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COMPLEMENTED c_0-SUBSPACES OF A NON-SEPARABLE $C(K)$-SPACE

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ABSTRACT. The non-separable Banach space of right continuous functions with left hand limits and the supremum norm is investigated to find the isomorphic types of complemented subspaces. It is shown that every isometric isomorph of c_0 is complemented in this space which may be identified as a non-separable $C(K)$ space. Sufficient conditions are given for other isomorphs of c_0 to be complemented in the space and the complement of a c_0 subspace is characterized isomorphically.

0. **Introduction.** It is well known that $D[0,1]$ may be identified with the space of continuous functions, $C(K)$, where $K = [0,1] \times \{0\} \cup [0,1] \times \{1\}$, is endowed with the order topology induced by the lexicographic order [5]. Other properties of $C(K)$ are studied by G. Godefroy [2] and M. Talagrand [5]. The main problem studied here is as follows: if X is a subspace of $D[0,1]$ isomorphic to c_0, is X complemented in $D[0,1]$? If K is a compact metric space, then $C(K)$ is separable and every isomorph of c_0 is complemented. But if K is not metrizable as in the case with $D[0,1]$, then the space $C(K)$ is not separable and in general the isomorphic types of complemented subspaces are unknown.

1. **Preliminaries.** If we consider two copies of the unit interval, $I_0 = [0,1] \times \{0\}$ and $I_1 = [0,1] \times \{1\}$, then $TL = I_0 \cup I_1$ endowed with the order topology induced by the lexicographic order TL is a separable compact Hausdorff space which is not metrizable. Moreover, open sets in TL may be expressed as a countable union of open intervals in much the same way as in the unit interval. Since TL is not metrizable the space of continuous functions, $C(TL)$, is not separable. Other properties of $C(TL)$ become apparent when we identify it with its isometric image $D[0,1]$ where

$$D[a,b] := \{f : [a,b] \longrightarrow \Re \mid f \text{ is right continuous with left hand limits at each point of } [a,b]\}.$$

The open balls in $C(TL)$ are defined as follows: for each $(p,0) \in I_0$ an open ϵ-ball about $(p,0)$ is defined

$$B\big((p,0);\epsilon\big) = \{(x,0) : p-\epsilon < x \leq p\} \cup \{(x,1) : p-\epsilon < x < p\}$$

and an open ϵ-ball about a point $(q,1)$ is defined

$$B\big((q,1);\epsilon\big) = \{(x,0) : q < x < q+\epsilon\} \cup \{(x,1) : q \leq x < q+\epsilon\}.$$

The identification of $C(TL)$ with $D[0,1]$ is made using the following property which follows immediately from the structure of open balls in $C(TL)$.

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PROPERTY 1.1. *If $f \in C(TL)$, then the function $f_1\colon [0,1] \longrightarrow \Re$ defined by $f_1(x) = f(x,1)$ is right continuous and the left-sided limit $f_1(x^-) = f(x,0)$ for all $x \in [0,1]$.*

From Property 1.1 we have that each member of $C(TL)$ determines a member of $D[0,1]$. The mapping $T\colon C(TL) \longrightarrow D[0,1]$ defined by $Tf = f_1$, is clearly an isometry which can easily be shown to be onto. The identification of $C(TL)$ with $D[0,1]$ gives the following key properties of $C(TL)$ ([1], p. 110):

(i) If $f \in C(TL)$, then there are at most finitely many values of t where $|f(t,1)-f(t,0)|$ exceeds a given number.

(ii) A function f in $C(TL)$ can have at most countably many values of t such that $f(t,0) \neq f(t,1)$.

(iii) The step functions are dense in $C(TL)$. Let c_0 be the space of all sequences of real numbers converging to 0 in the supremum norm. In order to show that every subspace of $C(TL)$ isometric to c_0 is complemented, we need the following property.

