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# EIGHT CONSECUTIVE POSITIVE ODD NUMBERS NONE OF WHICH CAN BE EXPRESSED AS A SUM OF TWO PRIME POWERS

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**Abstract**

In this paper we prove the following result: there exists an infinite arithmetic progression of positive odd numbers such that for any term $k$ of the sequence and any nonnegative integer $n$, each of the 16 integers $k-2^n$, $k-2-2^n$, $k-4-2^n$, $k-6-2^n$, $k-8-2^n$, $k-10-2^n$, $k-12-2^n$, $k-14-2^n$, $k2^n-1$, $(k-2)2^n-1$, $(k-4)2^n-1$, $(k-6)2^n-1$, $(k-8)2^n-1$, $(k-10)2^n-1$, $(k-12)2^n-1$ and $(k-14)2^n-1$ has at least two distinct odd prime factors; in particular, for each term $k$, none of the eight integers $k$, $k-2$, $k-4$, $k-6$, $k-8$, $k-10$, $k-12$ or $k-14$ can be expressed as a sum of two prime powers.

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## 1. Introduction

By calculation, it is found that almost all positive odd numbers can be expressed in the form $2^n + p$, where $n$ is a positive integer and $p$ is prime. In 1934, Romanoff [9] proved that the set of positive odd numbers which can be expressed in the form $2^n + p$ has positive asymptotic density in the set of all positive odd numbers, where $n$ is a nonnegative integer and $p$ is prime. For a positive integer $n$ and an integer $a$, let $a(\text{mod } n) = \{a + nk \mid k \in \mathbb{Z}\}$. We say that $\{a_i(\text{mod } m_i)\}_{i=1}^{k}$ is a *covering system* if every integer $b$ satisfies $b \equiv a_i(\text{mod } m_i)$ for at least one value of $i$. By employing a covering system, Erdős [5] proved that there is an infinite arithmetic progression of positive odd numbers, each of which has no representation of the form $2^n + p$. Cohen and Selfridge [4] proved that there exist infinitely many odd numbers which are neither the sum nor the difference of two prime powers. In 2005, Chen [2] proved that there is an arithmetic progression of positive odd numbers such that for each of its terms $M$, none of the five consecutive odd numbers $M$, $M-2$, $M-4$, $M-6$ and $M-8$ can be expressed in the form $2^n \pm p^\alpha$, where $p$ is a prime and $n, \alpha$ are nonnegative

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integers. Recently, Chen and Tang [3] presented an explicit arithmetic progression of this type. Chen [1, Corollary 3 with $a = 1$] proved that there exists an infinite arithmetic progression of positive odd numbers such that for any term $k$ of the sequence and any nonnegative integer $n$, each of the ten integers $k - 2^n$, $k - 2 - 2^n$, $k - 4 - 2^n$, $k - 6 - 2^n$, $k - 8 - 2^n$, $k2^n - 1$, $(k - 2)2^n - 1$, $(k - 4)2^n - 1$, $(k - 6)2^n - 1$ and $(k - 8)2^n - 1$ has at least two distinct odd prime factors.

For related information, see the papers by Filaseta *et al.* [6], Luca and Stănică [8], and Guy [7, A19, B21, F13].

In this article, it will be shown that the 'ten' in [1] can be improved to 'sixteen'.

THEOREM 1.1. *There exists an infinite arithmetic progression of positive odd numbers such that for any term $k$ of the sequence and any nonnegative integer $n$, each of the 16 integers $k - 2^n$, $k - 2 - 2^n$, $k - 4 - 2^n$, $k - 6 - 2^n$, $k - 8 - 2^n$, $k - 10 - 2^n$, $k - 12 - 2^n$, $k - 14 - 2^n$, $k2^n - 1$, $(k - 2)2^n - 1$, $(k - 4)2^n - 1$, $(k - 6)2^n - 1$, $(k - 8)2^n - 1$, $(k - 10)2^n - 1$, $(k - 12)2^n - 1$ and $(k - 14)2^n - 1$ has at least two distinct odd prime factors. In particular, for each term $k$, none of the eight integers $k$, $k - 2$, $k - 4$, $k - 6$, $k - 8$, $k - 10$, $k - 12$ or $k - 14$ can be expressed as a sum of two prime powers.*

REMARK 1.2. The key to dealing with this kind of problem is to find suitable covering systems so that the Chinese remainder theorem can be applied. In [1, Theorem] (see also [2, Theorem 1]), conditions are given that these covering systems should satisfy. In this paper we will successfully find eight covering systems which satisfy the conditions of [1, Theorem]; the construction of covering systems is a very difficult topic.

