}^m)$ and $Gr(n,\mathbb{C}^m)$, with maps

$$Gr^{\text{split}}(n,\mathbb{C}^m) \to Fl(1,\ldots,n;\mathbb{C}^m) \to Gr(n,\mathbb{C}^m),$$

sending $(L_1,\ldots,L_n)$ to the flag with $V_i = L_1 \oplus \cdots \oplus L_i$, and projecting a flag $V_\bullet$ to $V = V_n \subseteq \mathbb{C}^m$.

Exercise 2.5.3. Show that $Gr^{\text{split}}(n,\mathbb{C}^m)$ is a locally trivial affine bundle, so the pullback map induces a ring isomorphism $\Lambda_B \xrightarrow{\sim} \Lambda_T$.

The isomomorphism in this exercise is part of a general phenomenon, as we will see in the next chapter, since the inclusion $T \hookrightarrow B$ is a deformation retract. On the other hand, one can also compute directly that $H^*Fl(1, \ldots, n; \mathbb{C}^m)$ is generated by the Chern classes $t_i = c_1(\mathbb{S}_i/\mathbb{S}_{i-1})$, with relations in degrees $2(m-n+1), \ldots, 2m$, so that $\Lambda_B = \mathbb{Z}[t_1, \ldots, t_n]$.

Exercise 2.5.4. Let $\chi_i \colon B \to \mathbb{C}^*$ be the character which picks out the ith diagonal entry of a matrix in B, and let $\mathbb{C}_{\chi_i}$ be the corresponding representation. Show that $t_i = c_1^B(\mathbb{C}_{\chi_i})$.

For other groups, the rings Λ_G can be much more complicated. For instance, the answer for PGL_n is not completely known!

In the case of the symplectic group $G = Sp_{2n} \subseteq GL_{2n}$, with its standard representation $V = \mathbb{C}^{2n}$, there is a simple answer:

$$\Lambda_{Sp_{2n}} = \mathbb{Z}[c_2, c_4, \ldots, c_{2n}], \quad \text{where} \quad c_{2k} = c_{2k}^G(V),$$

so here again Λ_G is generated by the Chern classes of the standard representation.

Here is the easy half of this computation. Using $\mathbb{E}_m = M^\circ_{m,n}$, we find

$$\mathbb{B}_m = M^\circ_{m,2n}/Sp_{2n} = \left\{ (V, \omega) \;\middle|\; \begin{array}{c} V \subseteq \mathbb{C}^m \text{ has dimension } 2n, \text{ and} \\ \omega \text{ is a symplectic form on } V \end{array} \right\}.$$

Using the projection to $M^\circ_{m,2n}/GL_{2n} = Gr(2n, \mathbb{C}^m)$, one can pull back the tautological bundle $\mathbb{S}$. On $\mathbb{B}_m$, this pullback bundle acquires a tautological symplectic form, identifying it with its dual. So whenever i is odd, $2c_i \mapsto 0$ under the map $\Lambda_{GL_{2n}} \to \Lambda_{Sp_{2n}}$. This comes from the general fact that $c_i(E) + (-1)^i c_i(E^\vee) = 0$, for any bundle E. To complete the argument, one must show that $H^*\mathbb{B}_m$ has no torsion, and that $\Lambda_{GL_{2n}} \to \Lambda_{Sp_{2n}}$ is surjective. (See Example 15.5.2.)

Similar arguments show that $\Lambda_{GL_n} \to \Lambda_{SO_n}$ sends $2c_i$ to 0 for i odd, but in this case it is not true that $c_i \mapsto 0$ (there is 2-torsion on Λ_{SO_n}), and the map is not surjective in general.

Exercise 2.5.5. Show that $\Lambda_{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Z}[t]/(2t)$, where t is a class in degree 2. For the additive group $\mathbb{Z}$, show that $\Lambda_{\mathbb{Z}} = \mathbb{Z}[t]/(t^2)$, where t has degree 1.

