MAXIMAL SUBGROUPS OF INFINITE DIMENSIONAL GENERAL LINEAR GROUPS

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Abstract

Let κ be an infinite cardinal, F a field, and let $GL(\kappa, F)$ be the group of all non-singular linear transformations on a κ-dimensional vector space V over F. Various examples are given of maximal subgroups of $GL(\kappa, F)$. These include
  (i) stabilizers of families of subspaces of V which are like filters or ideals on a set,
  (ii) almost stabilizers of certain subspaces of V,
  (iii) almost stabilizers of a direct decomposition of V into two κ-dimensional subspaces.
It is also noted that $GL(\kappa, F)$ is not the union of any chain of length κ of proper subgroups.

1. Introduction

Throughout this paper κ will denote an infinite cardinal, F a field, V a vector space of dimension κ over F, and $G := GL(\kappa, F)$ will be the group of all non-singular linear transformations from V onto itself. This paper is mainly concerned with maximal subgroups of G, and its goal is to extend to G certain results from [1], [7] and [8] on maximal subgroups of infinite symmetric groups. The examples given of maximal subgroups are very far from exhaustive, and there are several further natural candidates for maximal subgroups which remain to be checked. Also, it would be natural to extend the results further to infinite dimensional classical groups preserving a form. I have not attempted this.

In Section 2 we define notions of filter and ideal on the lattice of subspaces

of V. The main theorem (Theorem 2.3) of this paper is that stabilisers in G of maximal filters or ideals are maximal in G. In fact, the action induced by G on any orbit of maximal ideals or filters is 2-transitive. The proof is straightforward. It is similar to the result of Richmann [9] that if S is the symmetric group on an infinite set Ω then the stabiliser in S of a maximal filter or ideal on Ω is maximal in S. We also note, in Theorem 2.12, that, in an appropriate sense, the almost stabiliser of any subspace of V is maximal in G. Another maximal subgroup of G, given in Theorem 2.13, is the almost stabiliser of a direct sum decomposition of V into two κ-dimensional subspaces. In Section 3 we add some miscellaneous remarks about G. In particular, we note in Theorem 3.1 that G is not the union of a κ-chain of proper subgroups (this is similar to a result from [8] about symmetric groups).

It is convenient to recall here the classification of the normal subgroups of G, due to Rosenberg [10]. Let F^* denote the multiplicative group of F. For each infinite cardinal $\lambda \leq \kappa$, let

$$G_\lambda := \{g \in G : \mathrm{Dim}(V/\mathrm{Fix}(g)) < \lambda\}$$

(here $\mathrm{Fix}(g)$ is the subspace of V consisting of vectors fixed by g). Note that $g \in G_\lambda$ if and only if $\mathrm{Dim}(\mathrm{Im}(g-1)) < \lambda$. Then the normal subgroups of G which are not contained in G_ω are precisely the groups of the form $G_\lambda \cdot Z$, where $Z \leq F^*$. We put $G_\lambda^* := G_\lambda \cdot F^*$. The normal subgroups of G which are contained in G_ω are also described in [10], and were classified by Dieudonné in [2] (they are the natural analogues of the normal subgroups of $\mathrm{GL}(n, F)$).

We will use the following standard permutation group notation. If H is a permutation group on a set Y, and $A \subseteq Y$, then $H_{\{A\}}$, $H_{(A)}$ denote respectively the setwise and pointwise stabilisers of A in H.

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2. Maximal subgroups of $\mathrm{GL}(\kappa, F)$

First, we record the following lemma of Peter Neumann, given in the present form in [8].

LEMMA 2.1 [8, LEMMA 2.1]. *If Ω is an infinite set, $S := \mathrm{Sym}(\Omega)$, and Γ, Δ are subsets of Ω with $|\Gamma \cap \Delta| = \mathrm{Min}\{|\Gamma|, |\Delta|\}$, then*

$$S_{(\Omega\setminus(\Gamma\cup\Delta))} = \langle S_{(\Omega\setminus\Gamma)}, S_{(\Omega\setminus\Delta)} \rangle.$$

This lemma has the following linear analogue, which we use repeatedly. If $U \leq W \leq V$ we say that U is a *moietous* subspace of W if $\mathrm{Dim}\, U = \mathrm{Dim}\, W = \mathrm{Dim}\, W/U$.

PROPOSITION 2.2. *Let V be the direct sum of moietous subspaces U_1, U_2, W, and put $H_1 := G_{(U_2),\{W+U_1\}}$, $H_2 := G_{(U_1),\{W+U_2\}}$. Then $G = \langle H_1, H_2 \rangle$.*

PROOF. Let $\mathscr{B}_1 = (u_{1\lambda} : \lambda < \kappa)$, $\mathscr{B}_2 = (u_{2\lambda} : \lambda < \kappa)$ and $\mathscr{B}_W = (w_\lambda : \lambda < \kappa)$ be ordered bases for U_1, U_2, and W respectively, and let $\mathscr{B}$ be the basis $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_W$ of V. Note that by Lemma 2.1, $\langle H_1, H_2 \rangle$ induces the full symmetric group on $\mathscr{B}$, so we need only show it is transitive on the set of ordered bases of V. Let $\mathscr{B}'$ be a basis of V. Also let π_1, π_2 and π_W be the natural projections of V onto U_1, U_2 and W.

CLAIM 1. There is $h_1 \in \langle H_1, H_2 \rangle$ such that $\mathscr{B}'h_1 \supseteq \mathscr{B}_1$.

