

# SubRiemannian Geometry on the Sphere $\mathbb{S}^3$

Ovidiu Calin, Der-Chen Chang, and Irina Markina

*Abstract.* We discuss the subRiemannian geometry induced by two noncommutative vector fields which are left invariant on the Lie group  $\mathbb{S}^3$ .

## 1 Introduction

The study of step 2 subRiemannian manifolds has the Heisenberg group as a prototype. This is a noncommutative Lie group with the base manifold  $\mathbb{R}^3$  and endowed with a nonintegrable distribution spanned by two of the noncommutative left invariant vector fields. This structure also enjoys the property of being a contact structure or a CR-manifold. The study of the subRiemannian geodesics on the Heisenberg group started with the work of Gaveau [9]. One trend in the literature has been to use the geometry of the Heisenberg group to describe the Heisenberg Laplacian and its heat kernel, see Beals, Gaveau, and Greiner [1–4]. Later, this structure led to generalizations of the Heisenberg group as can be seen in Calin, Chang, and Greiner [5, 6] and Chang and Markina [7]. For more fundamental issues on subRiemannian geometry, see Strichartz [10].

Is there another 3-dimensional noncommutative Lie group besides the Heisenberg group? The answer is positive and this paper deals with it. The sphere  $\mathbb{S}^3$  and the group  $SU(2)$  have these properties. More important, this is the first compact subRiemannian manifold has been under consideration.

The present paper starts with an introduction to quaternions and then defines the 3-dimensional sphere as the set of quaternions of length one. The quaternion group induces on  $\mathbb{S}^3$  a structure of a noncommutative Lie group. This group is compact and the results obtained in this case are very different from those obtained in the case of the Heisenberg group, which is a noncompact Lie group. As in the Heisenberg group case, we introduce a nonintegrable distribution on the sphere and a metric on it using two of the noncommutative left invariant vector fields. This way  $\mathbb{S}^3$  becomes a subRiemannian manifold. It is known that the group  $SU(2)$  is isomorphic with the sphere  $\mathbb{S}^3$  and represents an example of subRiemannian manifold where the elements are matrices. The main issue here is to study the connectivity by horizontal curves and geodesics on this manifold. In this paper, we are using Lagrangian methods to

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study the connectivity theorem on  $S^3$  by horizontal curves with minimal arc-length. We show that for any two points in  $S^3$ , there exists such a geodesic joining these two points.

### 2 The Quaternion Group $\mathbf{Q}$

William R. Hamilton introduced quaternions in 1873. They are obtained as a non-commutative extension of complex numbers. Let  $\mathbf{Q}$  be the set of matrices  $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$  with complex number entries  $\alpha, \beta \in \mathbb{C}$ . With the usual matrix multiplication law, the set  $\mathbf{Q}$  becomes a group called the *quaternion group*. The identity element is  $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and the inverse of  $q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$  is  $q^{-1} = \frac{1}{\Delta} \begin{pmatrix} \bar{\alpha} & -\beta \\ \beta & \alpha \end{pmatrix}$ , where  $\Delta = |\alpha|^2 + |\beta|^2$ . Let

$$(2.1) \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{I}$ ,  $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$ ,  $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ , and  $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$ . A real number  $a \in \mathbb{R}$  can be identified with  $a\mathbf{i}$  by the one-to-one homomorphism  $\varphi: \mathbb{R} \rightarrow \mathbf{Q}$ ,  $\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ . If  $\alpha = a_0 + a_1\mathbf{i}$  and  $\beta = b_0 + b_1\mathbf{i} \in \mathbb{C}$ , then an elementary computation yields

$$q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = a_0\mathbf{I} + a_1\mathbf{i} + b_0\mathbf{j} + b_1\mathbf{k}.$$

The set  $\{\mathbf{I}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$  is a basis of  $\mathbf{Q}$ . Let  $q = a\mathbf{I} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$  and  $q' = a'\mathbf{I} + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$  be two quaternions.

A straightforward computation shows

$$qq' = (aa' - bb' - cc' - dd')\mathbf{I} + (ab' + ba' + cd' - dc')\mathbf{i} + (ac' + ca' + db' - bd')\mathbf{j} + (ad' + da' + bc' - cb')\mathbf{k}.$$

The *conjugate* of  $q = a\mathbf{I} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$  is  $\bar{q} = a\mathbf{I} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ . A simple computation shows that  $q\bar{q} = a^2 + b^2 + c^2 + d^2 = \Delta$ ,  $q^{-1} = \frac{1}{\Delta}\bar{q}$ . The modulus of  $q$  is  $|q| = \sqrt{\Delta}$ . Since  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  do not commute, the multiplication of quaternions is noncommutative.

### 3 $S^3$ as a Noncommutative Lie Group

Consider  $S^3 = \{q \in \mathbf{Q}; |q| = 1\}$ . For any  $q_1, q_2 \in S^3$ ,  $|q_1^{-1}q_2| = \frac{|q_2|}{|q_1|} = 1$ , i.e.,  $S^3$  is a subgroup of  $\mathbf{Q}$ . The group law on  $S^3$  is

$$(a, b, c, d) * (a', b', c', d') = (aa' - bb' - cc' - dd', ab' + ba' + cd' - dc', ac' + ca' + db' - bd', ad' + da' + bc' - cb'),$$

with the identity element  $(1, 0, 0, 0)$  and the inverse  $(a, b, c, d)^{-1} = (a, -b, -c, -d)$ . In this way  $(S^3, *)$  becomes a compact, noncommutative Lie group.

We shall work out the left invariant vector fields with respect to the left translation  $L_q: \mathbb{S}^3 \rightarrow \mathbb{S}^3, L_q h = q * h$ . These are vector fields on  $\mathbb{S}^3$  such that

$$dL_q(X_h) = X_{q*h}, \quad \forall q, h \in \mathbb{S}^3.$$

The set of all left invariant vector fields on  $\mathbb{S}^3, \mathcal{L}(\mathbb{S}^3) = \{X; dL_q X = X, \forall q \in \mathbb{S}^3\}$  is called the Lie algebra of  $\mathbb{S}^3$ . Thus  $\mathcal{L}(\mathbb{S}^3)$  is a subalgebra of  $\mathcal{L}(Q)$ , i.e., it is a subset closed under the Lie bracket  $[\cdot, \cdot]$ .

In order to find a basis for  $\mathcal{L}(\mathbb{S}^3)$ , we shall first find a basis for  $\mathcal{L}(Q)$ . Let  $X$  be a left invariant vector field on  $Q$ . Then  $X_q = (dL_q)X_e$ , where  $e = (1, 0, 0, 0)$  is the identity of  $Q$ . If write

$$X_q = \sum_{i=1}^4 X_q^i \frac{\partial}{\partial x_i},$$

the components are  $X_q^i = (dL_q)X_e(x_i) = X_e(x_i \circ L_q), i = 1, \dots, 4$ , where  $x_i$  is the  $i$ -th coordinate.

