Notes: (1) On a Geometrical Problem.
(2) On an Algebraical Equation of Professor Cayley's.

By Professor STEGGALL.

On the Fundamental Principles of Quaternions and other Vector Analyses.

By Dr WILLIAM PEDDIE.

When a student of mathematics commences the study of a subject which involves the assimilation of what are, to him, fundamentally new ideas, his progress is, as a rule, slow at first. And, even after he has become accustomed to these ideas, he may still require a long course of laborious practice, before he can attain to that mastery of the method which would enable him to use it as a powerful aid to research. Thus students, familiar with geometrical methods, when first commencing the study of Cartesian analysis, require much practice before they can call up mentally the geometrical figure corresponding to a given equation. And, the more general the new method is, the greater is the difficulty felt to be. So, in Hamilton's system of quaternions, the difficulty of assimilation is greater than it is in the Cartesian analysis. And it seems as if it were for this reason that, in recent years, attempts have been made, by men of known mathematical ability, to smooth the paths.

Practically, all these attempts consist in using, instead of Hamilton's, another system of quaternions, cut up into parts; the parts of that system being used because they are imagined to be superior to the corresponding parts of Hamilton's system in respect of naturalness. Subsequently, I shall say somewhat regarding the reasonableness (or unreasonableness) of this claim; but, whatever conclusion be accepted on this point, M'Auley's appeal to the spoon-feeders, to "provide spoon-meat of the same *kind* as the other physicians" (*Nature*, Dec. 15, 1892), is most appropriate.

Some of the strictures recently passed on quaternions refer rather to the way in which the subject is presented in the standard treatises than to quaternions themselves. Heaviside (*Electrician*, Nov. 18, 1892) refers to three special "sticking-points" in Tait's treatise. One of these is the investigation of Hamilton's cubic

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in Chap. V. There is no special difficulty in the investigation, though the process may have been difficult enough to *discover*; and, curiously enough, when one turns with great expectations to Heaviside's alternative process, it is with genuine disappointment that it is found to be, *step-for-step*, Hamilton's with $c'Vmn$ written instead of $V\lambda\mu$, and with other corresponding surface changes.

Another special "sticking-point" is said to be in Chap. IV., "where the reader may be puzzled to find out why the usual simple notion of differentials is departed from, although the departure is said to be obligatory." Surely the fact that, in this chapter, the usual notion is freely used, should produce reflection rather than misconception.

Chief of all the "sticking-points" is "the fundamental Chap. II., wherein the rules for the multiplication of vectors are made to depend upon the difficult mathematics of spherical conics, combined with versors, quaternions, and metaphysics." It is somewhat puzzling to find Heaviside speaking of the mathematics of spherical conics—at least so far as they are used there—as difficult. The "metaphysics" evidently refers to Hamilton's speculation, which Tait takes care to call a *quasi*-metaphysical speculation. His conclusions are the necessary logical results of his postulates, which, in so far as they refer to the nature of space, express the results of experience, and cannot be called metaphysical. One of the chief merits of this chapter, from a student's point of view, lies in the wealth of alternative proofs which it contains. Doubtless, by assuming the fundamental rules of vector multiplication, the identification of unit vectors and quadrantal versors might have been more directly made. But it seems as if Heaviside had failed to notice that Tait's method shows that such sweeping assumptions are unnecessary—that *partial* assumption of certain of the rules only is needed.

Another bone of contention is the minus sign which appears in the square of a vector or in a scalar product. Gibbs says (*Nature*, April 2, 1891), "When we come to functions having an analogy to multiplication, the product of the length of two vectors and the cosine of the angle which they include, from any point of view except that of the quaternionist, seems more simple than the same quantity taken negatively." Macfarlane (*Proc. Amer. Ass.*, 1891) says, that "a student of physics finds a difficulty in the principle of quaternions which makes the square of a vector negative"; and

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Heaviside (*Electrician*, Dec. 9, 1892) writes, that "the vector having to submit to the quaternion, leads to the extraordinary result that the square of every vector is a *negative* scalar. This is merely because it is true for quadrantal versors, and the vector has to follow suit. The reciprocal of a vector, too, goes the wrong way, merely to accomodate versors and quaternions."