Remark. For a finite group G, there is another construction, which gives rise to an explicit cochain complex computing $\Lambda_G = H^*_G(\mathrm{pt})$. Let $C^\bullet = C^\bullet(G, \mathbb{Z})$ be the complex with

$$C^i = \{\text{functions } \varphi\colon G^i \to \mathbb{Z}\}$$

and for $\varphi \in C^i$ define the differential $d\varphi \in C^{i+1}$ by

$$\begin{aligned} d\varphi(g_0, \dots, g_i) = \varphi(g_1, \dots, g_i) & \\ &+ \sum_{j=0}^{i-1} (-1)^{j+1} \varphi(g_1, \dots, g_{j-1}, g_j g_{j+1}, \dots, g_i) \\ &+ (-1)^{i+1} \varphi(g_0, \dots, g_{i-1}). \end{aligned}$$

Then $H^k_G(\mathrm{pt})$ is the cohomology $H^k(C^\bullet)$ of this complex. One way to prove this goes through the *Milnor construction* for the universal principal bundle $\mathbb{E}G \to \mathbb{B}G$.

In the context of group theory, $H^*_G(\mathrm{pt}) = H^*\mathbb{B}G = H^*(G, \mathbb{Z})$ is known as the *group cohomology* of G with coefficients in the trivial G-module $\mathbb{Z}$.

2.6 Projective space

Let G be any group acting linearly on an n-dimensional vector space V. Then G also acts on the projective space $\mathbb{P}(V)$, as well as the tautological subbundle $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$. Let $\zeta = c^G_1(\mathcal{O}(1))$ be the Chern class in $H^2_G\mathbb{P}(V)$.

Proposition 2.6.1. *We have*

$$H^*_G\mathbb{P}(V) = \Lambda_G[\zeta]/(\zeta^n + c_1\zeta^{n-1} + \cdots + c_n),$$

where $c_i = c^G_i(V)$ are the Chern classes of the given representation.

Proof This is a special case of the general formula computing the cohomology of a projective bundle in terms of that of the base. In our circumstance, the relevant identification is

$$\begin{tikzcd} \mathbb{E} \times^G \mathbb{P}(V) \arrow[r, equal] \arrow[d] & \mathbb{P}(\mathbb{E} \times^G V) \arrow[d] \\ \mathbb{B} \arrow[r, equal] & \mathbb{B}, \end{tikzcd}$$

compatibly with identifications of $\mathcal{O}(1)$. Thus ζ is the hyperplane class for the projective bundle, and $c_i^G(V) = c_i(\mathbb{E} \times^G V)$ are the Chern classes of this vector bundle on $\mathbb{B}$. □

Example 2.6.2. For $G = GL(V)$, we have

$$H_G^* \mathbb{P}(V) = \mathbb{Z}[c_1, \ldots, c_n][\zeta]/(\zeta^n + c_1\zeta^{n-1} + \cdots + c_n).$$

For $T = (\mathbb{C}^*)^n$ acting on V via the standard action, we have

$$H_T^* \mathbb{P}(V) = \mathbb{Z}[t_1, \ldots, t_n][\zeta]/\prod_{i=1}^{n}(\zeta + t_i).$$

(This comes from the computation $c_i^T(V) = e_i(t_1, \ldots, t_n)$.)

Notes

Our definition of equivariant cohomology, using approximations by algebraic varieties, is modelled on the analogous construction for Chow groups. This technique was pioneered by Totaro (1999) and further developed by Edidin and Graham (1998), who defined equivariant Chow groups.

Algebraic versions of Lemma 2.2.1 appear in Totaro's construction of the Chow ring of a classifying space; see (Totaro, 1999, Remark 1.4) or (Totaro, 2014, §2). In algebraic geometry, the method of proving Proposition 2.2.2 (establishing independence of choice of approximation) was used by mathematicians studying invariant theory; see especially Bogomolov's definition of the Brauer group (Bogomolov, 1987, §3). In topology, this argument goes back to Borel's foundational papers; see (Borel, 1953, §18).

An alternative argument for Proposition 2.2.6 showing that the quotient $\mathbb{E} \times^G X = (\mathbb{E} \times X)/G$ is a complex analytic space can be given using a general statement about analytic structures on quotients, proved by Cartan (1957) and generalized by Holmann (1960).

