QUASIDISTINGUISHED COUNTABLE ENLARGEMENTS OF NORMED SPACES

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1. Introduction. If E is a Hausdorff locally convex space and M is an $\aleph_0$-dimensional subspace of the algebraic dual E^* that is transverse to the continuous dual E', then, according to [**7**], the Mackey topology $\tau(E, E' + M)$ is a *countable enlargement* (CE) *of* $\tau(E, E')$ [or *of* E]. Much is still unknown as to when CEs preserve barrelledness (cf. [**14**]). E is *quasidistinguished* (QD) if each bounded subset of the completion $\hat{E}$ is contained in the completion of a bounded subset of E [**12**]. Clearly, each normed space is QD, and Tsirulnikov [**12**] asked if each CE of a normed space must be a QDCE, i.e., must preserve the QD property. Since CEs preserve metrizability (but not normability), her question was whether metrizable spaces so obtained must be QD, and was moderated by Amemiya's negative answer (cf. [**5**, p. 404]) to Grothendieck's query, who had asked if *all* metrizable spaces are QD, having proved the separable ones are [**4**].

And although Catalán and Tweddle [**1**] showed that every infinite-dimensional normed space admits a QDCE, they answered her question in the negative by their Proposition 2, which is transparently and precisely equivalent to the second part of the following.

FACT. *A sufficient condition in order that every CE of a normed space E be a QDCE is that every* (*Hausdorff*) *quotient* E/H *with* $\dim(E/H) \leq c$ *be separable. Moreover* (*Catalán and Tweddle* [**1**, *Proposition* 2]), *the condition is necessary if we assume the Continuum Hypothesis* (*CH*).

As pointed out in [**1**], $E = l^\infty = l^\infty/\{0\}$ now yields a specific negative answer.

This note's contribution is twofold: (1) We twice answer Tsirulnikov's question without assuming CH or any other non-ZFC axiom. If one subsequently assumes CH, our answers become that of Catalán and Tweddle above [**1**, Proposition 2].

(2) We improve Propositions 1 and 3 of [**1**] so as to provide the first part of the above fact, again without assuming CH or any other non-ZFC axiom.

2. Results. In the remainder of the paper E will denote an infinite-dimensional normed space, B its closed unit ball, $\hat{B}$ the closed unit ball in the completion $\hat{E}$ of E, and a bar over a set will indicate closure in the normed space E unless otherwise indicated as, for example, in the equation $\hat{B} = \bar{B}^{\hat{E}}$. Denote by M an arbitrary $\aleph_0$-dimensional subspace of E^* such that $M \cap E = \{0\}$, with Hamel basis $\{f_n : n \in \mathbb{N}\}$. Tsirulnikov [**12**, Lemma] showed that the mapping θ which takes $x \in E$ into $(x, f_1(x), f_2(x), \ldots)$ is an isomorphism from $(E, \tau(E, E' + M))$ onto a dense subspace D of the Fréchet space $\hat{E} \times \omega$, where ω is the product of $\aleph_0$ copies of the scalar field $\mathbb{K}$.

We can use Catalán and Tweddle's characterization of QDCE ((ii) below) to find simpler ones.

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LEMMA. *The following are equivalent.*

(i) *The countable enlargement* $\tau(E, E'+M)$ *is a QDCE.*

(ii) *There is a bounded subset* A *of* D *such that* $\bar{A}^{\hat{E}\times\omega} \supseteq \hat{B}\times\{0\}$.

(iii) *There is a bounded subset* A *of* D *such that* $\bar{A}^{E\times\omega} \supseteq B\times\{0\}$.

(iv) *There is a* $\tau(E, E'+M)$*-bounded subset* A *of* E *such that* $\bar{A} \supseteq B$.

(v) *There is a* $\sigma(E, M)$*-bounded subset* A *of* E *such that* $\bar{A} \supseteq B$.

Proof. (i)$\Leftrightarrow$(ii) is due to [**1**, §2], and (ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v) is clear.

[(v)$\Rightarrow$(iv).] Assume A is a $\sigma(E, M)$-bounded subset of E with $\bar{A} \supseteq B$. Then $A\cap 2B$ is a $\tau(E, E'+M)$-bounded subset of E and $\overline{A\cap 2B} \supseteq B$, since given $x\in B$ there exists some sequence $\{x_n : n\in\mathbb{N}\}$ in A converging to x, so that $\|x_n - x\| \le 1$ for sufficiently large n; i.e., $x_n \in A\cap 2B$.

