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AN N-PARAMETER CHEBYSHEV SET WHICH IS NOT A SUN

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Recently, Dunham has given examples for 1-parameter and 2-parameter Chebyshev sets which are not suns. In this note n-parameter sets with these properties are described.

1. **Introduction.** When studying the old problem whether Chebyshev sets are always convex, Klee [10] introduced certain sets which were called suns by Efimov and Stechkin [7]. Recently, in two shorts notes Dunham [4, 5] has given examples of 1-parameter- and 2-parameter-sets which are Chebyshev sets but not suns (cf. also [3]). The examples refer to Chebyshev sets in $\mathscr{C}[0, 1]$ containing an isolated point.

Combining Dunham's idea with some more advanced techniques, in this note we will construct Chebyshev sets in $\mathscr{C}[0, 1]$ which are the union of an n-dimensional manifold with boundary and an isolated point. Since every sun is a connected set [4], the constructed set is not a sun.

2. **The underlying set.** The construction is started by introducing the following convex cone in $\mathscr{C}[0, 1]$:

$$K = \left\{h : h(x) = \sum_{j=1}^{n} \frac{a_j}{x+j}, \quad a_j \geq 0, \quad j = 1, 2, \ldots, n\right\}. \tag{2.1}$$

Observe that $K\backslash\{0\}$ belongs to the set of positive functions:

$$C^+ = \{h \in \mathscr{C}[0, 1] : h(x) > 0, x \in [0, 1]\}. \tag{2.2}$$

Moreover, the cone K has the Haar property [1].

Definition. Let $u_1, u_2, \ldots, u_n \in \mathscr{C}[0, 1]$ and $0 \leq m \leq n$. The convex cone

$$\left\{h : h(x) = \sum_{j=1}^{n} a_j u_j(x);\ a_j \in \mathbb{R}, j = 1, 2, \ldots, m;\ a_j \geq 0, j = m+1, \ldots, n\right\}$$

has the Haar property, if the functions $\{u_j\}_{j\in}J$ span a Haar subspace whenever

$$\{1, 2, \ldots, m\} \subset J \subset \{1, 2, \ldots, n\}.$$

More generally, we get cones with the Haar property contained in $C^+ \cup \{0\}$,

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when in (2.1) the terms $(x+j)^{-1}$ are replaced by $\gamma(j, x)$ with γ being an arbitrary totally positive kernel [9].

The function

$$\varphi(x, y) = e^{y-(x/y)}, \qquad 0 \le x \le 1, y > 0, \tag{2.3}$$

is strictly increasing in y, if x is considered fixed. Hence, φ induces a continuous mapping:

$$\psi: C^+ \to C^+,$$
$$(\psi h)(x) = \varphi(x, h(x)).$$

We will consider the approximation in the transformed family

$$G = \psi(K \backslash \{0\}) \cup \{0\}.$$

Since $g(0) > 1$ for each $g \in G$, $g \neq 0$, zero is an isolated point in G.

3. **Existence.** Let $\mathscr{C}[0, 1]$ be endowed with the uniform norm:

$$\|f\| = \sup\{|f(x)| : x \in [0, 1]\}.$$

An element g^* in a non-void subset $G \subset \mathscr{C}[0, 1]$ is called a best approximation to f in G, if $\|f-g\| \ge \|f-g^*\|$ for all $g \in G$.

To prove that there is a best approximation in G to each $f \in \mathscr{C}[0, 1]$ consider a minimizing sequence $\{g_\nu\}$ satisfying

$$\lim_{\nu \to \infty} \|f-g_\nu\| = \eta := \inf\{\|f-g\| : g \in G\}.$$

Without loss of generality we may assume $g_\nu \neq 0$. Let $g_\nu = \psi(h_\nu)$. By standard arguments $\{g_\nu\}$ is bounded. This implies boundedness of $g_\nu(0)$ and $h_\nu(0)$. From the representation (2.1) of the elements in K it follows that $\|h_\nu\|$ is also bounded. Select a subsequence of $\{h_\nu\}$ which converges to some $h^* \in K$. If $h^* \neq 0$, then the corresponding subsequence of $\{g_\nu\}$ converges uniformly to $g^* = \psi(h^*)$, which is a best approximation. If on the other hand $h^* = 0$, then the subsequence converges to $g^* = 0$ uniformly on each compact subinterval of (0, 1). This implies optimality of g^* by simple arguments (cf. [5]).

4. **Varisolvency of transformed Haar subspaces.** Assume that $u_1, u_2, \ldots, u_d \in \mathscr{C}[0, 1]$ span a d-dimensional subspace. With these functions a mapping

$$F: \mathbb{R}^d \to \mathscr{C}[0, 1],$$
$$F(a_1, a_2, \ldots, a_d) = \sum_i a_i u_i(x)$$

is defined. Let A be an open subset of $\mathbb{R}^d$ such that $H = F(A)$ is contained in C^+, the set of positive functions. Then $V = \psi(H)$ is a well defined family which will be investigated now.