PROOF OF CLAIM. First, since $\langle \mathscr{B}'\pi_1 \rangle = U_1$ there is $k_1 \in H_1$ such that $\mathscr{B}'k_1\pi_1 \supseteq \mathscr{B}_1$. For each $\lambda < \kappa$ choose $x_\lambda \in \mathscr{B}'k_1$ such that $x_\lambda\pi_1 = u_{1\lambda}$, and write $x_\lambda = u_{1\lambda} + v_\lambda$ where $v_\lambda \in W \oplus U_2$. There is a moiety A of κ such that $\langle v_\lambda : \lambda \in A \rangle$ has codimension κ in $W \oplus U_2$. Choose $k_2 \in \langle H_1, H_2 \rangle$ so that $\{u_{1\lambda}, v_\lambda : \lambda \in A\}k_2$ spans a moietous subspace of U_1. Then choose $k_3 \in H_1$ so that $x_\lambda k_2 k_3 = u_{1\lambda}$ for all $\lambda \in A$. Finally, choose $k_4 \in \langle H_1, H_2 \rangle$ so that $\{u_{1\lambda} : \lambda \in A\}k_4 = \{u_{1\lambda} : \lambda < \kappa\}$, and put $h_1 = k_1 k_2 k_3 k_4$. This proves the claim.

Next, since $\langle H_1, H_2 \rangle$ induces the symmetric group on $\mathscr{B}$ there is $h_2 \in \langle H_1, H_2 \rangle$ such that $\mathscr{B}'h_1 h_2 \supseteq \mathscr{B}_1 \cup \mathscr{B}_W$. Thus,

$$\mathscr{B}'h_1 h_2 = \mathscr{B}_1 \cup \mathscr{B}_W \cup \{\sum_{\mu < \kappa} a_{\lambda\mu}u_{1\mu} + \sum_{\mu < \kappa} b_{\lambda\mu}w_\mu + u'_\lambda : \lambda < \kappa\}$$

where $(u'_\lambda : \lambda < \kappa)$ is a basis for U_2, and $a_{\lambda\mu}, b_{\lambda\mu} \in F$ for all $\lambda, \mu < \kappa$. Clearly there is $h_3 \in H_2$ taking $\mathscr{B}_W \cup \{\sum_{\mu<\kappa} a_{\lambda\mu}u_{1\mu} + \sum_{\mu<\kappa} b_{\lambda\mu}w_\mu + u'_\lambda : \lambda < \kappa\}$ onto $\mathscr{B}_W \cup \{\sum_{\mu<\kappa} a_{\lambda\mu}u_{1\mu} + u_{2\lambda} : \lambda < \kappa\}$. Next, choose $h_4 \in H_1$ so that $u_{1\lambda}h_4 = w_\lambda$ and $w_\lambda h_4 = u_{1\lambda}$. Then

$$\mathscr{B}'h_1 h_2 h_3 h_4 = \mathscr{B}_1 \cup \mathscr{B}_W \cup \{\sum_{\mu < \kappa} a_{\lambda\mu}w_\mu + u_{2\lambda} : \lambda < \kappa\}.$$

Now there is $h_5 \in H_2$ so that $\mathscr{B}'h_1 h_2 h_3 h_4 h_5 = \mathscr{B}$, as required.

The notations of *filter* and *ideal* on a set have natural vector space analogues. An *ideal* on V is a family $\mathscr{I}$ of subspaces of V such that
   (i) $\{0\} \in \mathscr{I}$ and $V \notin \mathscr{I}$.
   (ii) if $U \in \mathscr{I}$ and $W \leq U$ then $W \in \mathscr{I}$,

(iii) if U_1, $U_2 \in \mathscr{I}$ then $U_1 + U_2 \in \mathscr{I}$.

Similarly, a *filter* on V is a family $\mathscr{F}$ of subspaces of V such that

(i) $V \in \mathscr{F}$, $\{0\} \notin \mathscr{F}$,

(ii) if $U \in \mathscr{F}$ and $U \leq W \leq V$ then $W \in \mathscr{F}$,

(iii) if U_1, $U_2 \in \mathscr{F}$ then $U_1 \cap U_2 \in \mathscr{F}$.

Note that, by Zorn's lemma, every ideal (respectively filter) on V is contained in a maximal ideal (respectively filter) on V. Note too that there is no natural duality between ideals and filters on V. If $\mathscr{S}$ is any family of subspaces of V then

$$G_{\{\mathscr{S}\}} := \{g \in G : \text{ for all } U \leq V, \ U \in \mathscr{S} \text{ if and only if } Ug \in \mathscr{S}\}.$$

Note that G has a natural action on the set of all maximal ideals on V, and on the set of all maximal filters on V. By the next theorem, this yields a large family of maximal subgroups of G.

THEOREM 2.3. *If $\mathscr{S}$ is either a maximal ideal or a maximal filter on V then G acts 2-transitively on the set of all G-translates of $\mathscr{S}$, and hence $G_{\{\mathscr{S}\}}$ is a maximal subgroup of G.*

LEMMA 2.4. (i) *Let $\mathscr{I}_1$, $\mathscr{I}_2$ be maximal ideals on V such that for every moietous subspace W of V, $W \in \mathscr{I}_1$ if and only if $W \in \mathscr{I}_2$. Then $\mathscr{I}_1 = \mathscr{I}_2$.*

(ii) *The same as* (i) *holds with filters replacing ideals throughout.*

(iii) *Every maximal ideal or filter on V contains a moietous subspace of V.*

PROOF. (i) It suffices to show that if $U \in \mathscr{I}_1$ and $\mathrm{Dim}\, U < \kappa$ then there is some moietous subspace W of V with $U \leq W$ and $W \in \mathscr{I}_1$. So let W_0 be moietous in V. If $W_0 \notin \mathscr{I}_1$ then there is $W_1 \in \mathscr{I}_1$ with $W_0 \oplus W_1 = V$. As U, $W_1 \in \mathscr{I}_1$, $U + W_1 \in \mathscr{I}_1$. Clearly $U + W_1$ is moietous in V.