Next we compute  $x_i \circ L_q, i = 1, 2, 3, 4$ . Let  $q = (a, b, c, d)$  and  $q' = (a', b', c', d')$ . Then

$$(3.1) \quad (x_1 \circ L_q)(q') = x_1(L_q q') = aa' - bb' - cc' - dd'.$$

Using

$$\begin{aligned} a &= x_1(q), & a' &= x_1(q'), & b &= x_2(q), & b' &= x_2(q'), \\ c &= x_3(q), & c' &= x_3(q'), & d &= x_4(q), & d' &= x_4(q'), \end{aligned}$$

equation (3.1) becomes

$$(x_1 \circ L_q)(q') = x_1(q)x_1(q') - x_2(q)x_2(q') - x_3(q)x_3(q') - x_4(q)x_4(q').$$

Dropping the argument  $q'$  yields

$$x_1 \circ L_q = x_1(q)x_1 - x_2(q)x_2 - x_3(q)x_3 - x_4(q)x_4.$$

By a similar computation we find

$$\begin{aligned} x_2 \circ L_q &= x_1(q)x_2 + x_2(q)x_1 + x_3(q)x_4 - x_4(q)x_3, \\ x_3 \circ L_q &= x_1(q)x_3 + x_3(q)x_1 + x_4(q)x_2 - x_2(q)x_4, \\ x_4 \circ L_q &= x_1(q)x_4 + x_4(q)x_1 + x_2(q)x_3 - x_3(q)x_2. \end{aligned}$$

The first component becomes

$$\begin{aligned} X_q^1 &= X_e(x_1 \circ L_q) \\ &= x_1(q)X_e(x_1) - x_2(q)X_e(x_2) - x_3(q)X_e(x_3) - x_4(q)X_e(x_4) \\ &= x_1(q)X_e^1 - x_2(q)X_e^2 - x_3(q)X_e^3 - x_4(q)X_e^4. \end{aligned}$$

Dropping  $q$  and denoting  $\xi^i = X_e^i$ ,  $i = 1, 2, 3, 4$ , the previous relation yields

$$X^1 = x_1\xi^1 - x_2\xi^2 - x_3\xi^3 - x_4\xi^4.$$

In a similar way we obtain

$$X^2 = x_1\xi^2 + x_2\xi^1 + x_3\xi^4 - x_4\xi^3,$$

$$X^3 = x_1\xi^3 + x_3\xi^1 + x_4\xi^2 - x_2\xi^4,$$

$$X^4 = x_1\xi^4 + x_4\xi^1 + x_2\xi^3 - x_3\xi^2.$$

Left invariant vector fields  $Z$  can be written as

$$\begin{aligned} Z = \sum_{i=1}^4 Z^i \frac{\partial}{\partial x_i} &= (x_1\xi^1 - x_2\xi^2 - x_3\xi^3 - x_4\xi^4) \frac{\partial}{\partial x_1} \\ &\quad + (x_1\xi^2 + x_2\xi^1 + x_3\xi^4 - x_4\xi^3) \frac{\partial}{\partial x_2} \\ &\quad + (x_1\xi^3 + x_3\xi^1 + x_4\xi^2 - x_2\xi^4) \frac{\partial}{\partial x_3} \\ &\quad + (x_1\xi^4 + x_4\xi^1 + x_2\xi^3 - x_3\xi^2) \frac{\partial}{\partial x_4}. \end{aligned}$$

Reordering yields

$$\begin{aligned} Z &= \xi^1 \left[ x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \right] \\ &\quad + \xi^2 \left[ -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \right] \\ &\quad + \xi^3 \left[ -x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4} \right] \\ &\quad + \xi^4 \left[ -x_4 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} \right] \\ &= \xi^1 N - \xi^2 X - \xi^3 T - \xi^4 Y, \end{aligned}$$

where  $N = \sum_{i=1}^4 x_i \frac{\partial}{\partial x_i}$  is the normal vector field to  $\mathbb{S}^3$ , and

$$X = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4},$$

$$Y = x_4 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4},$$

$$T = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_4}.$$

Using the  $4 \times 4$ -matrix representations:

$$(3.2) \quad I_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$I_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

and the standard inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^4$ , we also can write

$$N = \langle xU, \nabla_x \rangle, \quad X = -\langle xI_1, \nabla_x \rangle, \quad Y = -\langle xI_3, \nabla_x \rangle, \quad T = -\langle xI_2, \nabla_x \rangle,$$

where  $U$  is the  $4 \times 4$  identity matrix.

It is easy to see that  $\{N, X, Y, T\}$  is a basis for  $\mathcal{L}(\mathbf{Q})$ . In order to find a basis for  $\mathcal{L}(\mathbb{S}^3)$  we shall consider the vector fields which are tangent to  $\mathbb{S}^3$ . Since a computation provides  $\langle N, X \rangle = \langle N, Y \rangle = \langle N, T \rangle = 0$ , it follows that  $X, Y, T$  are tangent to  $\mathbb{S}^3$ . Hence any left invariant vector field on  $(\mathbb{S}^3, *)$  is a linear combination of  $X, Y$  and  $T$ . Since the matrix of coefficients

$$\begin{pmatrix} x_2 & -x_1 & -x_4 & x_3 \\ x_4 & -x_3 & x_2 & -x_1 \\ x_3 & x_4 & -x_1 & -x_2 \end{pmatrix}$$

has rank 3 at every point, it follows that  $X, Y$  and  $T$  are linearly independent and hence form a basis of  $L(\mathbb{S}^3)$ . It is worth noting that the above rank property is not true without the constraint  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ .

We also note that  $\{X, Y, T, N\}$  form an orthonormal system with respect to the usual inner product of  $\mathbb{R}^4$ . Then  $\mathcal{L}(\mathbf{Q}) = \mathcal{L}(\mathbb{S}^3) \oplus \mathbb{R}N$ .

### 3.1 The Horizontal Distribution

Let  $\mathcal{H} = \text{span}\{X, Y\}$  be the distribution generated by the vector fields  $X$  and  $Y$ . Since  $[Y, X] = 2T \notin \mathcal{H}$ , it follows that  $\mathcal{H}$  is not involutive. We can write  $\mathcal{L}(\mathbb{S}^3) = \mathcal{H} \oplus \mathbb{R}T$ . The distribution  $\mathcal{H}$  will be called the *horizontal distribution*. Any curve on the sphere which has the velocity vector contained in the distribution  $\mathcal{H}$  will be called a *horizontal curve*.

**Remark 3.1** Notice that this situation differs from the Heisenberg group since the choice of the horizontal distribution is not unique. Because of  $[X, T] = -2Y$  and  $[Y, T] = 2X$  we could chose  $\mathcal{H} = \text{span}\{X, T\}$  or  $\mathcal{H} = \text{span}\{Y, T\}$ . The geometries defined by different horizontal distributions are cyclically symmetric, so we restrict our attention to  $\mathcal{H} = \text{span}\{X, Y\}$ .

The following result deals with a characterization of horizontal curves. It will make sense for later reasons to rename the variables  $y_1 = x_3$  and  $y_2 = x_4$ .