PROPERTY 1.2. *If Z is a closed subset of TL, then there is a linear isometry, T, from $C(Z)$ into $C(TL)$ so that for every $f \in C(Z)$ the restriction of Tf to Z is equal to f.*

PROOF. Since Z is closed $Z' = \bigcup_{n=1}^{\infty} \langle (a_n, j_n), (b_n, k_n)\rangle$ where any two of these intervals are either disjoint or coincide at all points. Define $T\colon C(Z) \longrightarrow C(TL)$ by $Tf = f$ on Z and on Z' define Tf as follows:

$$(Tf)(x,i) = \begin{cases} f(a_n,j_n) + \frac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n), & \text{if } a_n \le x \le b_n \text{ and } j_n = k_n = 1 \\ f(a_n,j_n) + \frac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n), & \text{if } a_n \le x \le b_n \text{ and } j_n = k_n = 0 \\ f(a_n,j_n) + \frac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n), & \text{if } a_n \le x \le b_n \text{ and } j_n = 1, k_n = 0 \\ f(a_n,b_n) + \frac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n), & \text{if } a_n \le x \le b_n \text{ and } j_n = 0, k_n = 1. \end{cases}$$

Note that the four cases above correspond to the four types of open intervals of TL and in each case give the natural extension on each open interval of the complement of the domain of f; that is, the image of f in $D[0,1]$ is extended linearly on the complement of its domain.

2. **Projections of $C(TL)$ onto c_0.**

In this section, we give results on the classification of complemented c_0-subspaces of $C(TL)$.

It is a standard theorem due to Sobczyk [4], that if X is a separable Banach space and $Y \subset X$, Y isomorphic to c_0, then Y is complemented in X. With Property 1.2, the same argument used in Veech's proof of Sobczyk's theorem [6] may be used to prove the following result.

THEOREM 2.1. *If F is a subspace of $C(TL)$ which is isometrically isomorphic to c_0 then F is complemented in $C(TL)$.*

PROOF. Let T be an isometric isomorphism mapping c_0 onto F. For each $n \in N$, let $f_n = Te_n$ where $\{e_i\}_{i=1}^{\infty}$ is the standard unit vector basis of c_0. Since $\|f_n\| = 1$, for each n

there is a number $t_n \in [0,1]$ and $i_n \in \{0,1\}$ such that $|f_n(t_n, i_n)| = 1$. Let Z be the set of limit points of the sequence $\{(t_n, i_n)\}$. Then $\|f_n + f_m\| = \|T(e_n \pm e_m)\| = 1$ for all $n \neq m$ which implies that $f_n(t_m, i_m) = 1$ if $n = m$ and zero otherwise. Hence if $(t,i) \in Z$ then for each $n \in N f_n(t,i) = \lim_{x\to\infty} f_n(t_{m_k}, i_{m_k}) = 0$.

Since $\{f_n\}_{n=1}^{\infty}$ is a basis for F the above statements means $f(t,i) = 0$ for all $(t,i) \in Z$ and $f \in F$. Now let Z^0 be the subspace of $C(TL)$ consisting of all functions vanishing at each point of Z. Define:

$$P\colon Z^0 \longrightarrow F \quad \text{by} \quad Px = \sum_{n=1}^{\infty} x(t_n, i_n)\,\operatorname{sgn}\big(f_n(t_n, i_n)\big) f_n.$$

Since $Pf_n = f_n$ for all $n \in N$, P is a projection of Z^0 onto F. P is bounded since

$$\|Px\| = \Big\|\sum_{n=1}^{\infty} x(t_n, i_n)\,\operatorname{sgn} f_n(t_n, i_n) e_n\Big\|_{c_0} = \sup_n |x(t_n, i_n)| \leq \|x\|.$$

To finish the proof a projection $Q\colon C(TL) \longrightarrow Z^0$ must be defined and then the map PQ will be the sought after projection of $C(TL)$ onto F. To this end let T be the extension map defined in Lemma 2.1. Clearly T is an isometry. Define $Qg = g - TRg$ where $Rg = g|_Z$. Then Q is clearly a bounded linear operator and if $g \in Z^0$ then $Rg = 0$ which implies $TRg = 0$ so that $Qg = g$ for all $g \in Z^0$.

The following result shows that not only c_0 but spaces close to c_0 are complemented in $C(TL)$.