[(iv)$\Rightarrow$(iii).] Suppose A_1 is a $\tau(E, E'+M)$-bounded subset of E with $\bar{A}_1 \supseteq B$. Since closures and continuous linear images of bounded sets are bounded, there is a closed bounded subset C of ω such that

$$\overline{\theta(A_1)}^{E\times\omega} \subseteq E\times C.$$

Let π denote the projection of $E\times\omega$ onto E along ω. Since C is compact,

$$\pi(\overline{\theta(A_1)}^{E\times\omega}) \supseteq \bar{A}_1 \supseteq B.$$

Thus $\overline{\theta(A_1)}^{E\times\omega} - (\{0\}\times C) \supseteq B\times\{0\}$. Now there exists some $\tau(E, E'+M)$-bounded subset A_2 such that

$$\overline{\theta(A_2)}^{E\times\omega} \supseteq \{0\}\times C$$

by [**6**, p. 133, Lemma 2]. Thus $A_1 - A_2$ is $\tau(E, E'+M)$-bounded, and $\theta(A_1 - A_2)$ is bounded in D, with

$$\overline{\theta(A_1 - A_2)}^{E\times\omega} \supseteq \overline{\theta(A_1)}^{E\times\omega} - \overline{\theta(A_2)}^{E\times\omega} \supseteq B\times\{0\}. \quad \square$$

Density character is the same for the quotient by a subspace as by its closure, and if uncountable, cannot exceed dimension. Thus a simple rephrasing of [**1**] yields

(Catalán and Tweddle [**1**, Proposition 2].) *Assume CH. There is a CE of E that is not a QDCE if there is a (normed) quotient of E with dimension and density character* $\mathfrak{c}$.

Without assuming CH, we twice replace $\mathfrak{c}$, once with the bounding cardinal $\mathfrak{b}$ and once with the dominating cardinal $\mathfrak{d}$. These cardinals, defined in 1939 and 1960 (cf. [**2**]), were recently given locally convex characterizations ([**9**], [**10**], [**11**]); e.g., $\mathfrak{b}$ is the least infinite-dimensionality for metrizable barrelled spaces, and $\mathfrak{d}$ is the least size of a fundamental system of bounded sets for a given non-normable metrizable locally convex space. They satisfy $\aleph_1 \le \mathfrak{b} \le \mathfrak{d} \le \mathfrak{c}$. Since for any infinite cardinal κ it is a trivial matter to find a normed space of dimension and density character κ, we thus twice answer Tsirulnikov's question ((i) and (ii) below), and without assuming CH. Clearly, if we afterward assume CH then (i) and (ii) below become precisely the Catalán-Tweddle answer. To avoid using CH, we rely on two CH-free reconstructions $\psi_{\mathfrak{b}}$ and $\psi_{\mathfrak{d}}$ [**9**] of Tweddle's space ψ [**13**]. These are dense barrelled subspaces of ω with dimension $\mathfrak{b}$ and $\mathfrak{d}$, respectively, whose bounded sets have lesser dimension [**9**, Theorems 6 and 7].

THEOREM. (i) *There is a CE of E which is not a QDCE if there is a quotient E/H with dimension and density character* $\mathfrak{b}$.
(ii) *There is a CE of E which is not a QDCE if there is a quotient E/H with dimension and density character* $\mathfrak{d}$.
(iii) *Each CE of E is a QDCE if every quotient of dimension* $\leq \mathfrak{c}$ *is separable.*

Proof. (i) Let $\{x_\beta + H : \beta \in \mathfrak{b}\}$ be a basis of E/H, and define $g_n \in E^*$ such that $g_n(H) = \{0\}$ and $\{(g_n(x_\beta))_{n=1}^\infty : \beta \in \mathfrak{b}\}$ is a predetermined basis for $\psi_\mathfrak{b}$.

Let $L = \mathrm{sp}(\{g_n : n \in \mathbb{N}\})$ and let M be an algebraic complement of $L \cap E'$ in L, so that $\dim(M) \leq \aleph_0$. Any $\tau(E, E' + M)$-bounded subset A of E is $\sigma(E, L)$-bounded; i.e., each $g_n(A)$ is a bounded scalar set. Therefore $\{(g_n(y))_{n=1}^\infty ; y \in A\}$ is a bounded subset of $\psi_\mathfrak{b}$, thus of dimension less than $\mathfrak{b}$, so the projection $\pi(A)$ of A into $\mathrm{sp}(\{x_\beta : \beta \in \mathfrak{b}\})$ along H also has dimension less than $\mathfrak{b}$. Now $\mathrm{sp}(\pi(A)) + H$ contains A and has codimension $\mathfrak{b}$ in E. Consequently, M is infinite-dimensional; otherwise the choice $A = B \cap M^0$ would have finite-codimensional span in E. Hence $\tau(E, E' + M)$ is a CE. On the other hand, $\bar{A}$ is contained in $\mathrm{sp}(\pi(A)) + H$, a proper subspace of E since the density character of E/H is $\mathfrak{b}$. Therefore $\bar{A} \not\supseteq B$, and by [(iv)$\Leftrightarrow$(i)] of the Lemma, $\tau(E, E' + M)$ is not a QDCE.

