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Note on four circles touching a common circle.

By Professor Allardice.

One of the proofs of the theorem that the three escribed circles and the inscribed circle of a triangle touch a common circle, depends upon the following well-known property of four circles that touch a common circle:—The common tangents of four such circles satisfy the relation

$$12 \,.\, 34 + 14 \,.\, 23 = 13 \,.\, 24\,;$$

where 12 denotes the common tangent to the circles 1 and 2, etc. The converse part of the theorem, however, namely, that when the above relation holds good, the four circles touch a common circle, is generally assumed; the object of this note is to supply a demonstration of that part of the theorem. It should be noted that if the circles 1 and 2 touch the fifth circle either both externally or both internally, 12 denotes the direct common tangent; while if one of them touches externally and the other internally, 12 denotes the transverse common tangent. Further, the length of the direct common tangent to two circles remains unaltered if the radius of each be diminished or be increased by the same amount; while the length of the transverse common tangent remains unaltered, if the radius of one be increased and that of the other be diminished by the same amount. It is obvious, moreover, that if two circles, 1 and 2, touch a circle A externally, and if the radii of 1 and 2 be diminished and the radius of A increased by the same amount, the contact still holds good; and all the other cases may be easily considered.

Suppose now that the relation $12 \,.\, 34 + 14 \,.\, 23 = 13 \,.\, 24$ is given.

We shall assume that if 12 and 13 are both direct common tangents then 23 must be a direct common tangent; and so on. The cases where this does not hold good will be referred to later on.

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There are two cases to be considered :—

1st. When there are not as many as three circles in which all the common tangents are direct.

2nd. When there are as many as three such circles.

1st. Let the common tangent to 1 and 2 (hence also to 3 and 4) be direct.

Invert the circles so that 1 and 2 become equal, and diminish the radii so that these circles become points. Then the radii of circles 3, 4, and 5 may be either increased or diminished, so that the contact holds good, and so that the lengths of the common tangents under consideration remain unaltered.

Let A and B be the two points, C and D the two circles in the figure thus obtained (Fig. 4).

Describe a circle to pass through A and B and touch D externally, and another circle to pass through A and touch C and D, so that C and D both touch this circle internally. Let these circles intersect in P; and invert the whole figure with respect to P. The result of this inversion is given in Fig. 5.

Make $D'S' = TS$; and draw a circle to touch AT and AD′ at S and S′. Then we have

$$AB \cdot S'D' + AD' \cdot BU = AS' \cdot BD'$$

$$\text{and} \quad AB \cdot S'D' + AD' \cdot BS' = AS' \cdot BD'.$$

Hence $BU = BS'$; and therefore B lies on the radical axis of C and the circle that touches AT and AD′ at S and S′, namely, AT. This is impossible unless AT and AD′ coincide, or the circle C coincides with the circle that touches AT and AD′ at S and S′. Thus the theorem is proved for this case.

2nd. We have now to consider the case in which there are as many as three circles in which all the common tangents are direct. Let these be the circles 1, 2, and 3. Invert the figure so that these three circles become equal, and diminish the radii so that they reduce to points. Construct the circle that passes through these three points, and invert with reference to a point on it. We thus have a figure (Fig. 6), consisting of three points A, B, C, on a straight line and a circle D; and we have to show that the straight line touches the circle.

Draw a tangent AT from A to the circle; then AT is 14.

With A as centre and AT as radius construct a circle; this will pass through the limiting point Q; and the same is true of the corresponding circles with B and C as centres.

Thus we have $\quad BC \,.\, AQ + AB \,.\, CQ = AC \,.\, BQ.$

Let $\quad AQ = x,\ BQ = y,\ CQ = z,\ AB = l,\ BC = m.$

Then $\quad mx + lz = (l+m)y$

also $\quad mx^2 + lz^2 = (l+m)y^2 + lm(l+m)$ by Apollonius's Theorem

$$\therefore \quad (l+m)(mx^2+lz^2) - (mx+lz)^2 = lm(l+m)^2$$

$$\therefore \quad lm(x^2 - 2xz + z^2) = lm(l+m)^2$$

$$\therefore \quad x \sim z = l + m\ ;$$

and hence ACQ is a flat triangle.

This is otherwise obvious by Ptolemy's Theorem; for from the relation $12 \,.\, 34 + 14 \,.\, 23 = 13 \,.\, 24$, we see that ABCQ is a cyclic quadrilateral; and as A, B, C lie on a straight line, Q must lie on the same straight line.

It should be noted that when the condition $12.34 + 14.23 = 13.24$ is given, and when, for instance, the tangents to 1 and 2 and to 3 and 4 are direct, while all the others are transverse, there is no means of determining whether the circles 1 and 2 touch the circle 5 internally while the circles 3 and 4 touch it externally, or the circles 1 and 2 touch the circle 5 externally while the circles 3 and 4 touch it internally; and, in general, only one of these will be the case.

The cases where the above condition holds good, but where the condition (for instance) that if 12 and 13 are direct then 23 must be direct, does not hold, are not included in the above investigation. It may be that four such circles cannot exist; but if they can exist, they will not, in general, touch a common circle.