Similarly, there exists an infinite arithmetic progression of positive odd numbers such that for any term $k$ of the sequence and any nonnegative integer $n$, each of the 16 integers $k + 2^n$, $k + 2 + 2^n$, $k + 4 + 2^n$, $k + 6 + 2^n$, $k + 8 + 2^n$, $k + 10 + 2^n$, $k + 12 + 2^n$, $k + 14 + 2^n$, $k2^n + 1$, $(k + 2)2^n + 1$, $(k + 4)2^n + 1$, $(k + 6)2^n + 1$, $(k + 8)2^n + 1$, $(k + 10)2^n + 1$, $(k + 12)2^n + 1$ and $(k + 14)2^n + 1$ has at least two distinct odd prime factors. Many other results parallel to those in [1] also hold true; we omit the details.

## 2. Proofs

LEMMA 2.1 [1, Theorem]. *Let $k_1, k_2, \ldots, k_{u+v}$ be integers, let $\{a_{ij} \pmod{m_{ij}}\}_{j=1}^{t_i}$ $(i = 1, 2, \ldots, u + v)$ be covering systems with $a_{ij} \geq 0$, and let $p_{ij}$ $(j = 1, 2, \ldots, t_i, i = 1, 2, \ldots, u + v)$ be positive primes such that*

$$p_{ij} \mid 2^{m_{ij}} - 1 \quad \forall\, i, j.$$

*Let $r_{ij}$ be integers such that $0 \leq r_{ij} < p_{ij}$ and*

$$r_{ij} \equiv 2^{a_{ij}} - k_i \pmod{p_{ij}}, \qquad j = 1, 2, \ldots, t_i, \; 1 \leq i \leq u,$$
$$r_{ij} \equiv -2^{a_{ij}} - k_i \pmod{p_{ij}}, \qquad j = 1, 2, \ldots, t_i, \; u + 1 \leq i \leq u + v.$$

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*Suppose that if $p_{ij} = p_{uv}$, then $r_{ij} = r_{uv}$. Then there exist two positive integers $M$ and $M_0$ with $2 \mid M$ and $2 \nmid M_0$ such that if $k \equiv M_0 (\bmod\ M)$, then for any nonnegative integer $n$, each of*

$$k + k_i - 2^n\ (1 \le i \le u), \quad (k + k_i)2^n - 1\ (1 \le i \le u),$$
$$k + k_i + 2^n\ (u + 1 \le i \le u + v), \quad (k + k_i)2^n + 1\ (u + 1 \le i \le u + v)$$

*has at least two distinct odd prime factors.*

PROOF OF THEOREM 1.1 For completeness, we give a full proof. Let $k_1 = 0$, $k_2 = -2$, $k_3 = -4$, $k_4 = -6$, $k_5 = -8$, $k_6 = -10$, $k_7 = -12$ and $k_8 = -14$.

Take

$$\begin{aligned}\{a_{1j}\ (\text{mod}\ m_{1j})\}_{j=1}^{8} = \{&0\ (\text{mod}\ 2),\ 3\ (\text{mod}\ 4),\ 5\ (\text{mod}\ 8),\\ &9\ (\text{mod}\ 16),\ 17\ (\text{mod}\ 32),\ 33\ (\text{mod}\ 64),\\ &1\ (\text{mod}\ 128),\ 65\ (\text{mod}\ 128)\},\end{aligned}$$

$$\begin{aligned}\{a_{2j}\ (\text{mod}\ m_{2j})\}_{j=1}^{7} = \{&1\ (\text{mod}\ 2),\ 0\ (\text{mod}\ 4),\ 6\ (\text{mod}\ 8),\\ &10\ (\text{mod}\ 16),\ 18\ (\text{mod}\ 32),\ 34\ (\text{mod}\ 64),\\ &2\ (\text{mod}\ 64)\},\end{aligned}$$

$$\begin{aligned}\{a_{3j}\ (\text{mod}\ m_{3j})\}_{j=1}^{26} = \{&0\ (\text{mod}\ 3),\ 2\ (\text{mod}\ 4),\ 3\ (\text{mod}\ 5)\\ &1\ (\text{mod}\ 10),\ 4\ (\text{mod}\ 12),\ 2\ (\text{mod}\ 15),\\ &1\ (\text{mod}\ 18),\ 7\ (\text{mod}\ 20),\ 8\ (\text{mod}\ 24),\\ &19\ (\text{mod}\ 25),\ 24\ (\text{mod}\ 25),\ 11\ (\text{mod}\ 36),\\ &23\ (\text{mod}\ 36),\ 25\ (\text{mod}\ 40),\ 25\ (\text{mod}\ 45),\\ &40\ (\text{mod}\ 45),\ 20\ (\text{mod}\ 48),\ 44\ (\text{mod}\ 48),\\ &9\ (\text{mod}\ 50),\ 39\ (\text{mod}\ 50),\ 37\ (\text{mod}\ 60),\\ &35\ (\text{mod}\ 72),\ 4\ (\text{mod}\ 75),\ 5\ (\text{mod}\ 120),\\ &29\ (\text{mod}\ 150),\ 215\ (\text{mod}\ 360)\},\end{aligned}$$