(ii) This is similar to (i).

(iii) Let $\mathscr{I}$ be a maximal ideal on V, and let W be a moietous subspace of V. If $W \notin \mathscr{I}$ then by maximality of $\mathscr{I}$, W has a complement U in V with $U \in \mathscr{I}$. Since U is moietous in V, this yields the result for ideals. The proof for filters is similar.

The following lemmas are analogues for vector spaces of results which are immediate for symmetric groups.

LEMMA 2.5. *Let $\mathscr{I}$ be an ideal on V, and put $G_{(\mathscr{I})} := \bigcup \{G_{\{U\},(V/U)} : U \in \mathscr{I}\}$. Then $G_{(\mathscr{I})}$ is a normal subgroup of $G_{\{\mathscr{I}\}}$.*

PROOF. CLAIM 1. If $U \in \mathscr{I}$ and $W \leq V$ with $U \oplus W = V$, then $G_{(W),\{U\}} \leq G_{\{\mathscr{I}\}}$.

PROOF OF CLAIM. Let $T \in \mathscr{I}$ and say T has a basis $(t_\lambda : \lambda < \mu)$ with $t_\lambda = u_\lambda + w_\lambda$ where $u_\lambda \in U$ and $w_\lambda \in W$, for all $\lambda < \mu$. Now if $g \in G_{(W),\{U\}}$ then $t_\lambda g = u_\lambda g + w_\lambda$ for all $\lambda < \mu$. Since $T + U \in \mathscr{I}$, and $\langle w_\lambda : \lambda < \mu \rangle \leq T + U$, $\langle w_\lambda : \lambda < \mu \rangle \in \mathscr{I}$. Hence, as $u_\lambda g \in U$ for all $\lambda < \mu$, $\langle t_\lambda g : \lambda < \mu \rangle \leq U + \langle w_\lambda : \lambda < \mu \rangle$ so $Tg \in \mathscr{I}$, as required.

CLAIM 2. In the notation of Claim 1, $G_{\{\mathscr{I}\},(U),(W/U)}$ is transitive on the set of complements of U in V.

PROOF OF CLAIM 2. Suppose $U \oplus W_1 = U \oplus W_2 = V$, and let W_1 have basis $\{w_\lambda : \lambda < \mu\}$. Then W_2 has a basis $\{u_\lambda + w_\lambda : \lambda < \mu\}$ where all the u_λ lie in U. Let g be the element of G fixing U pointwise and mapping each w_λ to $u_\lambda + w_\lambda$. Since $W_1 g = W_2$, it suffices to show $g \in G_{\{\mathscr{I}\}}$. So suppose $T \in \mathscr{I}$. Then T has a basis $(u'_\lambda + \sum_{\nu < \mu} a_{\lambda\nu} w_\nu : \lambda < \mu, \ u'_\lambda \in U, \ a_{\lambda\nu} \in F)$. Now clearly $T_W := \langle \sum_{\nu < \mu} a_{\lambda\nu} w_\nu : \lambda < \mu, \ a_{\lambda\nu} \in F \rangle \in \mathscr{I}$, since it is a subspace of $U + T$. Also, Tg has a basis $(u'_\lambda + \sum_{\nu < \mu} a_{\lambda\nu}(u_\nu + w_\nu) : \lambda < \mu, \ u'_\lambda \in U, \ a_{\lambda\nu} \in F)$. Hence $Tg \leq U + T_W$, so as $U + T_W \in \mathscr{I}$, $Tg \in \mathscr{I}$. A similar argument shows that if $R \in \mathscr{I}$ then $Rg^{-1} \in \mathscr{I}$. Hence $g \in G_{\{\mathscr{I}\}}$ as required.

It follows easily from Claims 1 and 2 that for each $U \in \mathscr{I}$, $G_{\{U\},(V/U)} \leq G_{\{\mathscr{I}\}}$. Thus, the lemma will follow if we can show that $G_{(\mathscr{I})}$ is a group. However, it is easily verified that if $U_1, U_2 \in \mathscr{I}$ and $g_1 \in G_{\{U_1\},(V/U_1)}$, $g_2 \in G_{\{U_2\},(V/U_2)}$ then $g_1 g_2 \in G_{\{U_1 + U_2\},(V/(U_1 + U_2))}$, and as $U_1 + U_2 \in \mathscr{I}$ this shows that $g_1 g_2 \in G_{(\mathscr{I})}$.

LEMMA 2.6. Let $\mathscr{F}$ be a filter on V, and let $G_{(\mathscr{F})} := \{g \in G : \mathrm{Fix}(g) \in \mathscr{F}\}$. Then $G_{(\mathscr{F})}$ is a normal subgroup of $G_{\{\mathscr{F}\}}$.

PROOF. This is immediate.

REMARK. Lemmas 2.5 and 2.6, together with Theorem 6.4 of [8] (also proved earlier in [9]), suggest the following question. If $\mathscr{S}$ is a maximal filter or maximal ideal on V, do we have $G_{(\mathscr{S})} = G_{\{\mathscr{S}\}}$?

LEMMA 2.7. Let $\mathscr{S}$ be a maximal filter (respectively maximal ideal) on V, and let U be moietous subspace of V lying in $\mathscr{S}$. Then there is $W \in \mathscr{S}$ such that W is a moietous subspace of U (respectively, U is a moietous subspace of W, and W is a moietous subspace of V).