THEOREM 2.2. *Let f_n be a sequence from $C(TL)$ such that*

(i) there exists $\lambda \in \Re$ such that $1 < \lambda < \lambda_0$ and

$$\frac{1}{\lambda}\Big\|\sum_{n=1}^{\infty} a_n e_n\Big\|_{c_0} \leq \Big\|\sum_{n=1}^{\infty} a_n f_n\Big\| \leq \lambda\Big\|\sum_{n=1}^{\infty} a_n e_n\Big\|_{c_0} \quad \textit{for all } (a_n) \in c_0$$

with λ_0 being a root of $x^6 - x^4 - 1$ greater than 1 ($\lambda_0 \approx 1.2106077944$).

(ii) there is a sequence (t_n, i_n) in TL with

(a) $f_n(t_n, i_n) > \frac{1}{\lambda}$

(b) $\lim_{n\to\infty} f_k(t_n, i_n) = 0$ for all k.

Then $F := \overline{\operatorname{span}}\{f_n\}$ (the closure of the linear span of $\{f_n\}_{n=1}^{\infty}$) is complemented in $C(TL)$.

PROOF. Let $X := \{f \in C(TL) : \lim_{n\to\infty} f(t_n, i_n) = 0\}$ where (t_n, i_n) is chosen as in (ii). Clearly X is a subspace of $C(TL)$. Next consider the map $Q\colon X \longrightarrow F$ defined by $Qg = \sum_{n=1}^{\infty} g(t_n, i_n) f_n$. Since $\lim_{n\to\infty} g(t_n, i_n) = 0$ for all $g \in X$ we have $\{g(t_n, i_n)\}_{n=1}^{\infty} \in c_0$ which implies that $Qg \in F$ and

$$\|Qg\| = \sup_{(t,1)}\Big|\sum_{n=1}^{\infty} g(t_n, i_n) f_n(t,i)\Big| \leq \lambda \sup_n |g(t_n, i_n)| \leq \|g\|,$$

thus Q is a bounded linear map of X into F. Let $\tilde{Q}$ be the restriction of Q to F. It shall be shown below that $\tilde{Q}$ is invertible whenever (i) holds. First, an invertible map $\tilde{D}$ will

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be defined and then it shall be shown that $\|\tilde{Q}-\tilde{D}\| \leq \|\tilde{D}^{-1}\|$ whenever (ii) holds from which it follows that $\tilde{Q}$ is also invertible. Define an operator $\tilde{D}: F \longrightarrow F$ by

$$\tilde{D}f = \sum_{n=1}^{\infty}\big(a_n f_n(t_n, i_n)\big) f_n \quad \text{where} \quad f = \sum_{j=1}^{\infty} a_j f_j \in F.$$

Then

$$\|\tilde{D}f\| \leq \lambda \sup_n |a_n f_n(t_n, i_n)| \leq \lambda \sup_n |a_n| \cdot \|f_n\| \leq \lambda^3 \|f\|.$$

The inverse of $\tilde{D}$ is the the operator $\tilde{D}^{-1}$ which is defined by

$$\tilde{D}^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{f_n(t_n, i_n)}$$

and

$$\|\tilde{D}^{-1}f\| \leq \lambda \sup_n \left|\frac{a_n}{f_n(t_n, i_n)}\right| \leq \lambda^3 \|f\|$$

so that $\tilde{D}^{-1}$ is also a bounded linear operator. Also

$$\begin{aligned}\|\tilde{Q}f - \tilde{D}f\| &= \left\|\sum_{n=1}^{\infty}\Big(\sum_{j\neq n} a_j f_j(t_n, i_n)\Big) f_n\right\| \\ &\leq \lambda \sup_n \left|\sum_{j\neq n} a_j f_j(t_n, i_n) f_n\right| \\ &\leq \lambda^2 \|f\| \sup_n |f_j(t_n, i_n)|\end{aligned} \tag{2.2.1}$$

