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SMALL AMPLITUDE LIMIT CYCLES FOR CUBIC SYSTEMS

V. GUÍÑEZ, E. SÁEZ AND I. SZÁNTÓ

ABSTRACT. In this article we study the simultaneous generation of limit cycles out of singular points and infinity for the family of cubic planar systems

$$\dot{x} = y(ax^2 + y^2 - 1) - 2cy^2 + \varepsilon xy(-4 + y)$$
$$\dot{y} = -x(x^2 + by^2 - 1) + 2dx^2 + \varepsilon xy(-4 + x).$$

With a suitable choice of parameters, the origin and four other singularities are foci and infinity is a periodic orbit. We prove that it is possible to obtain the following configuration of limit cycles: two small amplitude limit cycles out of the origin, a small amplitude limit cycle out of each of the other four foci, and a large amplitude limit cycle out of infinity. We also obtain other configurations with fewer limit cycles.

1. **Introduction.** In this paper we consider systems of the form

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1}$$

where P and Q are relatively prime real cubic polynomials. We are interested in the possible configurations of small and large amplitude limit cycles for systems of form (1). Here small amplitude limit cycles are limit cycles which bifurcate out of a critical point, and large amplitude limit cycles are limit cycles which do so out of infinity.

It is well known that small amplitude limit cycles can be obtained from a fine focus of order k (see below) with a sequence of perturbations of the coefficients of the system such that each perturbation reduces the order of the fine focus and reverses its stability. We recall that a critical point is a fine focus of (1), if it is a centre for the corresponding linearised system. We now define the order of a fine focus. For simplicity we assume that the origin is a fine focus. It is also well known that there is a function V defined in a neighbourhood of the origin such that $\dot{V}$, its rate of change along orbits, is of the form $\eta_2 r^2 + \eta_4 r^4 + \cdots$. The η_{2k} are the *focal values* and are polynomial functions of the coefficients in P and Q. The origin is a fine focus of order k if

$$\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0 \text{ and } \eta_{2k+2} \neq 0.$$

In general, given a class of systems with a fine focus at the origin we compute each η_{2k} modulo the ideal generated by $\{\eta_2, \ldots, \eta_{2k-2}\}$—thus we set $\eta_2 = \eta_4 = \cdots = \eta_{2k-2} = 0$

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in the expression for η_{2k}. The polynomials so obtained are called the *Liapunov quantites*, and denoted by $L(0), L(1), \ldots$ (*cf.* [9]).

There exists another method for computing the focal values for a fine focus which can be extended for infinity when it is a periodic orbit for the system. If the origin is a focus, system (1) can be written in polar coordinates (r, θ) in the form

$$\frac{dr}{d\theta} = R(r, \theta), \tag{2}$$

for r sufficiently small. The corresponding flow is of the form

$$r(\theta, r_0) = \sum \alpha_n(\theta) r_0^n,$$

where the $\alpha_n(\theta)$ are determined recursively by substitution into (2). The numbers $\eta_k = \alpha_k(2\pi)$ are called *focal values*, and the order of the fine focus is k if $\alpha_1(2\pi) = 1$, $\alpha_n(2\pi) = 0$ if $1 < n \leq 2k$, and $\alpha_{2k+1}(2\pi) \neq 0$; note that we necessarily have that $\alpha_{2l}(2\pi) = 0$ if $\alpha_n(2\pi) = 0$ for $n \leq 2l - 1$. More details can be found in [1].

When infinity is a periodic orbit of (1) focal values and Liapunov quantities are defined in the same way by using the coordinates (ρ, θ) with $\rho = r^{-1}$, and large amplitude limit cycles are similarly obtained.

Let $X = (P, Q)$ be the polynomial vector field associated to system (1). We say that the vector field X, or system (1), has the small configuration

$$\{k_1, \ldots, k_n\}$$

if X has n different foci $F_1, \ldots, F_n$ and k_i small amplitude limit cycles around F_i, for $i = 1, \ldots, n$. Moreover, if X has k_{n+1} large amplitude limit cycles created from infinity, we say that X has the small-large configuration

$$\{k_1, \ldots, k_n; k_{n+1}\}.$$

For quadratic systems the small-configurations are:

$$\{k\}, \text{ with } k = 1, 2, 3, \text{ and } \{1, 1\} \quad (cf.\ [2], [12])$$

and there are no large amplitude limit cycles.