(ii) Simply replace $\mathfrak{b}$ by $\mathfrak{d}$.

(iii) The third part of the Theorem is an immediate consequence of the Proposition below. Indeed, if $\tau(E, E' + M)$ is any CE, then $E/\overline{M^0}$ is algebraically isomorphic to a subspace of ω, which has dimension $\mathfrak{c}$, and thus $\dim(E/\overline{M^0}) \leq \mathfrak{c}$. □

PROPOSITION. *$\tau(E, E' + M)$ is a QDCE if $E/\overline{M^0}$ is separable.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence in E such that $\{x_n + \overline{M^0} : n \in \mathbb{N}\}$ is dense in $E/\overline{M^0}$. Each $E_n = \cap\{f_k^\perp : k \leq n\}$ is dense in E since $M \cap E' = \{0\}$, and there exists some u_n in E_n such that

$$\|x_n - u_n\| \leq 1/n.$$

Now $\{u_n + \overline{M^0} : n \in \mathbb{N}\}$ is also dense in $E/\overline{M^0}$, so that $\cup\{u_n + \overline{M^0} : n \in \mathbb{N}\}$ is dense in E, and thus so is $A = \cup\{u_n + M^0 : n \in \mathbb{N}\}$. For each $k \in \mathbb{N}$, $f_k(A) = \{f_k(u_n) : n \in \mathbb{N}\}$ is a finite set, hence bounded; i.e., A is $\sigma(E, M)$-bounded. Therefore [(v)$\Leftrightarrow$(i)] of the Lemma and $\bar{A} = E \supseteq B$ ensure $\tau(E, E' + M)$ is a QDCE. □

Our Proposition improves Propositions 1 and 3 of [**1**]. As indicated earlier, if one assumes CH, then (i) and (ii) of the Theorem becomes Proposition 2 of [**1**], and (iii) the converse. However, the converse to our Proposition is denied by the following

CLAIM. *If $\aleph_0 \leq \dim(E) \leq \mathfrak{c}$, then there is a QDCE $\tau(E, E' + M)$ with $M^0 = \{0\}$.*

Proof. Since E is fit, i.e., has a dense subspace F with algebraic complement G of the same dimension [**8**, Corollary 2], then E has a dense Hamel basis P. Specifically, there is a dense subset D of F with $|D| = \dim(G)$; let $\{H(x) : x \in D\}$ be a partition of a Hamel basis for G such that each $H(x)$ is a (linearly independent) countable set whose closure contains 0. (If necessary, replace members of a given Hamel basis for G with appropriately small positive multiples thereof.) The union of all the sets $x + H(x)$ is linearly independent and thus is contained in a Hamel basis P for E; each $x \in D$ is in the closure of $x + H(x)$, hence in $\bar{P}$. That is, $\bar{P}$ contains D, hence F, hence E.

Let $I = \{a \in \mathbb{K} : |a| \leqq 1\}$. Since separability is a $\mathfrak{c}$-multiplicative property [3, Corollary 2.3.16], there is a dense sequence S in the separable compact product space I^P. Now sp(S) is dense in $\mathbb{K}^P$, and $\mathbb{K}^P$ is naturally identified with $(E^*, \sigma(E^*, E))$ via $x^* \mapsto (x^*(x))_{x \in P}$. Under this identification $M = \text{sp}(S)$ is a dense subspace of E^* of dimension $\aleph_0$. Thus $M^0 = \{0\}$ and $\tau(E, E' + M)$ is a CE, since for any absolutely convex linear combination f of members of S,

$$\{f\}^0 \supseteq S^0 \supseteq P,$$

implying

$$\overline{\{f\}^0} \supseteq \bar{P} = E,$$

and f is either 0 or discontinuous, i.e., $M \cap E' = \{0\}$. Now $S^0 \supseteq P$ implies P is $\sigma(E, M)$-bounded, and from $\bar{P} = E \supseteq B$ it follows that $\tau(E, E' + M)$ is a QDCE, by the Lemma [(v)$\Leftrightarrow$(i)].

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