Now this point raises the whole question of the value of quaternions as such. Given that the quaternion is useless, or nearly so, in itself; and that scalar and vector products are only of use separately; no one will quarrel greatly with the advocates of the positive sign. Heaviside remarks that the physicist "is very much concerned with vectors, but not at all, or at any rate scarcely at all, with quaternions"; that "if the usual investigations of physical mathematics involved quaternions, then the physicist would no doubt have to use them. But they do not. If you translate physical investigations into vectorial language, you do not get quaternions; you get vector algebra instead." Gibbs remarks, that "the question arises whether the quaternionic product can claim a prominent and fundamental place in the system of vector analysis. It certainly does not hold any such place among the fundamental geometrical conceptions as the geometrical sum, the scalar product, or the vector product. The geometrical sum $\alpha + \beta$ represents the third side of a triangle as determined by the sides α and β. $V\alpha\beta$ represents in magnitude the area of the parallelogram determined by the sides α and β, and, in direction, the normal to the plane of the parallelogram. $S\gamma V\alpha\beta$ represents the volume of the parallelopiped, determined by the edges α, β, and γ. These conceptions are the very foundations of geometry." "I do not know of anything which can be urged in favour of the quaternionic product of two vectors as a *fundamental* notion in vector analysis, which does not appear trivial or artificial in comparison with the above considerations. The same is true of the quaternionic quotient, and of the quaternion in general."

Whatever be the case with regard to the mathematician, the statement that the physicist is scarcely, if at all, concerned with quaternions has surely been made without sufficient reflection. We may observe the velocities of a planet at two distinct instants, and merely describe the facts: or we may ask *how* the one became the other. The answer may be given in two ways—either by stating what vector quantity *added to* the one gives the other, or by stating what quantity *acting upon* the one gives the other. The former corresponds to the methods of pure vector analysis; the latter to those of quaternions, involving turning and lengthening. Both methods are of importance to the physicist. He sometimes wishes to consider the external addition to the changing quantity; sometimes to consider the internal changes as such. And one might just as rationally assert that he has to do only with quaternions as that he has to do only with vectors, since it is easy to use a notation expressing a vector in terms of a quaternion. The so-called vector analyst really uses a notation which expresses a quaternion in terms of vectors, and so his analysis simply bristles with quaternions: the quantity is none the less a quaternion because he chooses to shut his eyes to the fact, or at least not to use it as such. When he deals with $V\alpha\beta$, the quantity $\alpha\beta$ is the thing which turns the unit vector β into the unit vector $+\alpha$, or into $-\alpha$, according as we define the square of a unit vector to be equal to positive, or to negative unity, respectively—if we operate from right to left. But the vector analyst refuses to take advantage of what is in his power. Surely Heaviside would not have spoken of "wrong ways" if he had observed that, while in the quaternionic system $\alpha\beta$ turns β into $-\alpha$ and α/β turns β into $+\alpha$; in a similar system, in which the square of a unit vector is positive unity, $\alpha\beta$ would simply do what the quaternionic α/β does: and, if no fancied metaphysical necessity made the analyst regard the reciprocal of a direction as identical with the direction itself, α/β would do what the quaternionic $\alpha\beta$ does. The one method is as "natural" as the other. The choice of one must be ruled by expediency; the test of expediency being chiefly generality and applicability.

I believe most distinctly that students will prefer quaternion methods to those by which it is proposed to supersede them. The former develope a system naturally without any assumptions beyond those made fundamentally. In the latter, new definitions take the place of connecting links—as in the case of a working hypothesis which does not work well. An almost endless series of examples might be given of the singular inapplicability of the non-quaternionic systems to physical and other problems. Macfarlane is practically the only recent writer on the subject who does not arbitrarily exclude the quaternion from his system, which differs from ordinary quaternions in that the square of a unit vector is positive unity, and that he chooses to operate from left to right.