With respect to cubic systems, almost all of the small configurations for systems of form (1) are known when the origin is a focus and there are no quadratic terms. These are:

$$\{k\} \text{ with } k = 1, \ldots, 5 \quad (cf.\ [3], [10]),$$

$$\{1, 2, 1\}, \{k_1, 1, k_1\}, \{k_2, 1, 1, 1, k_2\} \text{ with } k_1 = 1, 2, 3 \text{ and } k_2 = 1, 2 \quad (cf.\ [9])$$

(in the above small configurations the central index corresponds to the origin).

For this class of cubic systems the origin is a fine focus of order at most five (*cf.* [3], [10]). For general cubic systems there are examples with the origin a fine focus of order eight and which generate the small configuration:

$$\{8\} \quad (cf.\ [7]).$$

With respect to small-large configurations, the following are known:

$$\{4;2\} \quad (cf.\ [4], [6]),$$
$$\{2,2,2,2;1\} \quad (cf.\ [8]).$$

In this paper we give examples of cubic systems having the following small-large configurations (see Figure 1)

$$\{1,2,1;1\}, \{1,1,2,1,1;1\} \text{ and } \{1,1,1,1,1\},$$

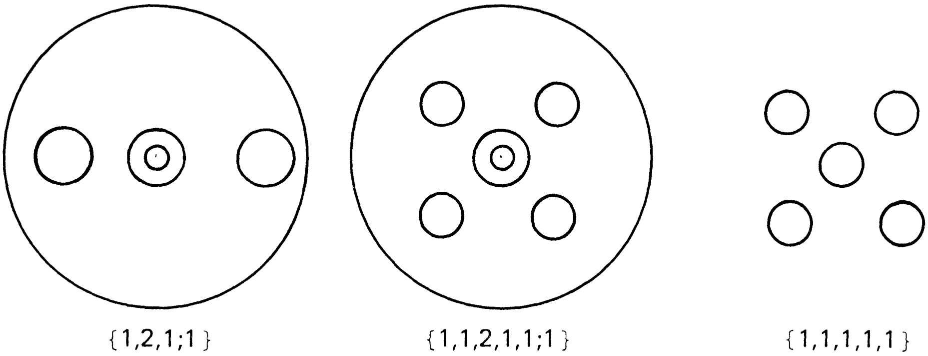

FIGURE 1

Section 2 is devoted to stating the main results of this paper, while the proofs of these results are given in Section 3.

Examples of polynomial vector fields of degree $2k+1$, with k large amplitude limit cycles may be found in [5].

2. **Main results.** Consider the vector field $X = (P, Q)$ where

$$(3) \qquad \begin{aligned} P(x,y) &= y(ax^2 + y^2 - 1) \\ Q(x,y) &= -x(x^2 + by^2 - 1), \end{aligned}$$

with $a > b > 1$. Since

$$(4) \qquad P(-x,y) = P(x,y), \quad Q(-x,y) = -Q(x,y),$$

$$(5) \qquad P(x,-y) = -P(x,y), \quad Q(x,-y) = Q(x,y),$$

the points $(0,0)$, $(\pm 1, 0)$, $(0, \pm 1)$ and infinity are centres for X. Moreover, the points $\left(\pm\sqrt{\frac{b-1}{ab-1}}, \pm\sqrt{\frac{a-1}{ab-1}}\right)$ are hyperbolic saddles. The phase portrait of X is shown in Figure 2.

Let $X_{c,d}$ be the family of vector fields given by

$$(6) \qquad X_{c,d}(x,y) = X(x,y) + (-2cy^2, 2dx^2).$$

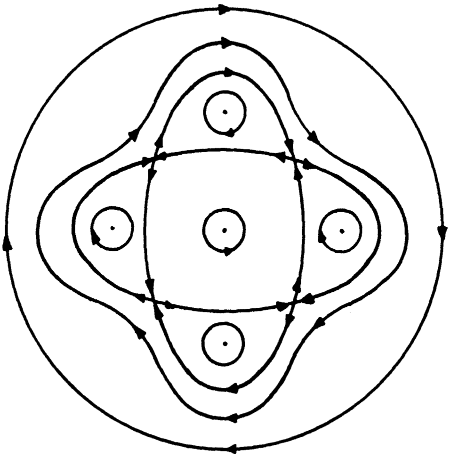

FIGURE 2

For c, d small enough, the critical points of $X_{c,d}$ with Poincaré index +1 are

$$(0,0),\ \big(A_-(d),0\big),\ \big(A_+(d),0\big),\ \big(0,A_-(c)\big) \text{ and } \big(0,A_+(c)\big),$$

where $A_-(z) = z - \sqrt{z^2+1}$ and $A_+(z) = z + \sqrt{z^2+1}$.

