A NOTE ON QUASI-FROBENIUS RINGS AND RING EPIMORPHISMS

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0. In this note, we characterize quasi-Frobenius rings by a weakened form of the usual condition, that every ideal is an annihilator ideal.

We then apply this result to pure rings in the sense of Cohn and to dominant rings, a concept arising in the study of ring epimorphisms. All rings considered have a unit element.

1. A ring A is called quasi-Frobenius, if it is left and right Artinian and if the conditions

$$(a_\ell) \qquad \ell(r(L)) = L \quad \text{and}$$

$$(a_r) \qquad r(\ell(R)) = R$$

are satisfied for all left ideals L and right ideals R, $\ell(X)$ and $r(X)$ denoting the left and right annihilators of a subset X of A.

Clearly, it suffices to assume the minimum condition only on one side. (The maximum condition would also do.)

On the other hand, Dieudonné has proved [3], that, for a left and right Artinian ring, it is sufficient to assume (a_ℓ) and (a_r) for all minimal left and right ideals only. We denote these modified conditions by (a_ℓ'') and (a_r''). Let us note the well-known fact, that (a_ℓ'') and (a_r'') hold automatically for the non-nilpotent minimal ideals, since they are generated by an idempotent.

We first show, that for Dieudonné's result, one needs only the minimum condition on the left.

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Recall, that a ring A is left perfect, it has the minimum condition on principal right ideals or, equivalently, if the Jacobson radical Rad A is T-nilpotent and A/Rad A is completely reducible (i.e. semisimple Artinian) [1]. In particular, a left Artinian ring is both left and right perfect.

PROPOSITION 1. *Let* A *be a left Artinian ring such that the conditions* (a_ℓ'') *and* (a_r'') *hold. Then* A *is a quasi-Frobenius ring.*

Proof. Consider the condition (d_ℓ): The dual $M^* = \mathrm{Hom}_A(M, A)$ (which is a right A-module in a canonical way) of any simple left A-module M is simple or zero. (d_r) is defined similarily.

Dieudonné [3, 3.4] has shown, that a left and right Artinian ring satisfying (d_ℓ) and (d_r) is quasi-Frobenius. He also proved [3, 3.4] that for A left Artinian (d_ℓ) implies the existence of a composition series for the right A-module $A_A \cong ({}_A A)^*$. Thus A is also right Artinian, and we only have to show, that the implications $(a_\ell'') \Rightarrow (d_r)$ and $(a_r'') \Rightarrow (d_\ell)$ hold for a left Artinian ring A.

This requires a slight modification of Dieudonné's arguments as follows: if M is simple, then $M \cong A/I$, where I is a maximal left ideal and $M^* \cong r(I)$. If r(I) were not simple or zero, then it would properly contain a minimal right ideal R, since A satisfies the minimum condition on principal right ideals. $0 \subset R \subset r(I)$ implies $A \supset \ell(R) \supset \ell(r(I)) = I$, and since I is maximal, this yields $\ell(R) = A$ or $\ell(R) = I$, but both cases are impossible, by virtue of (a_r''). Thus $(a_r'') \Rightarrow (d_\ell)$ and similarily $(a_\ell'') \Rightarrow (d_r)$.
q.e.d.

If A is merely Noetherian, then (a_ℓ'') and (a_r'') are of course not sufficient, even if there exist proper minimal ideals. In this case we have to assume, that (a_ℓ) and (a_r) are satisfied for all principal left and right ideals. We will denote these new conditions by (a_ℓ') and (a_r').

PROPOSITION 2. *Let* A *be a left Noetherian ring such that the conditions* (a_ℓ') *and* (a_r') *are satisfied. Then* A *is a quasi-Frobenius ring.*

Proof. Since A is left Noetherian, it satisfies the maximum condition for left annihilator ideals (i.e. left ideals I such that $\ell(r(I)) = I$) and hence the minimum condition for right annihilator

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ideals. But then, by (a_r'), A satisfies the minimum condition on principal right ideals and is therefore left perfect. It is well known, that a left Noetherian left (or right) perfect ring is left Artinian, and we can thus apply Prop. 1. q.e.d.

We note, that for this proof one needed only (a_ℓ'') and not (a_ℓ').

Since, in sections 2 and 3, we will consider only commutative rings, it may be useful to give a simple criterion for a commutative Artinian ring to be quasi-Frobenius.

The left socle $Soc_\ell(A)$ is defined to be the sum of all minimal left ideals of A. It is well known, that for an Artinian ring A one has $Soc_\ell(A) = r(\text{Rad } A)$.

PROPOSITION 3. A commutative Artinian ring A is a quasi-Frobenius ring if and only if its socle S is a principal ideal.