PROOF. Let $\mathscr{S}$ be a filter. Choose a moietous subspace X of U. If $X \notin \mathscr{S}$ then by maximality there is $Y \in \mathscr{S}$ with $X \cap Y = 0$. Now choose W to be any moietous subspace of U containing $Y \cap U$. The proof for ideals is similar.

LEMMA 2.8. *Let $\mathscr{S}$ be a maximal ideal or maximal filter on V. Then $G_{\{\mathscr{S}\}}$ is transitive on the set of moietous subspaces of V which lie in $\mathscr{S}$.*

PROOF. Suppose that $\mathscr{S}$ is a maximal ideal. Let U_1, U_2 be moietous subspaces of V lying in $\mathscr{S}$. If $U_1 + U_2$ is not moietous in V, then both U_1 and U_2 are moietous in $U_1 + U_2$; hence, there is $g \in G_{\{U_1+U_2\},(V/(U_1+U_2))}$ with $U_1 g = U_2$, and by Lemma 2.5 and the fact that $U_1 + U_2 \in \mathscr{S}$, $g \in G_{\{\mathscr{S}\}}$, as required. If $U_1 + U_2$ is moietous in V, then by Lemma 2.7 there is $W \in \mathscr{S}$ such that $U_1 + U_2$ is moietous in W and W is moietous in V; hence there is $g \in G_{\{W\},(V/W)}$ with $U_1 g = U_2$, and by Lemma 2.5 we have $g \in G_{\{\mathscr{S}\}}$, as required.

We omit the proof when $\mathscr{S}$ is a filter, since it is very similar (it uses Lemmas 2.6 and 2.7).

PROOF OF THEOREM 2.3. Let $\mathscr{S}$ be a maximal ideal, and let Ω be the set of G-translates of $\mathscr{S}$. Let $\mathscr{S}_1, \mathscr{S}_2, \mathscr{J}_1, \mathscr{J}_2 \in \Omega$ with $\mathscr{S}_1 \neq \mathscr{S}_2$ and $\mathscr{J}_1 \neq \mathscr{J}_2$. By Lemma 2.4(i) there is a moietous subspace U_1 of V with $U_1 \in \mathscr{S}_1 \backslash \mathscr{S}_2$, and as $U_1 \notin \mathscr{S}_2$ and by maximality of $\mathscr{S}$, U_1 has a complement U_2 in V with $U_2 \in \mathscr{S}_2 \backslash \mathscr{S}_1$. Similarly, there are moietous subspaces W_1, W_2 of V with $W_1 \in \mathscr{J}_1 \backslash \mathscr{J}_2$, $W_2 \in \mathscr{J}_2 \backslash \mathscr{J}_1$, and $V = W_1 \oplus W_2$. By Lemmas 2.5 and 2.8 there are $g_1, g_2 \in G$ such that for all $i, j \in \{1, 2\}$, $\mathscr{S}_j g_j = \mathscr{J}_j$ and $U_i g_j = W_i$ (here, 2.5 is used since we need that for any $W \in \mathscr{S}$, $G_{\{\mathscr{S}\},\{W\}}$ is transitive on the set of complements of W in V). Let g be the unique element of G inducing g_1 on U_2 and g_2 on U_1. Now $g g_1^{-1} \in G_{\{U_1\},(V/U_1)} \leq G_{\{\mathscr{J}_1\}}$ (by Lemma 2.5), so $\mathscr{S}_1 g = \mathscr{J}_1$. Similarly, $\mathscr{S}_2 g = \mathscr{J}_2$.

The proof when $\mathscr{S}$ is a filter is very similar, and we omit it.

For the final part of Theorem 2.3 recall that a transitive permutation group H on a set X is primitive if and only if, for every $x \in X$, H_x is maximal in H. Since 2-transitive permutation groups are primitive, this yields maximality of $G_{\{\mathscr{S}\}}$.

QUESTION. Is the degree of transitivity in the actions of Theorem 2.3 greater than two?

One would expect Theorem 2.3 to yield a large number of non-conjugate maximal subgroups. This is certainly true if the field is not too large. We call

an ideal (respectively filter) *uniform* if it contains no subspace of codimension in V (respectively dimension) less than κ.

LEMMA 2.9. *There are at least* $2^{2^{\kappa}}$ *uniform maximal ideals on* V, *and at least* $2^{2^{\kappa}}$ *uniform maximal filters on* V.

PROOF. We prove the result for ideals, the proof for filters being similar. Let $\mathscr{S}$ be the set of subspaces of V of dimension less than κ. Also let $\mathscr{B} = (v_{\lambda} : \lambda < \kappa)$ be an ordered basis of V. By a theorem of Pospisil (Jech [5; p. 256]) there are $2^{2^{\kappa}}$ distinct uniform maximal ideals on $\mathscr{B}$, $(\mathscr{I}_{\lambda} : \lambda < 2^{2^{\kappa}})$ say (here, "ideal" just means ideal of subsets of a set, and an ideal $\mathscr{I}$ on $\mathscr{B}$ is *uniform* if, for every $A \in \mathscr{I}$, $|\mathscr{B} \backslash A| = \kappa$). For each $\lambda < 2^{2^{\kappa}}$ let $\mathscr{I}_{\lambda}$ be the ideal on V generated by

$$\mathscr{S} \cup \{\langle A \rangle : A \in \mathscr{I}_{\lambda}\}$$

(note that each $\mathscr{I}_{\lambda}$ is an ideal, that is, does not contain V). By Zorn's lemma each $\mathscr{I}_{\lambda}$ is contained in a maximal ideal $\mathscr{I}_{\lambda}^{*}$ on V, and, as $\mathscr{S} \subseteq \mathscr{I}_{\lambda}$, $\mathscr{I}_{\lambda}^{*}$ is uniform. Clearly the ideals $\mathscr{I}_{\lambda}^{*}$ are all distinct, since the $\mathscr{I}_{\lambda}$ are distinct.