$$\begin{aligned}\{a_{4j}\ (\text{mod}\ m_{4j})\}_{j=1}^{9} = \{&0\ (\text{mod}\ 2),\ 1\ (\text{mod}\ 4),\ 7\ (\text{mod}\ 8),\\ &11\ (\text{mod}\ 16),\ 19\ (\text{mod}\ 32),\ 35\ (\text{mod}\ 64),\\ &67\ (\text{mod}\ 128),\ 3\ (\text{mod}\ 256),\ 131\ (\text{mod}\ 256)\}\end{aligned}$$

$$\begin{aligned}\{a_{5j}\ (\text{mod}\ m_{5j})\}_{j=1}^{13} = \{&1\ (\text{mod}\ 2),\ 2\ (\text{mod}\ 3),\ 2\ (\text{mod}\ 5),\\ &4\ (\text{mod}\ 9),\ 6\ (\text{mod}\ 10),\ 6\ (\text{mod}\ 12),\\ &10\ (\text{mod}\ 18),\ 0\ (\text{mod}\ 20),\ 24\ (\text{mod}\ 30),\\ &34\ (\text{mod}\ 36),\ 48\ (\text{mod}\ 60),\ 34\ (\text{mod}\ 90),\\ &88\ (\text{mod}\ 180)\},\end{aligned}$$

$$\begin{aligned}\{a_{6j}\ (\text{mod}\ m_{6j})\}_{j=1}^{60} = \{&1\ (\text{mod}\ 3),\ 3\ (\text{mod}\ 4),\ 1\ (\text{mod}\ 5)\\ &2\ (\text{mod}\ 7),\ 7\ (\text{mod}\ 10),\ 2\ (\text{mod}\ 11),\\ &3\ (\text{mod}\ 11),\ 0\ (\text{mod}\ 14),\ 12\ (\text{mod}\ 21),\\ &4\ (\text{mod}\ 22),\ 6\ (\text{mod}\ 27),\ 8\ (\text{mod}\ 28),\\ &20\ (\text{mod}\ 28),\ 5\ (\text{mod}\ 33),\ 18\ (\text{mod}\ 36),\end{aligned}$$

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$$
\begin{aligned}
&32 \pmod{42},\ 6 \pmod{44},\ 18 \pmod{44},\\
&45 \pmod{48},\ 30 \pmod{54},\ 32 \pmod{56},\\
&5 \pmod{60},\ 8 \pmod{66},\ 20 \pmod{66},\\
&21 \pmod{72},\ 9 \pmod{80},\ 42 \pmod{81},\\
&69 \pmod{81},\ 78 \pmod{81},\ 68 \pmod{84},\\
&80 \pmod{84},\ 33 \pmod{96},\ 81 \pmod{96},\\
&29 \pmod{100},\ 49 \pmod{100},\ 89 \pmod{100},\\
&66 \pmod{108},\ 102 \pmod{108},\ 4 \pmod{112},\\
&60 \pmod{112},\ 98 \pmod{132},\ 110 \pmod{132},\\
&38 \pmod{135},\ 69 \pmod{144},\ 117 \pmod{144},\\
&24 \pmod{162},\ 96 \pmod{162},\ 24 \pmod{168},\\
&108 \pmod{168},\ 53 \pmod{180},\ 113 \pmod{180},\\
&105 \pmod{240},\ 153 \pmod{240},\ 122 \pmod{264},\\
&254 \pmod{264},\ 83 \pmod{270},\ 209 \pmod{300},\\
&269 \pmod{300},\ 294 \pmod{324},\ 533 \pmod{540}\},
\end{aligned}
$$