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The quaternionic aspect of his system may be seen thus. Let α be any vector whatsoever, and let i be any unit vector. In Macfarlane's system αii represents a vector got by rotating α rigid-body-wise through two right angles round the axis of i: the corresponding vector in quaternions is $-i\alpha i$ or $i\alpha i^{-1}$. In Macfarlane's system $-i(\alpha i)$ [or $i \cdot \alpha i^{-1}$, if he did not fancy that the direction reciprocal to a given direction *should* be that direction itself] is the vector $-\alpha$: in quaternions this is αii. Certain results have been interchanged, and that is all.

It might not have appeared *a priori* that this was all, for in this system a restriction, which holds in quaternions, disappears. The associative law does not apply, and in this respect the new algebra might have been more general; for, as Kelland points out in the Preface to Kelland and Tait's *Introduction to Quaternions*, generality is attained by the removal of restrictions. In arithmetic, the treatment of fractions was impossible until multiplication ceased to be regarded as a series of additions; and algebra became possible when negative quantities were recognised. But, in algebra, the commutative law holds. Quaternions—the self-contained algebraic system most suitable to tri-dimensional space—became possible when it was denied. But, in quaternions, the associative law holds. It may be that, in some system free from this restriction, greater generality will be reached. But the essential identity, pair by pair, of the results of the two systems under consideration, precludes the idea in this case. And so, the new system being no more general than quaternions, and being distinctly less workable (for no one will maintain that a non-associative algebra is so workable as an associative algebra), expediency decides in favour of quaternions.

Macfarlane asks, "What reason do writers on quaternions give for taking $xx' + yy' + zz'$ negatively in the case of the product of two vectors?" and asserts that "the true reason for taking the expression negatively is to satisfy the rule of association." This is not so: for it is easy to prove that we may take the square of a unit vector as positive unity, and yet get the associative law; provided only that we take $ij = \sqrt{-1}k$, etc., where i, j, and k, are unit rectangular vectors, and ij or $\sqrt{-1}k$ is the quadrantal versor whose axis is k. But, in this case, the product of an even number of vectors is a linear function of the three unit rectangular versors,

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while the product of an odd number is linear in i, j, k. Thus odd and even products are fundamentally distinct, and simplicity is lost.

Another point, in regard to which quaternions have been attacked, is that of applicability to space of n-dimensions. Hyde (*Directional Calculus*, Preface), speaking of Grassmann's method, says, "It seems scarcely possible that any method can be devised, comparable with this, for investigating n-dimensional space;" and Macfarlane asserts that "the method of Hamilton appears to be restricted to space of three dimensions." Gibbs speaks more strongly. "As a contribution to analysis in general, I suppose that there is no question that Grassmann's system is of indefinitely greater extension [than Hamilton's] having no limitation to any particular number of dimensions" (*Nature*, May 28, 1891). "How much more deeply noted in the nature of things are the functions $S\alpha\beta$ and $V\alpha\beta$ than any which depend on the definition of a quaternion, will appear in a strong light if we try to extend our formulæ to space of four or more dimensions. It will not be claimed that the notions of quaternions will apply to such a space, except, indeed, in such a limited and artificial manner as to rob them of their value as a system of geometrical algebra. But vectors exist in such a space, and there must be a vector analysis for such a space. The notions of geometrical addition and the scalar product are evidently applicable in such a space. As we cannot define the direction of a vector, in space of four or more dimensions, by the condition of perpendicularity to two given vectors, the definition of $V\alpha\beta$, as given above, will not apply *totidem verbis* to space of four or more dimensions. But a little change in the definition, which would make no essential difference in three dimensions, would enable us to apply the idea at once to space of any number of dimensions" (*Nature*, April 2, 1891).

Fortunately, the "strong light" of which Gibbs speaks shines the other way. The notions of quaternions *are* applicable to space of four or any number of dimensions. The general system should give a definition of $V\alpha\beta$, perfectly definite in space of any dimensions, and reducing to the usual one when the dimensions are limited to three. And it does.

The problem is to find a general system involving quantities $i, j, k, l, \ldots \ldots$, which represent unit rectangular vectors in cyclical order, and obey the laws $i^2 = j^2 = \ldots\ldots = -1$; $ij = -ji$, $\ldots\ldots$; and

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also the associative law. And the system must reduce to quaternions when only three of these vectors exist.