LEMMA 2.10. *Distinct maximal ideals on* V *have distinct stabilisers, and distinct maximal filters on* V *have distinct stabilisers.*

PROOF. We prove the result for ideals, the proof for filters being similar. It suffices by Lemma 2.4 to show that if $\mathscr{I}$ is a maximal ideal then $\mathscr{I}$ contains every moietous subspace U of V for which $G_{\{\mathscr{I}\}} \geq G_{\{U\},(V/U)}$. So suppose $W \notin \mathscr{I}$ where W is moietous in V and $G_{\{W\},(V/W)} \leq G_{\{\mathscr{I}\}}$. As $W \notin \mathscr{I}$ and $\mathscr{I}$ is maximal, there is $U \in \mathscr{I}$ with $U \oplus W = V$. Furthermore, by Lemma 2.7 there is $X \in \mathscr{I}$ such that U is moietous in X and X is moietous in V. Proposition 2.2 now shows that $G_{\{\mathscr{I}\}} = G$, a contradiction.

COROLLARY 2.11. *Provided the cardinal* $(\kappa \cdot |F|^{\kappa})$ *is less than* $2^{2^{\kappa}}$, G *has* $2^{2^{\kappa}}$ *distinct non-conjugate maximal subgroups.*

PROOF. First, note that since $|G|$ is equal to the number of ordered bases of V, $|G| = (\kappa \cdot |F|^{\kappa})$. The result now follows from Theorem 2.3 and Lemmas 2.9 and 2.10.

QUESTION. If the assumption in Corollary 2.11 fails, how many maximal subgroups does G have up to conjugacy? In particular, how many maximal ideals and maximal filters are there?

REMARK. In the above material on filters and ideals, we have essentially had to give the same proof twice, once for filters and once for ideals. It would be interesting to know if there is any way to make these proofs uniform.

We now obtain some further maximal subgroups of G. As the proofs are similar to earlier arguments, we omit some details. If $\lambda \leq \kappa$ is a cardinal and U, W are subspaces of V of dimension λ, then we write $U =_\lambda W$ if $\mathrm{Dim}((U + W)/U) + \mathrm{Dim}((U + W)/W) < \lambda$. Note that $=_\lambda$ is an equivalence relation (to see transitivity, note that if $X =_\lambda Y =_\lambda Z$ then $\mathrm{Dim}((X + Y + Z)/Y) < \lambda$ so $\mathrm{Dim}((X+Y+Z)/(X+Y)) < \lambda$; hence, as $\mathrm{Dim}((X+Y)/X) < \lambda$, $\mathrm{Dim}((X+Y+Z)/X) < \lambda$, so $\mathrm{Dim}((X+Z)/X) < \lambda$ and a similar argument gives $\mathrm{Dim}((X + Z)/Z) < \lambda$).

THEOREM 2.12. *Let $U \leq V$ with $\mathrm{Dim}\, U = \lambda \leq \kappa$ and $\mathrm{Dim}(V/U) = \kappa$. Let $H := \{g \in G : Ug =_\lambda U\}$. Then H is a group, and is maximal in G.*

REMARK. We may regard H as the almost stabiliser of the subspace U. Note that $H \geq G_{\{U\}} \cdot G_\lambda^*$, with equality if and only if $\lambda > \aleph_0$.

PROOF. We omit the proof that H is a group. Suppose first that $\lambda < \kappa$. Let $g \in G\backslash H$. By replacing g by g^{-1} if necessary, we may suppose that $\mathrm{Dim}((U + Ug)/U) = \lambda$, so U is moietous in $U + Ug$. It is easily seen that there is $k \in \langle H, g \rangle$ such that $U \cap Uk = 0$. Choose moietous subspaces V_1, V_2, W of V such that $V = (U \oplus V_1) \oplus W \oplus (Uk \oplus V_2)$. Then $\langle H, g \rangle$ contains both $G_{(U \oplus V_1), \{Uk \oplus V_2 \oplus W\}}$ and $G_{(Uk \oplus V_2), \{U \oplus V_1 \oplus W\}}$. Hence, by Proposition 2.2, $G = \langle H, g \rangle$, as required.

Suppose now that $\lambda = \kappa$. Then we may assume (by adjusting by elements of H if necessary) that U is moietous in Ug, or Ug is moietous in U, or $U \cap Ug$ is moietous in each of U, Ug, or U and Ug generate their direct sum which is moietous in V, or $U \oplus Ug = V$. The first three possibilities can be reduced to the fourth, and in the fourth, much as in the last paragraph, an application of Proposition 2.2 shows that $\langle H, g \rangle = G$. So suppose that the fifth case holds, that is $U \oplus Ug = V$. Let $Ug = W_1 \oplus W_2$ be a decomposition into moietous subspaces, let $(u_\mu : \mu < \kappa)$, $(w_\mu : \mu < \kappa)$ be ordered bases of U, W_2 respectively, and let $W_3 := \langle u_\mu + w_\mu : \mu < \kappa \rangle$. Clearly $\langle H, g \rangle \geq G_{(U), \{W_1 + W_2\}}$. Also, there is $h \in H$ with $Ugh = W_1 \oplus W_3$; hence, as $W_3 \oplus (W_1 \oplus W_2) = V$ and H is transitive on the set of complements of U, there is $k \in \langle H, g \rangle$ with $Uk = W_2$. It follows that $\langle H, g \rangle \geq G_{(W_2), \{U + W_1\}}$. Proposition 2.2 applied to U, W_1, W_2 now gives that $\langle H, g \rangle = G$.