**Proposition 3.2** *Let  $\gamma(s) = (x_1(s), x_2(s), y_1(s), y_2(s))$  be a curve on  $\mathbb{S}^3$ . The curve  $\gamma$  is horizontal if and only if  $\langle \dot{x}, y \rangle = \langle x, \dot{y} \rangle$ .*

**Proof** Since  $\{X, Y, T\}$  span the tangent space of the sphere  $\mathbb{S}^3$ , we have

$$\dot{\gamma} = aX + bY + cT.$$

Using that  $\{X, Y, T\}$  is an orthonormal system on  $\mathbb{S}^3$ , we have

$$c = \langle \dot{\gamma}, T \rangle = \dot{x}_1 y_1 + \dot{x}_2 y_2 - \dot{y}_1 x_1 - \dot{y}_2 x_2 = \langle \dot{x}, y \rangle - \langle x, \dot{y} \rangle.$$

The curve  $\gamma$  is horizontal if and only if  $c = 0$ , i.e.,  $\langle \dot{x}, y \rangle = \langle x, \dot{y} \rangle$ . ■

Hence a horizontal curve  $\gamma = (x_1, x_2, y_1, y_2): (0, \tau) \rightarrow \mathbb{R}^3$  has the velocity  $\dot{\gamma} = aX + bY$  where

$$(3.3) \quad a = \langle \dot{\gamma}, X \rangle = \dot{x}_1 x_2 - \dot{x}_2 x_1 - \dot{y}_1 y_2 + \dot{y}_2 y_1,$$

$$(3.4) \quad b = \langle \dot{\gamma}, Y \rangle = \dot{x}_1 y_2 - \dot{x}_2 y_1 + \dot{y}_1 x_2 - \dot{y}_2 x_1.$$

If we consider a metric on  $\mathcal{H}$  such that  $\{Y, X\}$  are orthonormal vector fields, then the length of the curve  $\gamma$  is

$$\ell(\gamma) = \int_0^T \sqrt{a(s)^2 + b(s)^2} ds.$$

The horizontal curves minimizing length will be treated in the next section. In the rest of this section we shall make a few considerations regarding the horizontal distribution given as the kernel of a one-form.

Consider the following one-form on  $\mathbb{R}^4$

$$\omega = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 = xdy - ydx.$$

One can easily check that

$$\omega(X) = -x_1 y_2 - y_1 x_2 + x_2 y_1 + x_1 y_2 = 0,$$

$$\omega(Y) = x_1 x_2 - y_1 y_2 - x_1 x_2 + y_1 y_2 = 0,$$

$$\omega(T) = -x_1^2 - y_1^2 - x_2^2 - y_2^2 = -1 \neq 0,$$

$$\omega(N) = x_1 y_1 - y_1 x_1 + x_2 y_2 - y_2 x_2 = 0,$$

where  $N = x_1 \partial_{x_1} + x_2 \partial_{x_2} + y_1 \partial_{y_1} + y_2 \partial_{y_2}$  is the unit normal to the sphere  $\mathbb{S}^3$ . Hence  $\ker \omega = \text{span}\{X, Y, N\}$ , and the horizontal distribution can be written as

$$\mathbb{S}^3 \ni p \rightarrow \mathcal{H}_p = \ker \omega \cap T_p \mathbb{S}^3.$$

Hence a vector  $\nu = (v, w) = (v_1, v_2, w_1, w_2)$  is in  $\mathcal{H}$  if and only if

$$\langle x, v \rangle + \langle y, w \rangle = 0 \quad (\iff \nu \perp N),$$

$$\langle v, y \rangle - \langle x, w \rangle = 0 \quad (\iff \omega(\nu) = 0),$$

$$|x|^2 + |y|^2 = 1 \quad (\iff (x, y) \in \mathbb{S}^3).$$

#### 4 Connectivity Theorem on $\mathbb{S}^3$

The present section deals with the global connectivity property by horizontal curves. Even if a result of Chow [8] states the connectivity by piece-wise horizontal curves, here we shall prove a globally smooth version of it. More precisely, the main result of this section states that given two points on  $\mathbb{S}^3$ , there is a horizontal smooth curve joining them.

Before proceeding to the proof, we need to introduce some terminology and notations. This deals with the parametrization of  $\mathbb{S}^3$  in terms of the Euler angles, which due to spherical symmetry are more suitable than the cartesian coordinates.

Let  $\varphi, \psi, \theta$  be Euler's angles and let  $\alpha = (\varphi + \psi)/2, \beta = (\varphi - \psi)/2$ . The sphere  $\mathbb{S}^3$  can be parametrized as

$$\begin{aligned}x_1 &= \cos \frac{\varphi + \psi}{2} \cos \frac{\theta}{2} = \cos \alpha \cos \frac{\theta}{2}, & x_2 &= \sin \frac{\varphi + \psi}{2} \cos \frac{\theta}{2} = \sin \alpha \cos \frac{\theta}{2}, \\y_1 &= \cos \frac{\varphi - \psi}{2} \sin \frac{\theta}{2} = \cos \beta \sin \frac{\theta}{2}, & y_2 &= \sin \frac{\varphi - \psi}{2} \sin \frac{\theta}{2} = \sin \beta \sin \frac{\theta}{2},\end{aligned}$$

with  $0 \leq \theta \leq \pi, -\pi \leq \alpha \leq \pi$ . In the following we shall write the restriction of the one-form  $\omega = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$  to  $\mathbb{S}^3$  using Euler's angles. Since

$$\begin{aligned}dx_1 &= -\sin \alpha \cos \frac{\theta}{2} d\alpha - \frac{1}{2} \cos \alpha \sin \frac{\theta}{2} d\theta & dx_2 &= \cos \alpha \cos \frac{\theta}{2} d\alpha - \frac{1}{2} \sin \alpha \sin \frac{\theta}{2} d\theta \\dy_1 &= -\sin \beta \sin \frac{\theta}{2} d\beta + \frac{1}{2} \cos \beta \cos \frac{\theta}{2} d\theta & dy_2 &= \cos \beta \sin \frac{\theta}{2} d\beta + \frac{1}{2} \sin \beta \cos \frac{\theta}{2} d\theta,\end{aligned}$$

we obtain

$$\begin{aligned}\omega &= x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 \\&= \left( \cos \beta \sin \alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2} - \sin \beta \cos \alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) d\alpha \\&\quad + \left( -\cos \alpha \cos \frac{\theta}{2} \sin \beta \sin \frac{\theta}{2} + \sin \alpha \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \beta \right) d\beta \\&\quad + \frac{1}{2} \left( \cos \alpha \cos \frac{\theta}{2} \cos \beta \cos \frac{\theta}{2} + \cos \alpha \cos \beta \sin^2 \frac{\theta}{2} \right. \\&\quad \quad \left. + \sin \alpha \sin \beta \cos^2 \frac{\theta}{2} + \sin \beta \sin \alpha \sin^2 \frac{\theta}{2} \right) d\theta \\&= \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin(\alpha - \beta) d\alpha + \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin(\alpha - \beta) d\beta + \frac{1}{2} \cos(\beta - \alpha) d\theta \\&= \frac{1}{2} \sin \theta \sin \psi (d\alpha + d\beta) + \frac{1}{2} \cos \psi d\theta \\&= \frac{1}{2} (\sin \theta \sin \psi d\varphi + \cos \psi d\theta).\end{aligned}$$

Hence  $\omega|_{\mathbb{S}^3} = \frac{1}{2}(\sin \theta \sin \psi d\varphi + \cos \psi d\theta)$ . The constraint  $\sin \theta \sin \psi d\varphi + \cos \psi d\theta = 0$  is nonholonomic since

$$2d\omega = \cos \theta \sin \psi d\theta \wedge d\varphi + \sin \theta \cos \psi d\psi \wedge d\varphi - \sin \psi d\psi \wedge d\theta \neq 0.$$

We finally arrived at the following characterization of the horizontal curves using Euler's angles.