But

$$\begin{aligned}\sum_{j\neq n} |f_j(t_n, i_n)| &= \sum_{j=1}^{\infty} |f_j(t_n, i_n)| - |f_n(t_n, i_n)| \\ &\leq \sup_t \sum_{j=1}^{\infty} |f_j(t, i)| - \frac{1}{\lambda} \qquad i = 0, 1 \\ &\leq \lambda - \frac{1}{\lambda}\end{aligned} \tag{2.2.2}$$

Now putting (2.2.1) and (2.2.2) together we have that

$$\|\tilde{Q}-\tilde{D}\| \leq \lambda^2\Big(\lambda - \frac{1}{\lambda}\Big) = \lambda^3 - \lambda \|x_n\|$$

Therefore, if $\lambda_0 > 1$ is chosen so that it is a root of $\lambda^6 - \lambda^4 - 1 = 0$ then

$$\|\tilde{Q}-\tilde{D}\| \leq \frac{1}{\|\tilde{D}^{-1}\|}.$$

For these values of λ, $\tilde{Q}$ will be an invertible operator and the map $\tilde{Q}^{-1}Q: \mathcal{X} \longrightarrow \mathcal{F}$ is a projection of $\mathcal{X}$ onto F. It is bounded and linear because both $\tilde{Q}$ and $\tilde{Q}^{-1}$ are bounded linear maps and clearly $\tilde{Q}^{-1}\tilde{Q} = \tilde{Q}\tilde{Q}^{-1} = I$ on F, is idempotent.

To finish the proof a projection P of $C(TL)$ onto $\mathcal{X}$ must be defined. The projection PQ will map $C(TL)$ onto F. Let Z = the cluster points of the set $\{(t_1, i_1), (t_2, i_2), \ldots\}$. Then Z is a closed subset of TL. For each g in $C(TL)$ let Rg be the restriction of g to z. TRg will be the linear extension of Rg to all of TL as defined in Lemma 2.1. Then $Pg = g - TRg$ is clearly a bounded linear map and $\lim_{n\to\infty}(Pg)(t_n, i_n) = 0$ because g and TRg agree on Z. Also if $g \in \mathcal{X}$, then $Rg = 0$ so that $Pg = g$ for all $g \in \mathcal{X}$, hence $P^2 g = Pg$ and P is a bounded projection of $C(TL)$ onto $\mathcal{X}$.

3. **Identifying the complement of c_0-subspaces of $C(TL)$.** If F is a complemented subspace of $C(TL)$ isomorphic to c_0 then we would like to identify the space G such that $C(TL) = F \oplus G$. The following lemma shall be instrumental in using the Pelczynski Decomposition method [3, p. 54] to show that if $F \subseteq C(TL)$ is an isometric copy of c_0 then the complement of F is isomorphic to $C(TL)$.

LEMMA 3.1. *$C(TL)$ is isomorphic to the infinite direct sum* $\big(C(TL) \oplus C(TL) \oplus \cdots\big)_{c_0}$.

PROOF. Let $X = \{g \in D[0, 1] : g(0) = 0\}$ and $J_n = [\frac{1}{2^n}, \frac{1}{2^{n-1}})$. Then

$$
\begin{aligned}
X &\sim \Big[\sum_{n=1}^{\infty} \oplus D(J_n)\Big]_{c_0} \\
&\sim \Big[\sum_{n=1}^{\infty} \oplus D[0,1)\Big]_{c_0} \qquad (3.1)\\
&\sim \Big[\sum_{n=1}^{\infty} \oplus D[0,1)\Big]_{c_0} \oplus \Big[\sum_{n=1}^{\infty} \oplus D[0,1)\Big]_{c_0} \\
&\sim X \oplus X
\end{aligned}
$$

Also if we define $T\colon D[0,1) \longrightarrow (X \oplus \Re)_\infty$ by $Tf = \big(f(x) - f(0), f(0)\big)$ we have that

$$D[0,1) \sim X \oplus \Re \qquad (3.2)$$

Clearly both T and T^{-1} are bounded linear maps. Next

$$X \sim X \oplus D[0,1) \qquad (3.3)$$

by the operator $f \longrightarrow (f|_{[0,1)}, f|_{[0,1)})$. Now putting (3.1), (3.2) and (3.3) together yields

$$
\begin{aligned}
X &\sim X \oplus D[0,1) \\
&\sim X \oplus (X \oplus \Re) \\
&\sim X \oplus \Re \\
&\sim D[0,1)
\end{aligned}
$$