$$
\begin{aligned}
\{a_{7j} \pmod{m_{7j}}\}_{j=1}^{90} = \{&0 \pmod{2},\ 1 \pmod{7},\ 1 \pmod{11},\\
&9 \pmod{11},\ 3 \pmod{12},\ 9 \pmod{13},\\
&7 \pmod{14},\ 11 \pmod{17},\ 5 \pmod{18},\\
&21 \pmod{26},\ 29 \pmod{34},\ 2 \pmod{35},\\
&31 \pmod{35},\ 6 \pmod{39},\ 33 \pmod{39},\\
&13 \pmod{49},\ 14 \pmod{51},\ 32 \pmod{51},\\
&47 \pmod{51},\ 5 \pmod{52},\ 17 \pmod{52},\\
&29 \pmod{52},\ 26 \pmod{63},\ 53 \pmod{63},\\
&25 \pmod{68},\ 51 \pmod{68},\ 59 \pmod{68},\\
&3 \pmod{70},\ 9 \pmod{70},\ 25 \pmod{77},\\
&15 \pmod{78},\ 12 \pmod{91},\ 19 \pmod{91},\\
&54 \pmod{91},\ 55 \pmod{98},\ 35 \pmod{102},\\
&53 \pmod{102},\ 101 \pmod{102},\ 49 \pmod{104},\\
&10 \pmod{105},\ 16 \pmod{105},\ 58 \pmod{105},\\
&107 \pmod{126},\ 97 \pmod{147},\ 20 \pmod{153},\\
&38 \pmod{153},\ 39 \pmod{154},\ 109 \pmod{154},\\
&151 \pmod{154},\ 105 \pmod{156},\ 117 \pmod{156},\\
&129 \pmod{156},\ 141 \pmod{156},\ 5 \pmod{182},\\
&89 \pmod{182},\ 131 \pmod{182},\ 27 \pmod{196},\\
&69 \pmod{196},\ 125 \pmod{196},\ 77 \pmod{204},\\
&95 \pmod{204},\ 179 \pmod{204},\ 197 \pmod{204},\\
&157 \pmod{210},\ 199 \pmod{210},\ 205 \pmod{210},\\
&4 \pmod{231},\ 46 \pmod{231},\ 125 \pmod{252},\\
&251 \pmod{252},\ 82 \pmod{273},\ 124 \pmod{273},\\
&166 \pmod{273},\ 139 \pmod{294},\ 181 \pmod{294},\\
&209 \pmod{306},\ 227 \pmod{306},\ 245 \pmod{306},
\end{aligned}
$$

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$$137 \pmod{308},\ 309 \pmod{312},\ 319 \pmod{462},\ 403 \pmod{462},\ 17 \pmod{504},\ 143 \pmod{504},\ 269 \pmod{504},\ 395 \pmod{504},\ 481 \pmod{546},\ 523 \pmod{546},\ 559 \pmod{588},\ 425 \pmod{612},\ 907 \pmod{924}\},$$

$$\{a_{8j} \pmod{m_{8j}}\}_{j=1}^{10} = \{1 \pmod 2,\ 2 \pmod 4,\ 0 \pmod 8,\ 12 \pmod{16},\ 20 \pmod{32},\ 36 \pmod{64},\ 68 \pmod{128},\ 132 \pmod{256},\ 260 \pmod{512},\ 4 \pmod{512}\}.$$

Noting that $\{a_j \pmod{m_j}\}_{j=1}^{k}$ is a covering system if and only if for every integer $n$ with $0 \le n < \mathrm{lcm}\{m_1, \ldots, m_k\}$ there exists a $j$ such that $n \equiv a_j \pmod{m_j}$, we can verify that each of the above $\{a_{ij} \pmod{m_{ij}}\}_{j=1}^{t_i}$ $(1 \le i \le 8)$ is a covering system (the first five systems are exactly as in the proof of [2, Theorem 2]). Now, for every $a_{ij} \pmod{m_{ij}}$, we appoint a prime $p_{ij}$ such that $m_{ij}$ is the order of $2 \pmod{p_{ij}}$ and such that if $p_{ij} = p_{uv}$, then