Our next lemma shows the different possibilities for these critical points.

LEMMA 1. *Let us consider the field $X_{c,d}$ with $|c|, |d| << 1$. Then:*

1) *If cd is positive (resp. negative), the origin is an expanding (resp. attracting) fine focus of order two, and if cd vanishes, the origin is a centre.*
2) *If c is positive (resp. negative), the point $\big(A_-(d),0\big)$ is an attracting (resp. expanding) fine focus of order one, and the point $\big(A_+(d),0\big)$ is an expanding (resp. attracting) fine focus of order one.*
3) *If d is positive (resp. negative), the point $\big(0,A_-(c)\big)$ is an attracting (resp. expanding) fine focus of order one, and the point $\big(0,A_+(c)\big)$ is an expanding (resp. attracting) fine focus of order one.*
4) *If c (resp. d) vanishes, the points $\big(A_-(d),0\big)$ and $\big(A_+(d),0\big)$ (resp. the points $\big(0,A_-(c)\big)$ and $\big(0,A_+(c)\big)$) are centres.*
5) *If $a > b > 1$ and $cd \neq 0$ and $\frac{17-7b}{8} < a < \frac{17-8b}{7}$ (resp. $\frac{17-8b}{7} < a < \frac{17-7b}{8}$), then infinity is a fine focus of order one which is attractive (resp. expansive) for cd negative and expansive (resp. attractive) for cd positive. If cd vanishes, infinity is a centre.*

Figure 3 gives a graphical summary of these results.

We now consider the family of vector fields $X_{c,d,\varepsilon} = X_{c,d} + Y_\varepsilon$ where

$$Y_\varepsilon(x,y) = \varepsilon xy \cdot (-4+y, -4+x).$$

In the next lemma we give the small-large configurations associated to $X_{c,d,\varepsilon}$.

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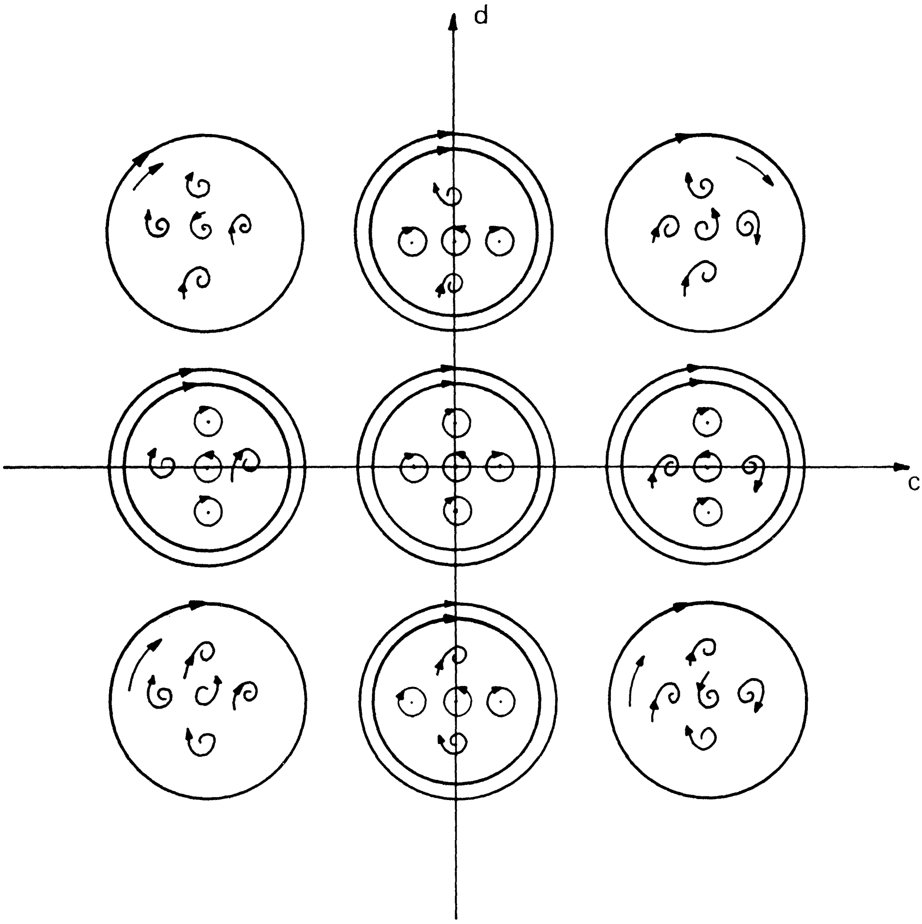