Thus $X \oplus \Re \sim D[0,1) \oplus \Re \sim D[0,1]$ and

$$
\begin{aligned}
C(TL) &\sim D[0,1] \sim D[0,1) \sim X \\
&\sim \Big[\sum_{n=1}^{\infty} \oplus D[0,1]\Big]_{c_0} \\
&\sim \Big[\sum_{n=1}^{\infty} \oplus D[0,1]\Big]_{c_0} \sim \Big[\sum_{n=1}^{\infty} \oplus C(TL)\Big]_{c_0}
\end{aligned}
$$

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LEMMA 3.2. *If F is a subspace of $C(TL)$ isometrically isomorphic to c_0 then $C(TL) = F \oplus G$ where G is isomorphic to $C(TL)$.*

PROOF. From Theorem 2.2 the composition map PQ will be a projection of $C(TL)$ onto F where $P\colon Z^{\perp} \longrightarrow F$ and $Q\colon C(TL) \longrightarrow Z^{\perp}$ are defined as

$$Pg = \sum_{n=1}^{\infty} g(t_n, i_n)\, \mathrm{sgn} f_n(t_n, i_n) f_n \quad \text{and} \quad Qg = g - URg.$$

Let $[a,b]$ be a subinterval of $[0,1]$ such that the set $\{(t_1,i_1),(t_2,i_2),\ldots\}$ and consider $D[a,b]$ which is isomorphic to $C(TL)$ and $P|_{D[a,b]_{C(TL)}}$ where

$$D[a,b]_{C(TL)} = \{f \in C(TL) : f(t,1) = g(t) \text{ and } f(t,0) = g(t^-) \text{ for some } g \in D[a,b]\}$$

will be the zero map. Thus $D[a,b]_{C(TL),g|_Z} = 0$ which means that $URg = 0$ and implies that $Qg = g$, therefore $(PQ)g = Pg = 0$.

Let R map $D[0,1]$ to $D[a,b]$ naturally. $R|_G$ mapping G onto $D[a,b]$ and thus onto $D[a,b]_{C(TL)}$ is a bounded linear projection and thus $D[a,b]_{C(TL)}$ is complemented in G.

At this point it has been shown that $C(TL) = F \oplus G$ where $F \sim c_0$ and $G = \ker(PQ)$ is isomorphic to $C(TL) \oplus W$ for a suitable Banach space W. Using the Pelczynski Decomposition method and Lemma 3.1

$$\begin{aligned} C(TL) \oplus G &\sim C(TL) \oplus \big(C(TL) \oplus W\big) \\ &\sim \big(C(TL) \oplus C(TL)\big) \oplus W \\ &\sim C(TL) \oplus W \\ \therefore \quad C(TL) \oplus G &\sim G \end{aligned}$$

and

$$\begin{aligned} C(TL) \oplus G &\sim \Big[\sum \oplus C(TL)\Big]_{c_0} \oplus G \\ &\sim \Big[\sum \oplus (F \oplus G)\Big]_{c_0} \oplus G \\ &\sim \Big[\sum \oplus F\Big]_{c_0} \oplus \Big[\sum \oplus G\Big]_{c_0} \oplus G \\ &\sim \Big[\sum \oplus F\Big]_{c_0} \oplus \Big[\sum \oplus G\Big]_{c_0} \\ &\sim \Big[\sum \oplus (F \oplus G)\Big]_{c_0} \\ &\sim C(TL). \end{aligned}$$

Thus, $C(TL) \sim C(TL) \oplus G \sim G$. Note that this is true not only for $G = \ker(PQ)$ where PQ is as defined above, but for any subspace G' where F is isometrically isomorphic to c_0 if $C(TL) = F \oplus G'$ then G' is isomorphic to $C(TL)$.

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