Let $h_1, h_2 \in H$, $h_1 \neq h_2$. By the Haar condition $h_1 - h_2$ has at most $d-1$ zeros in [0, 1]. It follows from the monotonicity of $\varphi(x, h)$ that $\psi(h_1) - \psi(h_2)$ has as many

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zeros as h_1-h_2. Consequently, for each pair $g_1, g_2 \in V$ the difference g_1-g_2 has at most $d-1$ zeros.

Let $x_1<x_2<\cdots<x_d$ be d distinct points in [0, 1]. We introduce the restriction mapping

$$R:\mathscr{C}[0,1] \to \mathbb{R}^d$$

$$R \cdot f = (f(x_1), f(x_2), \ldots, f(x_d)).$$

The preceding discussion shows that $R: V\to\mathbb{R}^d$ is a one-one mapping. Consequently the product map $R \circ \psi \circ F: A\to R(V) \subset \mathbb{R}^d$ is a homeomorphism. By virtue of Brouwer's theorem on the invariance of the domain [8], $R(V)$ is open in $\mathbb{R}^d$. This means that the set of vectors $(y_1, y_2 \ldots, y_d)$, for which the interpolation problem

$$g(x_i) = y_i, \qquad i = 1, 2, \ldots, d$$

has a solution $g \in V$, is open in d-space. Moreover, the solution is determined by the continuous mapping $R^{-1}=\psi \circ A \circ (R \circ \psi \circ A)^{-1}$. Hence, V is varisolvent [12, p. 3] with constant degree d.

Rice's theory of varisolvent families establishes that there is at most one best approximation in V. The gap in his theory discovered by Dunham [6], does not matter in this case, because the degree is a constant [2].

Finally, we notice that V is asymptotically convex [11, p. 163] and is an Haar embedded manifold [13]. The construction of sets with these properties from Haar subspaces in [11] and [13] is very similar.

5. **Uniqueness.** Now we are ready to prove uniqueness of the best approximation in the set G introduced in Section 2. Formally the proof is similar to the proof of uniqueness for cones with the Haar property [1].

Assume that $g_i=\psi(h_i)\neq 0$, $i=1, 2$, are two best approximations to f in G. Put $h^*=(h_1+h_2)/2$ and observe that $g^*=\psi(h^*)$ is another best approximation, because the monotonicity of φ implies that $h^*(x)$ lies between $h_1(x)$ and $h_2(x)$ for each $x \in [0, 1]$. Write $h^*(x)=\sum_{j=1}^{n} a_j^* \cdot (x+j)^{-1}$ and set $J=\{j: 1\leq j\leq n, a_j^*>0\}$ The manifold

$$H = \left\{h = \sum_{j\in J} a_j(x+j)^{-1}: a_j \in \mathbb{R}\right\} \cap C^+$$

is a subset of a Haar subspace and satisfies the conditions specified in the last section. Hence, there is at most one best approximation in the varisolvent family $\psi(H)$. Since $g_1, g_2 \in \psi(H)$, we have $g_1=g_2$. This proves uniqueness in $G\backslash\{0\}$.

Assume that $g_1=\psi(h_1)\neq 0$ and $g_2=0$ are two best approximations. Put $h_3=h_1/2$. From $g_2(x)=0<\psi(h_3)(x)<\psi(h_1)(x)$ we conclude that $g_3=\psi(h_3) \in G$ is another best approximation. This contradicts uniqueness in $G\backslash\{0\}$.

References

1. D. Braess, *Kritische Punkte bei der nichtlinearen Tschebyscheff-Approximation*. Math. Z. **132** (1973), 327–341.

2. ———, *On varisolvency and alternation*, J. Approximation Theory **12** (1974), 230–233.

3. B. Brosowski, F. Deutsch, J. Lambert and P. D. Morris, *Chebyshev sets which are not suns*, Math. Annalen **212** (1974), 89–101.

4. C. B. Dunham, *Chebyshev sets in* $\mathscr{C}[0, 1]$ *which are not suns*, Canad. Math. Bull. (1) **18** (1975), 35–38.

5. ———, *A 2-parameter Chebyshev set in* $\mathscr{C}[0, 1]$ *which is not a sun*, Proc. Amer. Math. Soc. (to appear).

6. ———, *Necessity of alternation*, Canad. Math. Bull. **11** (1968), 743–744.

7. N. V. Efimov and S. B. Stechkin, *Some properties of Chebyshev sets* (Russian) Dokl. Akad. Nauk SSSR **118** (1958), 17–19.

8. W. Hurewicz and H. Wallmann, *Dimension Theory*, Princeton University Press, Princeton 1948.

9. S. Karlin, *Total Positivity* I, Stanford University Press, Stanford 1968.

10. V. Klee, *A characterization of convex sets*, Amer. Math Monthly **56** (1949), 247–249.

11. G. Meinardus, *Approximation of Functions: Theory and Numerical Methods*, Springer, Berlin-Heidelberg-New York 1967.

12. J. R. Rice, *The Approximation of Functions* 2, *Nonlinear and Multivariate Theory*, Addison-Wesley, Reading 1969.

13. D. Wulbert, *Uniqueness and differential characterization of approximations from manifolds of functions*, Amer. J. Math. **93** (1971), 350–366.

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