$$2^{a_{ij}} - k_i \equiv 2^{a_{uv}} - k_u \pmod{p_{ij}}. \tag{2.1}$$

Let

$$p_{11} = p_{21} = p_{41} = p_{51} = p_{71} = p_{81} = 3,$$
$$p_{12} = p_{22} = p_{32} = p_{42} = p_{62} = p_{82} = 5,$$
$$p_{13} = p_{23} = p_{43} = p_{83} = 17, \quad p_{14} = p_{24} = p_{44} = p_{84} = 257,$$
$$p_{15} = p_{25} = p_{45} = p_{85} = 65537, \quad p_{16} = p_{26} = p_{46} = p_{86} = 641, \quad p_{27} = 6700417,$$
$$p_{31} = p_{52} = p_{61} = 7, \quad p_{33} = p_{53} = p_{63} = 31,$$
$$p_{34} = p_{55} = p_{65} = 11, \quad p_{35} = p_{56} = p_{75} = 13,$$
$$p_{37} = p_{57} = p_{79} = 19, \quad p_{38} = p_{58} = 41,$$
$$p_{3(12)} = p_{5(10)} = 109, \quad p_{3(13)} = p_{6(15)} = 37.$$
$$p_{3(18)} = p_{6(19)} = 97, \quad p_{3(17)} = 673,$$
$$p_{3(21)} = 1321, \quad p_{5(11)} = p_{6(22)} = 61,$$
$$p_{64} = p_{72} = 127, \quad p_{66} = p_{73} = 89,$$
$$p_{67} = p_{74} = 23, \quad p_{68} = p_{77} = 43.$$

We can verify that (2.1) holds for all of these cases.

Note that the Fermat numbers $F_6$, $F_7$ and $F_8$ are composite. Let $p_{18} = p_{47} = p_{87}$ and $p_{17}$ be two distinct prime divisors of $2^{64} + 1$, let $p_{48}$ and $p_{49} = p_{88}$ be two distinct prime divisors of $2^{128} + 1$, and let $p_{89}$ and $p_{8(10)}$ be two distinct prime divisors of $2^{256} + 1$. Then (2.1) follows from the fact that

$$2^{2^k+1} - 0 \equiv 2^{2^k+2} - (-2) \equiv 2^{2^k+3} - (-6) \equiv 2^{2^k+4} - (-14) \pmod{2^{2^k} + 1}.$$

If $m > 1$ and $m \neq 6$, then there exists at least one prime $p$ such that $m$ is the order of $2 \pmod p$ (see [10]; we may verify this directly by calculation). Thus, we may appoint a prime $p_{ij}$ for each of the $a_{ij} \pmod{m_{ij}}$ provided that the multiplicity of the modulus $m_{ij}$ is one. To complete the proof, it suffices to appoint $k$ distinct primes $p_1, p_2, \ldots, p_k$ to the modulus $m$ which has multiplicity $k$ except for the above cases. If $p_1, p_2, \ldots, p_k$ are primes and the order of each $2 \pmod{p_i}$ is $m$, then we write this as $m[p_1, p_2, \ldots, p_k]$. By calculation, we find that

| | |
|---|---|
| 25[601, 1801], | 28[29, 113], |
| 35[71, 122921], | 39[79, 121369], |
| 44[397, 2113], | 45[631, 23311], |
| 50[251, 4051], | 51[103, 2143, 11119], |
| 52[53, 157, 1613], | 63[92737, 649657], |
| 66[67, 20857], | 68[137, 953, 26317], |
| 70[281, 86171], | 72[433, 38737], |
| 81[2593, 71119, 97685839], | 84[1429, 14449], |
| 91[911, 112901153, 23140471537], | 96[193, 22253377], |
| 100[101, 8101, 268501], | 102[307, 2857, 6529], |
| 105[29191, 106681, 152041], | 108[246241, 279073], |
| 112[5153, 54410972897], | 132[312709, 4327489], |
| 144[577, 487824887233], | 153[919, 75582488424179347083438319], |
| 154[617, 78233, 35532364099], | 156[313, 1249, 3121, 21841], |
| 162[163, 135433, 272010961], | 168[3361, 88959882481], |
| 180[181, 54001, 29247661], | 182[224771, 1210483, 25829691707], |
| 196[197, 19707683773, 4981857697937], | 200[401, 340801, 2787601, 3173389601], |
| 204[409, 3061, 13669, 1326700741], | 210[211, 664441, 1564921], |
| 240[394783681, 46908728641], | 252[40388473189, 118750098349], |
| 264[7393, 1761345169, 98618273953], | 294[748819, 26032885845392093851], |

231[463, 4982397651178256151338302204762057],
273[108749551, 4093204977277417, 86977595801949844993],
300[1201, 63901, 13334701, 1182468601],
306[123931, 26159806891, 27439122228481],
462[14323, 70180796165277040349245703851057],
504[1009, 21169, 2627857, 269389009, 1475204679190128571777],
546[547, 105310750819, 292653113147157205779127526827].

Now, the first part of the theorem follows from Lemma 2.1. It is clear that if $l$ is an odd integer such that for any nonnegative integer $n$, $l - 2^n$ always has at least two distinct prime factors, then $l$ cannot be expressed as a sum of two prime powers. The second part of the theorem therefore follows from the first part. This completes the proof. □

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