Even when X is a nonsingular variety, the space $\mathbb{E} \times^G X$ may not exist as a scheme (although it is always an algebraic space). Some general criteria guaranteeing that it does exist are given by Edidin and Graham (1998, Proposition 23). Sufficient conditions include: X is quasi-projective, with a linearized G-action; or G is a *special* group such as GL_n, SL_n, a torus, or products of such groups.

We will work out the cohomology rings of Grassmannians and flag varieties in Chapter 4. Alternative arguments for the computation in Lemma 2.4.4 can be found in many algebraic topology texts – for example, the book by Dold (1980, Proposition 12.17).

Using coefficients in a field, there are classical computations of $H^*\mathbb{B}G$ by Borel (1953). Some computations of integral cohomology for orthogonal groups were carried out in (Brown, 1982; Feshbach, 1983).

In many other cases, the integral cohomology (or Chow) rings of $\mathbb{B}G$ are either unknown, or were computed rather recently. The Chow ring for SO_{2n} was computed by Field in her 2000 Ph. D. thesis (Field, 2012). Computations for PGL_p, with p prime, were done in both cohomology and Chow rings by Vistoli (2007), whose paper also serves as a good survey for other work on the subject. More recent progress can be found in (Gu, 2021).

The "Milnor construction" for $\mathbb{B}G$ was given in (Milnor, 1956); see also (Husemoller, 1975, §4.11).

Hints for Exercises

Exercise 2.1.1. For the fiber over $y \in Y$, the map to the tautological bundle of $Gr(d, E_y)$ is simply $(v_1, \ldots, v_d) \times (z_1, \ldots, z_d) \mapsto (\mathrm{span}\{v_1, \ldots, v_d\}, \sum z_i v_i)$.

Exercise 2.2.3. Use the triple product $\mathbb{E} \times \mathbb{E}' \times \mathbb{E}''$.

Exercise 2.2.4. The equivariant map φ induces a section of the projection $(\mathbb{E} \times \mathbb{E}') \times^G X \to \mathbb{E}' \times^G X$.

Exercise 2.3.1. For Chern classes, this just uses the pullback of the vector bundle E to $\mathbb{E} \times \mathbb{E}'$. For classes of subvarieties, one needs the smooth pullback property; see Appendix A, Proposition A.3.2.

Exercise 2.4.2. Use a Grassmannian correspondence to parametrize the kernel of such a matrix. See (Harris, 1992, Proposition 12.2).

Exercise 2.5.1. Use the same $\mathbb{E}_m$, and identify $\mathbb{E}_m/SL(V) \to \mathbb{E}_m/GL(V)$ with the variety $\mathrm{Iso}(\bigwedge^n \mathbb{S}, \mathbb{C})$ over $Gr(n, \mathbb{C}^m)$, parametrizing subspaces $V \subseteq \mathbb{C}^m$ equipped with an isomorphism $\bigwedge^n V \to \mathbb{C}$. Explicitly, this is the variety cut by Plücker equations in $\bigwedge^n \mathbb{C}^m \smallsetminus 0$. It is also the complement of the 0-section in the line bundle $\mathrm{Hom}(\bigwedge^n \mathbb{S}, \mathbb{C}) \cong \bigwedge^n \mathbb{S}^\vee$. Then one can use the following general fact, which is an easy application of the Gysin sequence: for a vector bundle E of rank r on X, if the homomorphism

$$H^{i-2r+1} X \xrightarrow{c_r(E)\cdot} H^{i+1} X$$

is injective, then $H^i(E \smallsetminus 0) = (H^i X)/(c_r(E) \cdot H^{i-2r} X)$. See (Milnor and Stasheff, 1974, Theorem 12.2).

Exercise 2.5.2. Use $\pi^* \mathbb{S} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$.

Exercise 2.5.5. For $\mathbb{Z}/2\mathbb{Z}$, use $\mathbb{E}_m = S^m$, so $\mathbb{B}_m = \mathbb{RP}^m$. For $\mathbb{Z}$, use $\mathbb{E} = \mathbb{R}$, which is already contractible, with $\mathbb{B} = S^1$.