This problem has been worked out by Clifford in a paper *On the Classification of Geometric Algebras.* He makes the above assumptions, and then seeks to find what assumption must be made analogous to the Hamilton law $ijk = -1$. The following method of procedure is perhaps more in accordance with Hamiltonian ideas.

In three dimensions, the product of two unit rectangular vectors is the remaining rectangular unit vector. Assume generally that the product of $n-1$ such units, in cyclical order, is a vector quantity representable by the remaining rectangular vector; so that

$$ijk\ldots\ldots m = -\psi_n n,$$

where $-\psi_n$ is the operator which transforms n into $ijk\ldots\ldots m$, and we get

$$ijk\ldots\ldots n = \psi_n;$$

and, if we put $\psi_n\psi_n^{-1} = \psi_n^{-1}\psi_n = 1$, we get

$$1 = \psi_n^{-1} ijk\ldots\ldots n.$$

Also $n\psi_n = nijk\ldots\ldots n = \pm ijk\ldots\ldots m$, according as n is even or odd; that is, $n\psi_n = \pm\psi_n n$ according as n is odd or even. And $n = \pm\psi_n n \psi_n^{-1}$ according as n is odd or even. In this way we see that, quite generally, ψ_n and ψ_n^{-1} are symbols commutative with vectors if n is odd, but non-commutative if n is even.

Whatever be the sign of ψ_n^2, the sign of ψ_{n+1}^2 must be similar or dissimilar according as n is odd or even: for ψ_{n+1}^2 is reduced to $\pm\psi_n^2$ by n interchanges of the vector $(n+1)$ with other vectors, together with the substitution of -1 for $(n+1)^2$. It follows that, in the case of even values *alone*, ψ^2 is positive and negative unity alternately; and the same rule holds in the case of odd values alone. In the special case of three dimensions, $\psi^2 = -1$, from which all other cases may be deduced

In an odd space of n-dimensions, we may put

$$\psi_n = \omega^{\frac{n+1}{2}},$$

where ω is a quantity whose square is negative unity. And, in an even space of n-dimensions, we may put

$$\psi_n = \omega^{\frac{n}{2}},$$

where ω^2 is also negative unity.

Except in the cases in which n is divisible by 4, we may suppose ω to be the imaginary of ordinary algebra; in these cases

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ψ would be positive or negative unity, so that the more general symbol should be retained because of the non-commutative nature of ψ in spaces of even dimensions.

In particular, in two dimensions, $ij = \sqrt{-1}$. Hence we get $j = -i\sqrt{-1} = \sqrt{-1}i$, and $i = -\sqrt{-1}j$. If $a = xi + yj$, the operator $\sqrt{-1}(\equiv \psi_2)$ gives $\sqrt{-1}a = xj - yi$. In this case there is no need to retain the symbols i, j; for $a = xi + yj = (x + y\sqrt{-1})i$, and i denotes a given direction, so that a may be completely denoted by $x + y\sqrt{-1}$. It appears, therefore, that complex algebra is a special case of this generalised quaternionic system. Ordinary arithmetic may be regarded as the special case $\psi_0 = 1$.

Thus, in respect of generality, as well as of simplicity, the quaternionic method has the advantage.

In four dimensions, from $ijkl = \psi$, we get $ijk = -\psi l$, $jkl = \psi i$, $kli = -\psi j$, $lij = \psi k$. It does not follow that the space is non-symmetrical, or that, as the condition of symmetry, we should have $ijk = -\psi l$, $jkl = -\psi i$, etc. For we have seen that, in the symmetrical two-dimensional space, we have $i = -\sqrt{-1}j$, $j = \sqrt{-1}i$, *not* $j = -\sqrt{-1}i$, as a necessary condition for symmetry.

In any space $V\alpha\beta$ represents a directed area in the plane of α, β. In three dimensions, it happens to be representable by a linear vector.

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JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

Early History of the Symmedian Point.

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In 1873, at the Lyons meeting of the French Association for the Advancement of the Sciences, Monsieur Emile Lemoine called attention to a particular point within a plane triangle which he called the centre of antiparallel medians. Since that time the

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