REMARK. It is very easily checked that the stabiliser in G of any finite-dimensional subspace of V is maximal in G. From Proposition 2.2, the same holds for subspaces of V of finite codimension in V. More generally,

if U is a subspace of V such that $\mathrm{Dim}\, U = \kappa$ and $\mathrm{Dim}(V/U) = \lambda$ where $\aleph_0 \leq \lambda < \kappa$, then $\{g \in G : Ug =_\lambda U\}$ is a maximal subgroup of G, again by Proposition 2.2.

If $V = V_1 \oplus \cdots \oplus V_n$ $(n < \omega)$ is a direct sum decomposition into moietous subspaces, then the *almost stabiliser* of the decomposition is

$$H := \left\{ \begin{array}{l} g \in G : \text{there is } \pi \in \mathrm{Sym}\{1, \ldots, n\} \text{ such that for} \\ \text{all } i = 1, \ldots, n, \ V_i g =_\kappa V_{i\pi^{-1}} \end{array} \right\}.$$

Then H is a group, and $H \geq G_\kappa^* \cdot (\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)) \cdot S_n$, with equality if and only if $\kappa > \aleph_0$.

THEOREM 2.13. *Let $V = V_1 \oplus V_2$, where V_1, V_2 are moietous subspaces of V, and let H be the almost stabiliser of the decomposition. Then H is maximal in G.*

PROOF. Choose $g \in G\backslash H$ and put $K := \langle H, g \rangle$. Also put $N := \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$.

CLAIM 1. There is $k \in K_{(V_1),(V/V_1)}$ such that $\mathrm{Dim}((V_2 + V_2 k)/V_2) = \kappa$.

SUBCLAIM 1. We may suppose there is no $h \in K$ such that $V_1 h$ is a moietous subspace of V_1.

PROOF OF SUBCLAIM. Suppose that there is such an h. Let $V_1 = V_1 h + Z$, and let $(z_\lambda : \lambda < \kappa)$ and $(v_\lambda : \lambda < \kappa)$ be ordered bases of Z and V_2 respectively. Then $V_2 h$ has a basis of the form $\{z_\lambda + s_\lambda, v_\lambda + t_\lambda : \lambda < \kappa\}$ where $s_\lambda, t_\lambda \in V_1 h$. Now choose k to be the linear transformation fixing V_1 pointwise and taking $v_\lambda + t_\lambda$ to $v_\lambda + t_\lambda + z_\lambda + s_\lambda$ for all $\lambda < \kappa$. Because k fixes $V_1 h$ and $V_2 h$ setwise, $hkh^{-1} \in N$, so $k \in K$. Hence k satisfies Claim 1.

From now on we make the assumption indicated in Subclaim 1. Now write $U_1 := V_1 \cap V_1 g$, $W_1 := V_2 \cap V_1 g$, $U_2 := V_1 \cap V_2 g$, $W_2 := V_2 \cap V_2 g$. There are cardinals $\mu, \nu < \kappa$ and linearly independent sets $\{x_{1\lambda} : \lambda < \mu\}$, $\{x_{2\lambda} : \lambda < \nu\} \subseteq V_1$ and $\{y_{1\lambda} : \lambda < \mu\}$, $\{y_{2\lambda} : \lambda < \nu\} \subseteq V_2$ such that

$$V_1 g = U_1 \oplus W_1 \oplus \langle x_{1\lambda} + y_{1\lambda} : \lambda < \mu \rangle,$$

$$V_2 g = U_2 \oplus W_2 \oplus \langle x_{2\lambda} + y_{2\lambda} : \lambda < \nu \rangle.$$

SUBCLAIM 2. We may assume that either

$$U_1 \oplus \langle x_{1\lambda} : \lambda < \mu \rangle =_\kappa V_1,$$

or

$$W_1 \oplus \langle y_{1\lambda} : \lambda < \mu \rangle =_\kappa V_2.$$

PROOF OF SUBCLAIM. Suppose that neither possibility in the subclaim holds. Clearly one of $\text{Dim}\,U_1$, $\text{Dim}\,W_1$, μ is equal to κ. Hence there is $f \in N$ with $V_1 gf$ moietous in $V_1 g$. It follows that $V_1 gfg^{-1}$ is moietous in V_1, contrary to our assumption after Subclaim 1.

SUBCLAIM 3. We may assume that $\mu = \kappa$.

PROOF OF SUBCLAIM. Assume for a contradiction that $\mu < \kappa$. Suppose first (by Subclaim 2) that $U_1 \oplus \langle x_{1\lambda} : \lambda < \kappa \rangle =_\kappa V_1$. Then either

(i) $\text{Dim}\,W_1 < \kappa$, and $V_1 g =_\kappa V_1$, or

(ii) $\text{Dim}\,W_1 = \kappa$, and $V_1 g$ contains a moietous subspace X with $X =_\kappa V_1$.

In case (i) there is $f \in K$ such that $V_1 gf = V_1$ and $\text{Dim}((V_2 gf + V_2)/V_2) = \kappa$. It follows that for some $n \in N$, the element $k := gfn$ satisfies Claim 1. Case (ii) cannot hold, since it would contradict our assumption after Subclaim 1 (recall that H contains the *almost* stabiliser of the decomposition $V_1 \oplus V_2$). If the second possibility in Subclaim 2 holds, then a similar argument (using an involution which interchanges V_1 and V_2) again justifies the assumption that $\mu = \kappa$.