FIGURE 3

LEMMA 2. *For c, d, ε small enough and $a > b > 1$ and $cd \neq 0$ and $\frac{17-8b}{7} < a < \frac{17-7b}{8}$, the vector field $X_{c,d,\varepsilon}$ has the following small-configurations:*

1) $\{1, 1, 1, 1, 1; 1\}$ *if c, d are positive and ε is negative;*

2) $\{1, 1, 1; 1\}$ *if d is negative and c, ε are positive;*

3) $\{1, 1, 1, 1\}$ *if c, d are negative and ε is positive.*

Finally, we consider the family of vector fields $X_{c,d,\varepsilon,\eta}$ with c, d, ε small and $0 < |\eta| << \min\{|c|, |d|, |\varepsilon|\}$ defined by

$$X_{c,d,\varepsilon,\eta}(x, y) = X_{c,d,\varepsilon}(x, y) + \eta(x, 0)$$

in the following

THEOREM. *For c, d, ε small enough and $a > b > 1$ and $cd \neq 0$ and $\frac{17-8b}{7} < a < \frac{17-7b}{8}$, the vector field $X_{c,d,\varepsilon,\eta}$ has the following small-configurations:*

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1) $\{1, 1, 2, 1, 1; 1\}$ *if* c, d, η *are positive and* ε *is negative;*
2) $\{1, 2, 1; 1\}$ *if* d, η *are negative and* c, ε *are positive;*
3) $\{1, 1, 1, 1, 1\}$ *if* c, d, η *are negative and* ε *is positive.*

3. **Proof of the main results.** Before proving our main results, we note that for the vector field X the closed small orbits encircling the origin have positive orientation, while the small closed orbits encircling the point $\big(A_\pm(d), 0\big)$, those encircling the point $\big(0, A_\pm(c)\big)$, and the large ones encircling infinity have negative orientation. We let $L_{d\pm}(k)$, $L_0(k)$ and $L_{c\pm}(k)$ denote the order k Liapunov quantities at the points, respectively, $\big(A_\pm(d), 0\big)$, $(0, 0)$ and $\big(0, A_\pm(c)\big)$, and $\alpha_k(-2\pi)$ will denote the derivative of order k of the Poincaré map at infinity.

PROOF OF LEMMA 1. For a general cubic system with a focus at the origin

$$\dot{x} = a_{10}x + y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$$
$$\dot{y} = -x + a_{10}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3$$

we have

$$L(0) = a_{10} \text{ and}$$
$$L(1) = -a_{11}(a_{20} + a_{02}) + b_{11}(b_{20} + b_{02}) + 2(a_{20}b_{20} - a_{02}b_{02}) + a_{12} + b_{21}$$

(*cf.* [9, Lemma 2.2]).

Therefore, for system (6) we have $L_0(0) = L_0(1) = 0$.

If $x_0 = A_\pm(d)$, by setting

$$x = u/M + x_0, \quad y = v/N,$$

with M, N positive such that $M^2 = 1/2(1 + x_0 d)$ and $N^2 = 1/(a - 1 + 2x_0 ad)$, system (6) becomes

$$\dot{u} = v + 2x_0 aMN^2uv - 2cN^3v^2 + aM^2N^2u^2v - x_0 bMN^2v^3$$
$$\dot{v} = -u + (3x_0 + 2d)M^3u^2 - x_0 bMN^2v^2 - M^4u^3 - bM^2N^2uv^2$$

and

$$L_{d\pm}(1) = 4A_\pm(d)c(a - b)MN^5.$$

Moreover, by setting

$$x = -v, \quad y = u$$

system (6) becomes

$$\dot{u} = v(bu^2 + v^2 - 1 + 2dv)$$
$$\dot{v} = -u(u^2 + av^2 - 1 - 2cu)$$

and therefore

$$L_{c\pm}(1) = 4A_\pm(c)d(a - b)\tilde{M}\tilde{N}^5,$$

with $\tilde{M}, \tilde{N}$ positive.