**Lemma 4.1** *Let  $c(s) = (\varphi(s), \psi(s), \theta(s))$  be a curve on  $\mathbb{S}^3$ . The curve  $c$  is horizontal if and only if  $\omega(\dot{c}) = 0$ , i.e.,*

$$(4.1) \quad \sin \theta \sin \psi \dot{\varphi} + \cos \psi \dot{\theta} = 0.$$

Before getting into the proof of the connectivity theorem, which is the main result of this section, a couple of lemmas are needed. The first one is a standard calculus exercise.

**Lemma 4.2** *Given the numbers  $\alpha, \beta, \gamma \in \mathbb{R}$ , there is a smooth function  $f: [0, 1] \rightarrow \mathbb{R}$  such that  $f(0) = 0$ ,  $f(1) = \alpha$ ,  $f'(0) = \beta$ ,  $f'(1) = \gamma$ .*

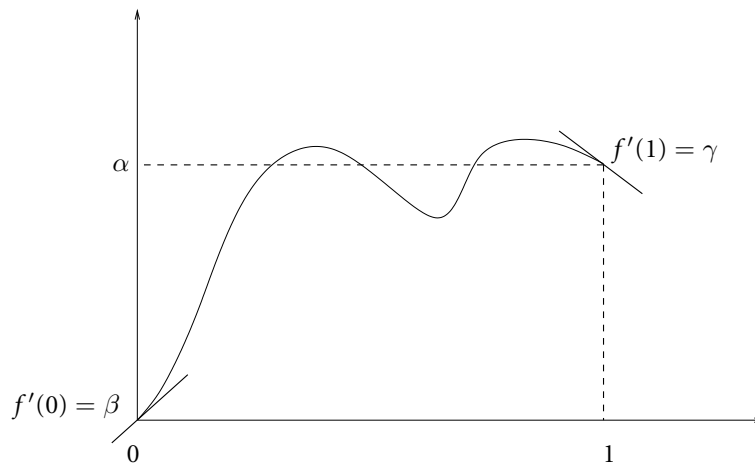


Figure 1

**Lemma 4.3** *Given  $q_0, q_1, I \in \mathbb{R}$ , there is a function  $q: [0, 1] \rightarrow \mathbb{R}$  such that*

$$q(0) = q_0, \quad q(1) = q_1, \quad \int_0^1 q(u) du = I.$$



**Proof** Applying Lemma 4.2, there is a function  $f: [0, 1] \rightarrow \mathbb{R}$  such that

$$f(0) = 0, \quad f(1) = I, \quad f'(0) = q_0, \quad f'(1) = q_1.$$

Let  $q(s) = f'(s)$ . Then  $f(s) = \int_0^s q(u) du$  and hence  $I = f(1) = \int_0^1 q(u) du$  and  $q(0) = f'(0) = q_0$  and  $q(1) = f'(1) = q_1$ . ■

The next result is the theorem of connectivity announced at the beginning of this section.

**Theorem 4.4** *Given two points  $P, Q \in \mathbb{S}^3$ , there is a smooth horizontal curve joining  $P$  and  $Q$ .*

**Proof** Let  $P(\varphi_0, \psi_0, \theta_0)$  and  $Q(\varphi_1, \psi_1, \theta_1)$  be the coordinates in the Eulerian angles. We shall find a horizontal curve  $c(s) = (\varphi(s), \psi(s), \theta(s))$  with  $c(0) = P$  and  $c(1) = Q$ . This is equivalent with finding functions  $\varphi(s), \psi(s), \theta(s)$  such that

$$(4.2) \quad \sin \theta(s) \sin \psi(s) \dot{\varphi}(s) + \cos \psi(s) \dot{\theta}(s) = 0,$$

satisfying the boundary conditions

$$\begin{aligned} \varphi(0) &= \varphi_0, & \psi(0) &= \psi_0, & \theta(0) &= \theta_0, \\ \varphi(1) &= \varphi_1, & \psi(1) &= \psi_1, & \theta(1) &= \theta_1. \end{aligned}$$

Assuming  $\cos \psi(s) \neq 0$ , (4.2) can be written as

$$(4.3) \quad \sin \theta(s) \tan \psi(s) \dot{\varphi}(s) + \dot{\theta}(s) = 0.$$

Let  $\theta(s) = \arcsin p(s)$ ,  $\psi(s) = \arctan q(s)$ , and  $\varphi(s) = \varphi_0 + s(\varphi_1 - \varphi_0)$  for some functions  $p(s), q(s)$  which will be determined later. Let  $k = \varphi_1 - \varphi_0$ . Then (4.3) yields

$$p(s) q(s) k + \frac{p'(s)}{\sqrt{1-p^2(s)}} = 0.$$

Separating and solving for  $p(s)$  yields

$$\frac{dp}{p\sqrt{1-p^2}} = -kq(s) ds$$

and integrating,

$$\begin{aligned} -\tanh^{-1} \frac{1}{\sqrt{1-p^2}} &= -k \left( \int_0^s q(u) du + C_1 \right) \\ \iff \frac{1}{\sqrt{1-p^2(s)}} &= \tanh \left[ k \left( \int_0^s q(u) du + C_1 \right) \right]. \end{aligned}$$

The constant  $C_1$  and the integral  $\int_0^1 q(u) du$  can be determined from the boundary conditions. We obtain

$$\frac{1}{\sqrt{1-p^2(0)}} = \tanh(kC_1) \implies C_1 = \frac{1}{k} \tanh^{-1}\left(\frac{1}{\sqrt{1-p^2(0)}}\right),$$

$$\frac{1}{\sqrt{1-p^2(1)}} = \tanh\left[k\left(\int_0^1 q(u) du + C_1\right)\right] \implies$$

$$\int_0^1 q(u) du = \frac{1}{k} \left[ \tanh^{-1} \frac{1}{\sqrt{1-p^2(1)}} - \tanh^{-1} \frac{1}{\sqrt{1-p^2(0)}} \right],$$

where  $p(0) = \sin \theta(0) = \sin \theta_0$  and  $p(1) = \sin \theta_1$ . Hence  $q(s)$  has to satisfy

$$(4.4) \quad \int_0^1 q(u) du = \frac{1}{k} \left[ \tanh^{-1} \frac{1}{|\cos \theta_1|} - \tanh^{-1} \frac{1}{|\cos \theta_0|} \right].$$

The boundary conditions also yield

$$(4.5) \quad q(0) = \tan \psi_0, \quad q(1) = \tan \psi_1.$$

Lemma 4.3 provides the existence of a function  $q: [0, 1] \rightarrow \mathbb{R}$  such that (4.4) and (4.5) are satisfied.

The function  $p(s)$  will be defined by the relation

$$\frac{1}{\sqrt{1-p^2(s)}} = \tanh \left[ k \int_0^s q(u) du + \tanh^{-1} \frac{1}{|\cos \theta_0|} \right].$$

Hence the curve  $c(s) = (\varphi(s), \psi(s), \theta(s)) = (\varphi_0 + ks, \arctan q(s), \arcsin p(s))$  is the desired horizontal curve joining the points  $P$  and  $Q$  on  $\mathbb{S}^3$ . ■

It is known that in the case of the Heisenberg group if  $(x_0, 0)$  and  $(x_1, 0)$  are two points in the  $\{t = 0\}$  plane, then the segment joining the points is a horizontal curve joining the points and lying in the  $\{t = 0\}$  plane. The following theorem is an analog for the sphere  $\mathbb{S}^3$ . The plane will be replaced by the sphere  $\mathbb{S}^2$ .