SUBCLAIM 4. We may assume that $\nu = \kappa$.

PROOF OF SUBCLAIM. This is virtually the same as the proof of Subclaim 3 (with the obvious adaptation of Subclaim 2).

By Subclaims 3 and 4, from now on we can assume that $\mu = \nu = \kappa$.

SUBCLAIM 5. We may assume $\text{Dim}\,U_1 = \text{Dim}\,U_2 = \kappa$.

PROOF OF SUBCLAIM. It is possible to find $f \in N$ such that there are disjoint $\nu_1, \nu_2 \subseteq \kappa$, each of size κ, with

$$(x_{1\lambda} + y_{1\lambda})f = \begin{cases} x_{1\lambda} + y_{1\lambda} & \text{if } \lambda \in \nu_1, \\ x_{2\lambda} + y_{2\lambda} & \text{if } \lambda \in \nu_2. \end{cases}$$

Clearly $V_1 gf \cap V_1 g$ and $V_1 gf \cap V_2 g$ both have dimension κ. Hence $V_1 gfg^{-1}$ intersects both V_1 and V_2 in subspaces of dimension κ. To obtain the subclaim, now replace g by gfg^{-1}.

Now, to prove Claim 1, let U_1, U_2 have bases $(u_{1\lambda} : \lambda < \kappa)$, $(u_{2\lambda} : \lambda < \kappa)$. Choose $h \in N$ to fix V_2, U_1 and $V_1/(U_1 + U_2)$ pointwise and to map $u_{2\lambda}$ to $u_{2\lambda} + u_{1\lambda}$ for all $\lambda < \kappa$. Then h fixes $V_1 g$ setwise, and $\text{Dim}((V_2 g + V_2 gh)/V_2 g) = \kappa$. There is $n \in g^{-1} N g$ such that hn fixes $V_1 g$ and $V/V_1 g$ pointwise and $\text{Dim}((V_2 g + V_2 ghn)/V_2 g) = \kappa$. Now $k := ghng^{-1}$ satisfies the claim.

CLAIM 2. $K \geq G_{\{V_1\}}$.

PROOF OF CLAIM. We argue as in the proof of [3, Proposition 2.3]. Note that $G_{\{V_1\}}$ is the semidirect product of $M := G_{(V_1),(V/V_1)}$ by N, and that, as an FN-module, M is naturally isomorphic to $\mathrm{Hom}(V_2, V_1)$ (here, if $(g_1, g_2) \in N$ and $\phi \in \mathrm{Hom}(V_2, V_1)$ then $\phi(g_1, g_2)$ sends $v_2 \in V_2$ to $(\phi(v_2 g_2^{-1})g_1$. Since $N \leq K$ and $k \in M \cap K$, we must show that any FN-invariant submodule A of $\mathrm{Hom}(V_2, V_1)$ containing the element corresponding to k is $\mathrm{Hom}(V_2, V_1)$. As $\mathrm{Dim}\, V_1 = \mathrm{Dim}\, V_2$ we may identify them, and as in [3], A becomes a 2-sided ideal in $\mathrm{Hom}(V_2, V_2)$. Such ideals are classified in [6] (see also [4, p. 258]), and since A contains an element corresponding to k (that is, an endomorphism of V_2 with infinite dimensional image), we have $A = \mathrm{Hom}(V_2, V_2)$, which proves the claim.

Given the claim it follows that $K \geq G_{\{V_2\}}$. Now by Theorem 2.12, $K = G$, as required.

Theorems 2.12 and 2.13 suggest that many of the maximal subgroups of the finite dimensional general linear groups may have infinite dimensional analogues. It would be interesting to investigate whether the almost stabilisers of other finite direct sum decompositions or of tensor decompositions, or the almost stabiliser of a symplectic form, if appropriately defined, ever give maximal subgroups. Note that if $\kappa = \omega$ and V is endowed with a symplectic form $(\, , \,)$, then the corresponding symplectic group $\mathrm{Sp}(V, F)$ also fixes the set $\mathscr{S}$ of all hyperplanes of V which are of the form $v^\perp := \{w \in V : (v, w) \in 0\}$ for some $v \in V$. Let $\{e_i, f_i : i < \omega\}$ be a standard basis of V with respect to $(\, , \,)$ (so for all $i, j < \omega$, $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$). Then $\mathscr{S}$ is the set of all those hyperplanes of V which contain all but finitely many elements of $\{e_i, f_i : i < \omega\}$. It follows easily that the full stabiliser H in G of $\mathscr{S}$ induces the general linear group on each finite dimensional subspace of V, so properly contains $\mathrm{Sp}(V, F)$. Hence, under any reasonable definition, the *almost stabiliser* of the form $(\, , \,)$ ought to contain H. (These observations were suggested by a conversation with Tim Penttila, and were made independently by Jon Hall.) There are other questions, such as the following, about large proper subgroups of G.

QUESTION. Is there a proper subgroup H of G such that for each moietous subspace U of V, $H^U = \mathrm{GL}(U)$?

3. Further remarks on G

We here give for G some further analogues of results proved in [8] for infinite symmetric group. Recall that if N, M are groups, $N \trianglelefteq M$ and

$H \leq M$, then H is a *supplement* for N in M if $NH = M$. In [8] (and also earlier, using GCH, in [11]) supplements of the normal subgroups of S (the symmetric group on a set of size κ) are classified, and this sheds some light on the maximal subgroups of S (see [8, Observation 6.1]). Also, in [8] we define the *cofinality* of an uncountable group H to be the smallest order type of a chain of proper subgroups of H with union H and it is shown that the cofinality of S is greater than κ. This is used in the classification of supplements of the normal subgroups of S. Here we obtain an analogous result on the cofinality, but do not have a similar classification.