To compute $L_0(2)$ we consider the function

$$\begin{aligned} V(x,y) = {} & \frac{1}{2}(x^2+y^2) + \frac{2}{3}(dx^3+cy^3) + \frac{1}{4}\big((a-b)x^4 + 2(1-b)x^2y^2\big) \\ & + \frac{2}{15}\big(d(3a-3b+5)x^5 + 5dx^3y^2 + 5c(a-b+1)x^2y^3 + c(2a-2b+5)y^5\big) \\ & + \frac{1}{36}\big((6a^2-16ac^2-9ab+6a+16bc^2-16c^2+3b^2-6b+16d^2)x^6 \\ & + 6cd(a-b)x^5y + 3(-16ac^2-3ab+6a+16bc^2-16c^2+3b^2-12b \\ & + 6)x^4y^2 + 16cd(a-b-2)x^3y^3 + 24(-8ac^2+8bc^2-8c^2-3b+3)x^2y^4 \\ & + 6cd(-a+b)xy^5\big). \end{aligned}$$

Its rate of change along orbits of system (6) is

$$\dot{V}(x,y) = \frac{1}{6}cd(a-b)(x^2+y^2)^3 + \cdots .$$

Therefore

$$L_0(2) = \frac{1}{6}cd(a-b)$$

and statements 1), 2), 3) and 4) of Lemma 1 are proven.

Concerning 5), let

$$x = \frac{1}{r}\cos(\theta) \text{ and } y = \frac{1}{r}\sin(\theta)$$

in system (6). Then

$$\frac{dr}{d\theta} = -r\frac{R_3(\theta)+rR_2(\theta)}{A_3(\theta)+rA_2(\theta)+r^2} = -r\sum_{k=0}^{\infty} r^{k-1}T_k(\theta)$$

where

$$\begin{aligned} A_3(\theta) &= -\cos^4(\theta) - \sin^4(\theta) - (a+b)\cos^2(\theta)\sin^2(\theta), \\ R_3(\theta) &= \cos(\theta)\sin(\theta)\big((a-1)\cos^2(\theta) - (b-1)\sin^2(\theta)\big), \\ A_2(\theta) &= 2\big(d\cos^3(\theta) - c\sin^3(\theta)\big), \\ R_2(\theta) &= 2\cos(\theta)\sin(\theta)\big(d\cos(\theta) - c\sin(\theta)\big); \end{aligned}$$

the first T_k are

$$\begin{aligned} T_1(\theta) &= -\frac{R_3(\theta)}{A_3(\theta)}, \\ T_2(\theta) &= -\frac{R_2(\theta)+A_2(\theta)T_1(\theta)}{A_3(\theta)}, \\ T_3(\theta) &= -\frac{A_2(\theta)T_2(\theta)+T_1(\theta)}{A_3(\theta)}. \end{aligned}$$

Therefore

$$\alpha_1(\theta) = \exp \int_0^\theta T_1(\epsilon)\, d\epsilon,$$
$$\alpha_2(\theta) = \alpha_1(\theta) \int_0^\theta \alpha_1(\epsilon) T_2(\epsilon)\, d\epsilon,$$
$$\alpha_3(\theta) = \alpha_1(\theta) \int_0^\theta \big(2\alpha_2(\epsilon)T_2(\epsilon) + \alpha_1^2(\epsilon)T_3(\epsilon)\big)\, d\epsilon.$$

For $c = 0$ or $d = 0$ system (6) verifies symmetries (4) or (5) and infinity is a centre. Since $T_1(\theta)$ does not depend on c and d, we have $\alpha_1(-2\pi) = 1$, thus $\alpha_2(-2\pi) = 0$; furthermore

$$\alpha_3(-2\pi) = -\int_0^{-2\pi} \frac{\alpha_1^2(\theta) A_2(\theta) T_2(\theta)}{A_3(\theta)}\, d\theta.$$

Straightforward calculations show that

$$\alpha_3(-2\pi) = \frac{4cd}{3} \int_0^{-2\pi} \frac{\alpha_1^2(\theta) f(\theta)}{A_3^3(\theta)}\, d\theta,$$

with

$$f(\theta) = \cos^4(\theta)\sin^4(\theta)\big((17 - 7a - 8b)\cos^2(\theta) - (17 - 8a - 7b)\sin^2(\theta)\big).$$

Since $f(\theta) > 0$ for $\frac{17-7b}{8} < a < \frac{17-8b}{7}$ and $f(\theta) < 0$ for $\frac{17-8b}{7} < a < \frac{17-7b}{8}$, the result now follows.