**Theorem 4.5** *Given  $(\varphi_0, \varphi_1)$  and  $(\theta_0, \theta_1)$ , there is a horizontal curve with  $\psi = \text{constant}$ , which joins the points with Euler coordinates  $(\varphi_0, \psi, \theta_0)$  and  $(\varphi_1, \psi, \theta_1)$ .*

**Proof** Let  $(\varphi, \psi, \theta)$  be a horizontal curve with  $\psi = \text{constant}$ . Then equation (4.1) can be written as

$$-\tan \psi \frac{d\varphi}{d\theta} = \frac{1}{\sin \theta}.$$

Integrating, yields

$$(4.6) \quad -\tan \psi \int_{\theta_0}^{\theta} \frac{d\varphi}{d\theta} d\theta = \int_{\theta_0}^{\theta} \frac{1}{\sin \theta} d\theta$$

$$\iff -\tan \psi (\varphi(\theta) - \varphi(\theta_0)) = \ln \tan \frac{\theta}{2} - \ln \tan \frac{\theta_0}{2}.$$

For  $\theta = \theta_1$  we obtain

$$(4.7) \quad \psi = \arctan\left(\ln \frac{\tan \theta_1/2}{\tan \theta_0/2} / (\varphi_0 - \varphi_1)\right).$$

Solving for  $\varphi$ , (4.6) yields

$$\varphi(\theta) - \varphi_0 = -\ln\left(\tan \frac{\theta}{2} / \tan \frac{\theta_0}{2}\right) / \tan \psi,$$

with  $\psi$  defined by (4.7). Hence the horizontal curve joining  $(\varphi_0, \psi, \theta_0)$  and  $(\varphi_1, \psi, \theta_1)$  is

$$(\varphi, \psi, \theta) = \left(\varphi_0 - \ln\left(\tan \frac{\theta}{2} / \tan \frac{\theta_0}{2}\right) / \tan \psi, \psi, \theta\right). \quad \blacksquare$$

Let  $S_{\psi_0} = \{(\varphi, \psi, \theta) \in \mathbb{S}^3; \psi = \psi_0\}$  be the subset of  $\mathbb{S}^3$  with the same Euler angle  $\psi$ . This corresponds to a 2-dimensional sphere.

**Corollary 4.6** *Given the points  $(\varphi_0, \psi_0, \theta_0)$  and  $(\varphi_1, \psi_1, \theta_1)$  on  $\mathbb{S}^3$ , with  $\psi_0 = \psi_1$ , there is a horizontal curve in  $S_{\psi_0}$  that joins the points.*

## 5 The SubRiemannian Geodesics

Given two points on the sphere  $\mathbb{S}^3$ , we already know that there are horizontal curves joining them. The horizontal curve with the shortest length is called a geodesic. Standard arguments show that an equivalent problem to finding the geodesics is to find the horizontal curves with minimum energy. This means to minimize the action integral

$$\int_0^T \frac{1}{2} [a^2(s) + b^2(s)] ds$$

subject to the horizontality constraint  $\langle \dot{x}, y \rangle = \langle x, \dot{y} \rangle$ . The above integral can be minimized by the solutions of the Euler–Lagrange system with Lagrangian

$$L(x, \dot{x}, y, \dot{y}) = \frac{1}{2}(a^2 + b^2) + \lambda(s)(x_1 \dot{y}_1 - y_1 \dot{x}_1 + x_2 \dot{y}_2 - y_2 \dot{x}_2),$$

where  $a$  and  $b$  depend on  $x, \dot{x}, y, \dot{y}$ , see formulas (3.3)–(3.4). The function  $\lambda(s)$  is a Lagrange multiplier function.

### 5.1 The Case $\lambda(s) = 0$

The curve that minimizes the energy in this case satisfies the Euler–Lagrange system with the Lagrangian  $L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}a^2 + \frac{1}{2}b^2$ . We have

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_1} &= ax_2 + by_2, & \frac{\partial L}{\partial x_1} &= -a\dot{x}_2 - b\dot{y}_2 \\ \frac{\partial L}{\partial \dot{x}_2} &= -ax_1 - by_1, & \frac{\partial L}{\partial x_2} &= a\dot{x}_1 + b\dot{y}_1 \\ \frac{\partial L}{\partial \dot{y}_1} &= -ay_2 + bx_2, & \frac{\partial L}{\partial y_1} &= a\dot{y}_2 - b\dot{x}_2 \\ \frac{\partial L}{\partial \dot{y}_2} &= ay_1 - bx_1, & \frac{\partial L}{\partial y_2} &= -a\dot{y}_1 + b\dot{x}_1. \end{aligned}$$

Then the Euler–Lagrange system

$$(5.1) \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1}, \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_2},$$

$$(5.2) \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{y}_1} = \frac{\partial L}{\partial y_1}, \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{y}_2} = \frac{\partial L}{\partial y_2},$$

becomes

$$\begin{aligned} a\dot{x}_2 + b\dot{y}_2 &= -2(a\dot{x}_2 + b\dot{y}_2), & a\dot{x}_1 + b\dot{y}_1 &= -2(a\dot{x}_1 + b\dot{y}_1), \\ a\dot{y}_2 - b\dot{x}_2 &= -2(a\dot{y}_2 - b\dot{x}_2), & a\dot{y}_1 - b\dot{x}_1 &= -2(a\dot{y}_1 - b\dot{x}_1). \end{aligned}$$

Multiplying the first equation by  $x_2$ , the second by  $x_1$ , the third by  $y_2$ , and the fourth by  $y_1$ , then adding, yields

$$\begin{aligned} &\dot{a}(x_1^2 + x_2^2 + y_1^2 + y_2^2) + \dot{b}(y_2x_2 + y_1x_1 - x_2y_2 - x_1y_1) \\ &= -2[a(x_2\dot{x}_2 + x_1\dot{x}_1 + y_2\dot{y}_2 + y_1\dot{y}_1) + b(x_2\dot{y}_2 + x_1\dot{y}_1 - y_2\dot{x}_2 - y_1\dot{x}_1)] \\ &\iff \dot{a} = -2b(y_2\dot{x}_2 + y_1\dot{x}_1 - x_2\dot{y}_2 - x_1\dot{y}_1) \iff \dot{a} = 0, \end{aligned}$$

where we have used the horizontality condition  $y_2\dot{x}_2 + y_1\dot{x}_1 - x_2\dot{y}_2 - x_1\dot{y}_1 = 0$  (see Proposition 3.2), the constraint  $x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1$ , and its derivative  $x_1\dot{x}_1 + x_2\dot{x}_2 + y_1\dot{y}_1 + y_2\dot{y}_2 = 0$ .

In a similar way, multiplying the first equation by  $y_2$ , the second by  $y_1$ , the third by  $-x_2$ , and the fourth by  $-x_1$ , then adding, yields  $\dot{b} = 0$ . We arrived at the following result, which provides four constants of motion (three angles and the energy).