THEOREM 3.1. *The group G has cofinality greater than κ.*

We use the following lemma.

LEMMA 3.2. *Let $H < G$ and suppose there is a decomposition $V = U \oplus W$ into moietous subspaces such that $H_{\{U\},\{W\}}$ induces $\mathrm{GL}(U)$ on U. Then*
(i) *there are $g, h \in G$ such that $\langle H, g, h \rangle = G$,*
(ii) *H is contained in a maximal subgroup of G.*

PROOF. (i) Choose $g \in G$ fixing W pointwise, U setwise, and, in its action on U, having fixed point space on U of codimension κ in U. Then by the results on normal subgroups of G given in the introduction $\langle g \rangle$ has normal closure $G_{\{U\},(W)}$ in $H_{\{U\},(W)}$, so $\langle H, g \rangle$ contains $G_{\{U\},(W)}$. Next, let $U = U_1 \oplus U_2$ be a decomposition into moietous subspaces and choose $h \in G$ with $Uh = U_1 \oplus W$, $Wh = U_2$. Then $\langle G_{(W),\{U_1+U_2\}}, G_{(U_2),\{U_1+W\}} \rangle \leq \langle H, g, h \rangle$, so by Proposition 2.2 we have $\langle H, g, h \rangle = G$.

(ii) This follows from (i) by Zorn's lemma, applied to the collection of all proper subgroups of G which contain H but omit at least one of g, h (cf. [8, Lemma 6.9]).

PROOF OF THEOREM 3.1. Suppose there is a chain $(H_\lambda : \lambda < \kappa)$ of proper subgroups of G with union G. Choose a direct sum decomposition $V = \bigoplus(V_\lambda : \lambda < \kappa)$ into moietous subspaces. For each λ, there is $g_\lambda \in \mathrm{GL}(V_\lambda)$ not induced by $H_{\lambda\{V_\lambda\},\{\bigoplus(V_\mu : \mu<\kappa, \mu\neq\lambda)\}}$ (for if this were false for some λ, then by Lemma 3.2 there would be $g, h \in G$ such that $\langle G_\lambda, g, h \rangle = G$; for some $\mu < \kappa$, we would have $g, h \in G_\mu$, and if $\nu = \mathrm{Max}\{\lambda, \mu\}$ then $G_\nu = G$, a contradiction). There is a unique $g \in G$ coinciding with g_λ on V_λ for each λ, and clearly $g \in G \setminus \bigcup(H_\lambda : \lambda < \kappa)$, a contradiction.

The next corollary was pointed out by Wilfrid Hodges.

COROLLARY 3.3. *Let E be the ring of F-endomorphisms of V. Then E is not the union of a κ-chain of proper subrings.*

PROOF. Suppose E is the union of a chain of subrings $(E_\lambda : \lambda < \kappa)$. Since G is a subgroup of the group of units of E, $G = \bigcup(G \cap E_\lambda : \lambda < \kappa)$. Hence, by Theorem 3.1, $G \le E_\lambda$ for some $\lambda < \kappa$. However, by Zelinsky [12] each element of E is a sum of two elements of G, so $E = E_\lambda$, as required.

Theorem 3.1 gives a little information about supplements of the normal subgroups G_λ. First, note the following lemma.

LEMMA 3.4. *If κ is regular then G_κ has cofinality at most κ.*

PROOF. Let $(v_\lambda : \lambda < \kappa)$ be an ordered basis of V, and for each $\lambda < \kappa$ let $U_\lambda := \langle v_\mu : \mu < \kappa \rangle$ and $H_\lambda := G_{\{U_\lambda\}, (V/U_\lambda)}$. Then $H_\lambda \le G_\lambda$ for all $\lambda < \kappa$, for if $h \in H_\lambda$, then $\mathrm{Im}(h - 1) \le U_\lambda$, so $\mathrm{Dim}(\mathrm{Im}(h - 1)) < \kappa$. Clearly also $H_\lambda \ne G_\kappa$ for all $\lambda < \kappa$. Finally, $\bigcup(H_\lambda : \lambda < \kappa) = G_\kappa$, for if $h \in G_\kappa$ then $\mathrm{Dim}(\mathrm{Im}(h - 1)) < \kappa$, so by the regularity of κ, $\mathrm{Im}(h - 1) \le U_\lambda$ for some $\lambda < \kappa$.

QUESTION. Does Lemma 3.4 hold if κ is singular?

QUESTION. Does G_κ every have cofinality less than κ?

Now suppose that K is a supplement for G_κ, that is, $K \le G$ and $G_\kappa \cdot K = G$. Then $(\langle H_\lambda, K \rangle : \lambda < \kappa)$ is a chain of length κ of subgroups of G with union G. Hence, by Theorem 3.1, there is $\lambda < \kappa$ such that $\langle H_\lambda, K \rangle = G$.

We conclude with an observation which must be well-known. We omit its proof, since it is routine.

LEMMA 3.5. *Let $E := \mathrm{End}_F(V)$. Then there is an inclusion-preserving bijection ϕ between the set of filters of subspaces of V and the set of proper right ideals of E, given by the rule: if $\mathscr{F}$ is a filter on V then $\phi(\mathscr{F}) := \{x \in E : \mathrm{Ker}(x) \in \mathscr{F}\}$.*

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