PROOF OF LEMMA 2. Clearly, the points

$$(0,0),\ \big(A_-(d),0\big),\ \big(A_+(d),0\big),\ \big(0,A_-(c)\big) \text{ and } \big(0,A_+(c)\big)$$

are critical points of the vector field $X_{c,d,\varepsilon}$. Moreover,

$$\operatorname{div} X_{c,d,\varepsilon}(x,y) = 2(a-b)xy + \varepsilon g(x,y),$$

with

$$g(x,y) = (x+2)^2 + (y+2)^2 - 8.$$

For c, d small enough, the function g is positive at the points $\big(A_+(d),0\big)$ and $\big(0,A_+(c)\big)$, negative at the points $\big(A_-(d),0\big)$ and $\big(0,A_-(c)\big)$, and vanishes at the origin (see Figure 4). Therefore we have that

$$\operatorname{div} X_{c,d,\varepsilon}(0,0) = 0$$

and that a small amplitude limit cycle is created around each of the singularities

1. $\big(A_\pm(d),0\big)$ and $\big(0,A_\pm(c)\big)$ if $0 < -\varepsilon << \min\{c,d\}$,
2. $\big(A_\pm(d),0\big)$ if $d < 0 < \varepsilon << c$, and

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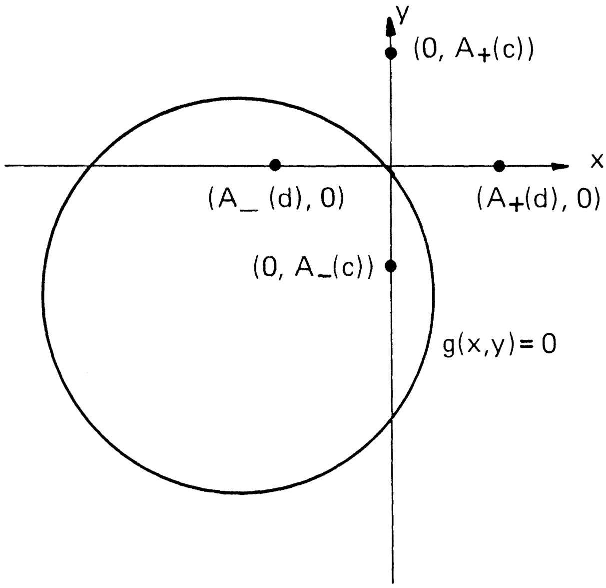

FIGURE 4

3. $\big(A_\pm(d),0\big)$ and $\big(0,A_\pm(c)\big)$ if $\max\{c,d\} << -\varepsilon < 0$.

Furthermore, for the vector field $X_{c,d,\varepsilon}$ we have

$$L_0(1) = 2\varepsilon(1-4c-4d) \text{ and } \alpha_1(-2\pi) = \exp h(\varepsilon),$$

with

$$h(\varepsilon) = -\int_0^{-2\pi} \frac{R_3(\theta) + 2\varepsilon\cos^2(\theta)\sin^2(\theta)}{A_3(\theta) + \varepsilon\cos(\theta)\sin(\theta)\big(\cos^2(\theta) - \sin^2(\theta)\big)}\, d\theta$$

where $A_3(\theta)$ and $R_3(\theta)$ are as in the proof of Lemma 1.

Since h vanishes at zero and its derivative with respect to ε at zero is

$$h'(0) = \int_0^{-2\pi} \frac{\cos^2(\theta)\sin^2(\theta)l(\theta)}{A_3^2(\theta)}\, d\theta,$$

with

$$l(\theta) = (a+1)\cos^4(\theta) + (b+1)\sin^4(\theta) + (a+b+2)\cos^2(\theta)\sin^2(\theta),$$

we have that $h'(0) < 0$ and thus $\alpha_1(-2\pi) < 1$ if ε is positive and $\alpha_1(-2\pi) > 1$ if ε is negative.

Therefore, for cases 1) and 2), a small amplitude limit cycle is created at the origin and a large one at infinity by Lemma 1 which completes the proof.

PROOF OF THEOREM. The limit cycles created in Lemma 2 are hyperbolic; the perturbation $\eta(x,0)$ makes the origin an attracting focus if η is negative, and an expansive focus if η is positive. Then, according to the value of $L_0(1)$ in the proof of Lemma 2, a new small amplitude limit cycle is created out of the origin for the vector field $X_{c,d,\varepsilon,\eta}$ if $|\eta| << \min\{|c|,|d|,|\varepsilon|\}$ and $\varepsilon\eta$ is negative and we obtain the result.

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