**Theorem 5.1** *Given two points  $P, Q \in \mathbb{S}^3$ , let  $\gamma$  be a length minimizing curve among all the horizontal curves which join  $P$  and  $Q$ . Then*

- (i)  $|\dot{\gamma}(s)|$  is constant along the curve;

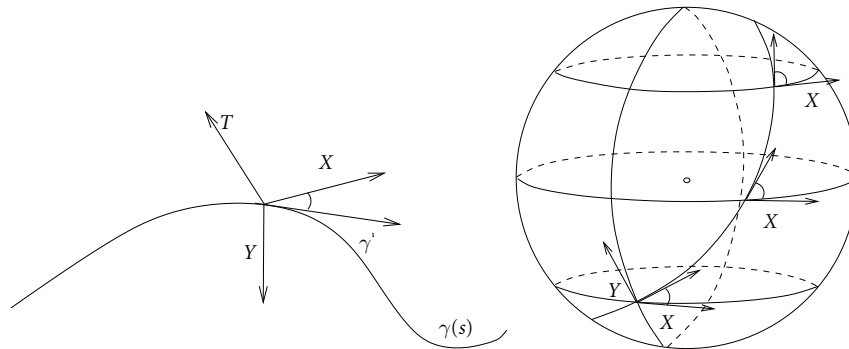


Figure 2: The angle between the velocity  $\dot{\gamma}(s)$  and the vectors  $X$  and  $Y$  is constant.

(ii) *the angles between the velocity  $\dot{\gamma}$  and the vector fields  $X, Y$  and  $T$  are constant along the curve.*

**Proof** (i) From  $\dot{a} = \dot{b} = 0$  it follows that  $a(s) = \text{const}$  and  $b(s) = \text{const}$  and hence  $|\dot{\gamma}(s)| = \sqrt{a(s)^2 + b(s)^2} = \text{const}$ .

Using (i), we have

$$\begin{aligned} \cos \widehat{\dot{\gamma}, X} &= \frac{\langle \dot{\gamma}, X \rangle}{|\dot{\gamma}| |X|} = \frac{a(s)}{\sqrt{a(s)^2 + b(s)^2}} = \text{const}, \\ \cos \widehat{\dot{\gamma}, Y} &= \frac{\langle \dot{\gamma}, Y \rangle}{|\dot{\gamma}| |Y|} = \frac{b(s)}{\sqrt{a(s)^2 + b(s)^2}} = \text{const}, \\ \cos \widehat{\dot{\gamma}, T} &= \frac{\langle \dot{\gamma}, T \rangle}{|\dot{\gamma}| |T|} = 0 \implies \widehat{\dot{\gamma}, T} = \pi/2. \end{aligned}$$

■

**Remark 5.2** This case when the Lagrange multiplier vanishes yields in the case of the Heisenberg group lines parallel to the  $x$ -space. These are particular cases of subRiemannian geodesics.

### 5.2 The Case $\lambda(s) \neq 0$

The Lagrangian in this case is  $L = \frac{1}{2}a^2 + \frac{1}{2}b^2 + \lambda(\dot{x}_1 y_1 + \dot{x}_2 y_2 - \dot{y}_1 x_1 - \dot{y}_2 x_2)$ . The Euler–Lagrange system (5.1)–(5.2) becomes

$$\begin{aligned} \dot{a}x_2 + \dot{b}y_2 &= -2(a\dot{x}_2 + b\dot{y}_2 + \lambda\dot{y}_1) - \lambda y_1 \\ \dot{a}x_1 + \dot{b}y_1 &= -2(a\dot{x}_1 + b\dot{y}_1 - \lambda\dot{y}_2) + \lambda y_2 \\ \dot{a}y_2 - \dot{b}x_2 &= -2(a\dot{y}_2 - b\dot{x}_2 + \lambda\dot{x}_1) - \lambda x_1 \\ \dot{a}y_1 - \dot{b}x_1 &= -2(a\dot{y}_1 - b\dot{x}_1 - \lambda\dot{x}_2) + \lambda x_2. \end{aligned}$$

Multiplying the first equation by  $x_2$ , the second by  $x_1$ , the third by  $y_2$  and the fourth by  $y_1$ , then adding, yields

$$\begin{aligned} & \underbrace{a(x_1^2 + x_2^2 + y_1^2 + y_2^2)}_{=1} + \underbrace{b(y_2x_2 + y_1x_1 - x_2y_2 - x_1y_1)}_{=0} \\ &= -2 \left( \underbrace{a(x_2\dot{x}_2 + x_1\dot{x}_1 + y_2\dot{y}_2 + y_1\dot{y}_1)}_{=0} + \underbrace{b(x_2\dot{y}_2 + x_1\dot{y}_1 - y_2\dot{x}_2 - y_1\dot{x}_1)}_{=0} \right. \\ & \quad \left. + \lambda \underbrace{(x_2\dot{y}_1 - x_1\dot{y}_2 + \dot{x}_1y_2 - \dot{x}_2y_1)}_{=b} \right) + \lambda \underbrace{(-x_2y_1 + x_1y_2 - x_1y_2 + x_2y_1)}_{=0} \\ \Leftrightarrow & \quad \dot{a} = -2\lambda b. \end{aligned}$$

In a similar way, multiplying the first equation by  $y_2$ , the second by  $y_1$ , the third by  $-x_2$  and the fourth by  $-x_1$ , then adding, we obtain

$$\begin{aligned} & \underbrace{a(x_2y_2 + x_1y_1 - x_2y_2 - x_1y_1)}_{=0} + \underbrace{b(y_2^2 + y_1^2 + x_2^2 + x_1^2)}_{=1} \\ &= -2 \left( \underbrace{a(\dot{x}_2y_2 + \dot{x}_1y_1 - \dot{y}_2x_2 - \dot{y}_1x_1)}_{=0} + b(y_2\dot{y}_2 + y_1\dot{y}_1 + x_2\dot{x}_2 + x_1\dot{x}_1) \right. \\ & \quad \left. + \lambda \underbrace{(\dot{y}_1y_2 - \dot{y}_2y_1 - \dot{x}_1x_2 + \dot{x}_2x_1)}_{=-a} \right) + \lambda \underbrace{(-y_1y_2 + y_2y_1 - x_1x_2 - x_2x_1)}_{=0} \\ \Leftrightarrow & \quad \dot{b} = 2\lambda a. \end{aligned}$$

Hence  $a$  and  $b$  satisfy the following system

$$(5.3) \quad \dot{a} = -2\lambda b$$

$$(5.4) \quad \dot{b} = 2\lambda a.$$

Multiplying the first equation by  $a$  and the second by  $b$  and add, yields  $a\dot{a} + b\dot{b} = 0$ , i.e.,  $a^2 + b^2 = r^2$  constant along the geodesics. This means that the energy is preserved and the velocity of the geodesics have constant length. Let  $\theta(s)$  be a function such that  $a(s) = r \cos \theta(s)$ , and  $b(s) = r \sin \theta(s)$ . Substituting in the equations (5.3) and (5.4), we obtain  $\sin \theta(s) \dot{\theta}(s) = 2\lambda(s) \sin \theta(s)$  and  $\cos \theta(s) \dot{\theta}(s) = 2\lambda(s) \cos \theta(s)$ , which after dividing the first equation by  $\sin \theta$  and the second by  $\cos \theta$  yields  $\dot{\theta}(s) = 2\lambda(s)$ . The solution is  $\theta(s) = 2\Lambda(s) + \theta_0$ , where  $\Lambda(s) = \int_0^s \lambda(u) du$ . Hence

$$(5.5) \quad a(s) = r \cos (2\Lambda(s) + \theta_0),$$

$$(5.6) \quad b(s) = r \sin (2\Lambda(s) + \theta_0).$$

Since  $\Lambda(0) = 0$ , then  $a(0) = r \cos \theta_0$  and  $b(0) = r \sin \theta_0$ , which provides  $\theta_0 = \tan^{-1}(b(0)/a(0))$ .