Proof. A is a product of a finite number of local Artinian rings and it is enough to prove the proposition for the local case.

If A is local with radical N and if it is quasi-Frobenius, then the annihilator of every minimal ideal is N; thus there exists only one minimal ideal and S is clearly principal.

Conversely, let S be a principal ideal generated by s. S is a finite direct sum of minimal ideals M_i $(i = 1, \ldots, n)$ and s has a unique decomposition $s = s_1 + \ldots + s_n$ $(s_i \in M_i, s_i \neq 0)$. If $n > 1$, then there must be an $a \in A$ such that $as = s_1$, i.e. $as_1 = s_1$ and $as_2 = \ldots = as_n = 0$. Since S annihilates N, $a \notin N$, but then a has an inverse and we have a contradiction. Thus $n = 1$ and A has a unique minimal ideal $M = S$, which is an annihilator. Dieudonné's result implies then, that A is a quasi-Frobenius ring. q.e.d.

Remarks.

(i) In a special case, the above proof can be generalized to the non-commutative case: Let A be a left and right Artinian ring which is a direct product of completely primary rings. Then A is quasi-Frobenius if and only if $Soc_\ell(A)$ is a principal left ideal and $Soc_r(A)$ is a principal right ideal. (Of course, one has then $Soc_\ell(A) = Soc_r(A)$.)

(ii) Statement (i) is not true for general Artinian rings. Nakayama [6] has proved, that a finite-dimensional algebra over a field is a Frobenius algebra if and only if $Soc_r(A)$ is a principal right ideal, and there exist quasi-Frobenius algebras which are not Frobenius.

(iii) Prop. 3 together with Nakayama's result implies that a commutative quasi-Frobenius algebra over a field is a Frobenius algebra. This fact has recently been noted by Wenger [8].

2. Let A be a subring of the commutative ring B. The dominion Dom(A,B) of A in B is the set of all $d \in B$ such that $f(d) = g(d)$ for all pairs of ring homomorphisms with domain B and common range, coinciding on A. One can show, that it doesn't matter, if the common range of f and g runs through all rings or through the commutative rings only. The dominion has, in a more general setting, been defined by Isbell [4].

The embedding $A \subset B$ is an epimorphism in the category of commutative rings (or, equivalently, of all rings) if and only if Dom (A,B) = B.

If Dom (A,B) = A for all B, then we call A dominant, thereby modifying Isbell's terminology.

It is known, that a self-injective ring is dominant [7, Kor. 5.4] and that in a dominant ring every principal ideal is an annihilator [7, Kor. 4.4]. Since a quasi-Frobenius ring is self-injective, Prop. 2 implies immediately.

PROPOSITION 4. A commutative Noetherian ring is dominant if and only if it is a quasi-Frobenius ring.

We say, that A is strongly dominant [7, Def. 3.2] if every homomorphic image of A (including A) is dominant. Levy [5] has shown, that every homomorphic image of a commutative Noetherian ring A is self-injective if and only if A is an Artinian principal ideal ring. This yields

PROPOSITION 5. A commutative Noetherian ring is strongly dominant if and only if it is an Artinian principal ideal ring.

3. A submodule N of a left A-module M is called pure in M if the sequence $0 \to P \otimes_A N \to P \otimes_A M$ is exact for every right A-module P. A ring A is called left pure if the left A-module A is a pure A-submodule of every ring B containing A. These definitions are due to Cohn [2]. Left self-injective rings for example are left pure.

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The notion of a commutative pure ring may be ambiguous, in that it may depend on the category of rings under consideration. In fact, we do not know, if there exists a commutative ring A which is a pure A-submodule of every commutative ring B ⊃ A, but which is not a pure A-submodule of a certain noncommutative ring C ⊃ A. (A similar problem arises for dominant rings).

However, in the case considered here, it happens, that no such ambiguity exists; in fact, we have

PROPOSITION 6. A commutative Noetherian ring is pure if and only if it is a quasi-Frobenius ring.

Proof. By [7, Lemma 5.3], we have the implication pure ⇒ dominant and by Prop. 4 we have dominant ⇒ quasi-Frobenius. q.e.d.

Thus a commutative Noetherian pure ring is self-injective and therefore has a property, which could be called "absolutely pure": it is a pure submodule of every module containing it. This justifies the remark preceding Prop. 6. By [7, Satz 6.1], commutative semiprime pure rings are also "absolutely pure" and so are all pure commutative rings, which are algebras over a field, by a remark of Cohn [2].

Finally, let us remark, that using Theorem 5.5. of [2], one easily proves, that a (not necessarily commutative) Noetherian algebra over a field is left pure if and only if it is a quasi-Frobenius algebra.

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