Let  $\gamma(s)$  be a subRiemannian geodesic. Since  $\{X, Y\}$  are orthonormal,

$$a = \langle \dot{\gamma}, X \rangle = \underbrace{|\dot{\gamma}|}_{=r} \cdot \underbrace{|X|}_{=1} \cos(\widehat{\dot{\gamma}, X}) = r \cos(\widehat{\dot{\gamma}, X}),$$

$$b = \langle \dot{\gamma}, Y \rangle = \underbrace{|\dot{\gamma}|}_{=r} \cdot \underbrace{|Y|}_{=1} \cos(\widehat{\dot{\gamma}, Y}) = r \cos(\widehat{\dot{\gamma}, Y}) = r \sin\left(\frac{\pi}{2} - \widehat{\dot{\gamma}, Y}\right).$$

Comparing with (5.5) and (5.6), we arrive at the following result.

**Proposition 5.3** *The angles under which the subRiemannian geodesics intersect the integral curves of the vector fields  $X$  and  $Y$  are given by the formulas*

$$\widehat{\dot{\gamma}(s), X_{\gamma(s)}} = 2\Lambda(s) + \theta_0, \quad \widehat{\dot{\gamma}(s), Y_{\gamma(s)}} = \frac{\pi}{2} - \widehat{\dot{\gamma}(s), X_{\gamma(s)}}.$$

The main goal now is to find  $\Lambda(s)$ . In order to do this we shall construct an equivalent variational problem. We need the following result which writes the energy in a more friendly way.

**Proposition 5.4** *If  $(x_1(s), x_2(s), y_1(s), y_2(s))$  is a subRiemannian geodesic, then we have  $\dot{x}_1^2(s) + \dot{x}_2^2(s) + \dot{y}_1^2(s) + \dot{y}_2^2(s) = a^2(s) + b^2(s)$ .*

**Proof** Using the definitions of  $a$  and  $b$ , the horizontality condition, and the holonomic constraint, we can write the following system

$$\begin{aligned} \dot{x}_1 x_2 + y_1 \dot{y}_2 - \dot{x}_2 x_1 - y_1 y_2 &= a(s), & \dot{x}_1 y_2 + x_2 \dot{y}_1 - \dot{x}_2 y_1 - y_2 x_1 &= b(s), \\ \dot{x}_1 y_1 + \dot{x}_2 y_2 - x_2 \dot{y}_2 - x_1 \dot{y}_1 &= 0, & \dot{x}_1 x_1 + \dot{x}_2 x_2 + \dot{y}_1 y_1 + \dot{y}_2 y_2 &= 0. \end{aligned}$$

This can be written in matrix form as

$$\underbrace{\begin{pmatrix} x_2 & -x_1 & -y_2 & y_1 \\ y_2 & -y_1 & x_2 & -x_1 \\ y_1 & y_2 & -x_1 & -x_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}}_{=M} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}.$$

Since  $\det M = 1$ ,  $M^{-1} = M^t$ ,  $M$  is an orthogonal matrix, and hence preserves the Euclidean length of vectors. It follows that  $\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2 = a^2 + b^2$  along any subRiemannian geodesic, which proves the statement. ■

**Remark 5.5** The above result holds because the Euclidean metric on  $\mathbb{R}^4$  restricted on  $\text{span}\{X, Y\}$  is the subRiemannian metric.

The next result will be useful later in the sequel, when we shall find the Lagrange multiplier.

**Proposition 5.6** *Let  $\gamma(s)$  be a subRiemannian geodesic. Then  $\langle \ddot{\gamma}(s), T_{\gamma(s)} \rangle = 0$ , for any  $s$ , i.e., the component of the acceleration along the missing direction  $T = [X, Y]$  vanishes.*

**Proof** Let  $\gamma(s) = (x_1(s), x_2(s), y_1(s), y_2(s))$ . Differentiating in the horizontality condition  $\dot{x}_1 y_1 + \dot{x}_2 y_2 - \dot{y}_1 x_1 - \dot{y}_2 x_2 = 0$  yields  $\ddot{x}_1 y_1 + \ddot{x}_2 y_2 - \ddot{y}_1 x_1 - \ddot{y}_2 x_2 = 0$ , which can be written as  $\langle (\ddot{x}_1, \ddot{x}_2, \ddot{y}_1, \ddot{y}_2), (y_1, y_2, -x_1, -x_2) \rangle = 0$  or  $\langle \ddot{\gamma}, T \rangle = 0$ , where

$$T = y_1 \partial_{x_1} + y_2 \partial_{x_2} - x_1 \partial_{y_1} - x_2 \partial_{y_2}. \quad \blacksquare$$

We shall consider another variational problem, where we replace the energy  $\frac{1}{2}(a^2 + b^2)$  by  $\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2)$  and consider the new Lagrangian

$$L^*(x, \dot{x}, y, \dot{y}) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2) + \lambda(s)(\dot{x}_1 y_1 + \dot{x}_2 y_2 - \dot{y}_1 x_1 - \dot{y}_2 x_2).$$

The Euler–Lagrange equations provided by the Lagrangian  $L^*$  describe the same subRiemannian geodesics as the equations associated with the initial Lagrangian  $L$ . The Euler–Lagrange equations for  $L^*$  are

$$\begin{aligned} \ddot{x}_1 &= 2\lambda \dot{y}_1 + \dot{\lambda} y_1, & \ddot{x}_2 &= 2\lambda \dot{y}_2 + \dot{\lambda} y_2, \\ \ddot{y}_1 &= -2\lambda \dot{x}_1 - \dot{\lambda} x_1, & \ddot{y}_2 &= -2\lambda \dot{x}_2 - \dot{\lambda} x_2. \end{aligned}$$

Multiplying the first equation by  $y_1$ , the second by  $y_2$ , the third by  $-x_1$ , and the fourth by  $-x_2$ , then adding, yields

$$\begin{aligned} y_1 \ddot{x}_1 + y_2 \ddot{x}_2 - x_1 \ddot{y}_1 - x_2 \ddot{y}_2 \\ = 2\lambda \underbrace{(y_1 \dot{y}_1 + y_2 \dot{y}_2 + x_1 \dot{x}_1 + x_2 \dot{x}_2)}_{=0} + \dot{\lambda} \underbrace{(y_1^2 + y_2^2 + x_1^2 + x_2^2)}_{=1}, \end{aligned}$$

which can be written as

$$\dot{\lambda}(s) = y_1 \ddot{x}_1 + y_2 \ddot{x}_2 - x_1 \ddot{y}_1 - x_2 \ddot{y}_2 = \langle (\ddot{x}_1, \ddot{x}_2, \ddot{y}_1, \ddot{y}_2), (y_1, y_2, -x_1, -x_2) \rangle = \langle \ddot{\gamma}, T \rangle.$$

Using Proposition 5.6, we obtain the following result.

**Proposition 5.7** *The Lagrange multiplier  $\lambda(s)$  is constant along any subRiemannian geodesics.*

If  $\lambda = c/2$ , with  $c$  constant, then Proposition 5.3 yields  $\widehat{\dot{\gamma}(s), X_{\gamma(s)}} = cs + \theta_0$ . This leads to the following characterization of the subRiemannian geodesics.

**Proposition 5.8** *The unit speed curve  $\gamma(s)$  is a subRiemannian geodesic on  $(\mathbb{S}^3, X, Y)$  if and only if the angle between its velocity and the direction of the vector field  $X$  increases linearly in  $s$ .*



### 5.3 The Group $SU(2)$

The special unitary group

$$SU(2) = \{x_1 + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{k}; x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1\}$$

is also a Lie group, which can be identified with the sphere  $\mathbb{S}^3$  by the isomorphism

$$\varphi: SU(2) \rightarrow \mathbb{S}^3, \quad \varphi(x_1I + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{k}) = (x_1, x_2, y_1, y_2).$$

Since

$$\begin{aligned} X_q &= x_2\partial_{x_1} - x_1\partial_{x_2} - y_2\partial_{y_1} + y_1\partial_{y_2} \\ &= (x_2, -x_1, -y_2, y_1) \cdot \nabla_q = (x_2 - x_1\mathbf{i} - y_2\mathbf{j} + y_1\mathbf{k}) \cdot \nabla_q \\ &= -(x_1I + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{k})\mathbf{i} \cdot \nabla_q = -(x_1, x_2, y_1, y_2)\mathbf{i} \cdot \nabla_q \\ &= -\langle (q\mathbf{i}), \nabla_q \rangle, \\ Y_q &= y_2\partial_{x_1} - y_1\partial_{x_2} + x_2\partial_{y_1} - x_1\partial_{y_2} \\ &= (y_2, -y_1, x_2, -x_1) \cdot \nabla_q = (y_2I - y_1\mathbf{i} + x_2\mathbf{j} - x_1\mathbf{k}) \cdot \nabla_q \\ &= -(x_1I + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{k})\mathbf{k} \cdot \nabla_q = -(x_1, x_2, y_1, y_2)\mathbf{k} \cdot \nabla_q \\ &= -\langle (q\mathbf{k}), \nabla_q \rangle, \\ T_q &= y_1\partial_{x_1} + y_2\partial_{x_2} - x_1\partial_{y_1} - x_2\partial_{y_2} \\ &= (y_1, y_2, -x_1, -x_2) \cdot \nabla_q = (y_1I + y_2\mathbf{i} - x_1\mathbf{j} - x_2\mathbf{k}) \cdot \nabla_q \\ &= -(x_1I + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{k})\mathbf{j} \cdot \nabla_q = -(x_1, x_2, y_1, y_2)\mathbf{j} \cdot \nabla_q \\ &= -\langle (q\mathbf{j}), \nabla_q \rangle, \end{aligned}$$

where  $q = (x_1, x_2, y_1, y_2)$ ,  $\nabla_q = (\partial_{x_1}, \partial_{x_2}, \partial_{y_1}, \partial_{y_2})$  and  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are defined in (2.1). It follows that the left invariant vector fields  $X, Y, T$ , which span the Lie algebra of  $\mathbb{S}^3$  correspond on  $SU(2)$  to  $-\mathbf{i}, -\mathbf{k}, -\mathbf{j}$ , respectively.

**Remark 5.9** An alternative approach to calculate geodesics is to use the Hamiltonian method. This method is very complicated in the case of studying geodesics on  $\mathbb{S}^3$ , but we can obtain some information from the superficial analysis. To match the notation with those we used for Lagrangian approach we shall use the representation (3.2) for the matrices (2.1). Then the vector fields  $X, Y$  and  $T$  can be written in the form

$$X = \langle -qI_1, \nabla_q \rangle, \quad Y = \langle -qI_3, \nabla_q \rangle, \quad T = \langle -qI_2, \nabla_q \rangle$$

where  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^4$ . Define the Hamiltonian function

$$H = \frac{1}{2}(X^2 + Y^2) = \frac{1}{2}(\langle qI_1, \xi \rangle^2 + \langle qI_3, \xi \rangle^2).$$

Then the Hamiltonian system is the following:

$$(5.7) \quad \begin{aligned} \dot{q} &= \frac{\partial H}{\partial \xi} \implies \dot{q} = \langle qI_1, \xi \rangle \cdot \langle qI_1 \rangle + \langle qI_3, \xi \rangle \cdot \langle qI_3 \rangle, \\ \dot{\xi} &= -\frac{\partial H}{\partial q} \implies \dot{\xi} = \langle qI_1, \xi \rangle \cdot \langle \xi I_1 \rangle + \langle qI_3, \xi \rangle \cdot \langle \xi I_3 \rangle. \end{aligned}$$

A geodesic is the projection of the solution of the Hamiltonian system to the  $q$ -space. Since  $\langle qI_1, q \rangle = \langle qI_2, q \rangle = \langle qI_3, q \rangle = 0$ , multiplying the first equation of (5.7) by  $q$  we get  $\langle \dot{q}, q \rangle = 0 \implies |q|^2 = \text{const}$ . We conclude that the solution of the Hamiltonian system belongs to a sphere. Taking the constant equal to 1, we get geodesics belonging to  $\mathbb{S}^3$ . Multiplying the first equation of (5.7) by  $qI_2$ , we get

$$(5.8) \quad \langle \dot{q}, qI_2 \rangle = 0,$$

by the role of multiplication between  $I_1$ ,  $I_2$ , and  $I_3$ . The reader can easily recognize the horizontality condition  $\langle \dot{x}, y \rangle = \langle x, \dot{y} \rangle$  in (5.8). It means that the solution of the Hamiltonian system is a horizontal curve. Multiplying the first equation of (5.7) by  $qI_1$  or by  $qI_3$ , we get  $\langle \xi, qI_1 \rangle = \langle \dot{q}, qI_1 \rangle$  and  $\langle \xi, qI_3 \rangle = \langle \dot{q}, qI_3 \rangle$ . We see that the Hamiltonian function can also be written in the form

$$H = \frac{1}{2} (\langle qI_1, \xi \rangle^2 + \langle qI_3, \xi \rangle^2) = \frac{1}{2} (\langle qI_1, \dot{q} \rangle^2 + \langle qI_3, \dot{q} \rangle^2).$$

If we multiply the first equation of (5.7) by  $\dot{q}$ , we then get

$$(5.9) \quad |\dot{q}|^2 = \langle qI_1, \xi \rangle^2 + \langle qI_3, \xi \rangle^2 = \langle qI_1, \dot{q} \rangle^2 + \langle qI_3, \dot{q} \rangle^2 = 2H.$$

Thus the Hamiltonian function gives the kinetic energy  $H = \frac{|\dot{q}|^2}{2}$  and it is a constant along the geodesics. Notice that  $\langle qI_1, \dot{q} \rangle = a$  and  $\langle qI_3, \dot{q} \rangle = b$  by (3.3) and (3.4). We conclude  $|\dot{q}|^2 = a^2 + b^2$  from (5.9) that corresponds to Proposition 5.4. This is only the beginning of a long-term project. Many problems remain open. For example, given any two points on  $\mathbb{S}^3$ , how many geodesics connect them? Is there any abnormal minimizer in this case? What is the action function? What is the volume element which is the solution of a transport equation? We will answer these questions in a forthcoming paper.

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*Department of Mathematics, Eastern Michigan University, Ypsilanti, MI, 48197, U.S.A.*  
*e-mail:* ocalin@emich.edu

*Department of Mathematics, Georgetown University, Washington, D.C., 20057, U.S.A.*

and

*National Centre for Theoretical Sciences, Mathematics Division, National Tsing Hua University, Hsinchu, 30013, Taiwan, ROC*  
*e-mail:* chang@georgetown.edu

*Department of Mathematics, University of Bergen, Johannes Brunsgate 12, Bergen 5008, Norway*  
*e-mail:* irina.markina@uib.no